Some remarks on expectations

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In [3] the notion of a $\mathcal{G}$-finite (see below) von Neumann algebra $\mathfrak{A}$ is developed and gives rise to an expectation which is a generalization of the concept of the center trace [2, III, § 5]. In § 2 we discuss ultraweakly closed ideals invariant under a group of automorphisms in a $\mathcal{G}$-finite algebra and certain normal state which serve to replace characters [2, p. 275]. We then remove a restriction from one of STØRMER's result [5] on expectations and examine the effect of this expectation on characterizing certain ideals.

1. In this paragraph we discuss consequences of the expectation $A \to A^{\mathcal{G}}$ as given in [3].

Definition. If $\mathfrak{A}$ is a von Neumann algebra and $\{\alpha_g | g \in \mathcal{G}\}$ is a group of automorphisms acting on $\mathfrak{A}$, then $\mathfrak{A}$ is said to be $\mathcal{G}$-finite if whenever $A \in \mathfrak{A}^+$, $A \neq 0$, there exists an invariant, normal state ϱ such that $\varrho(A) \neq 0$.

In [3] it is shown that if $\mathfrak{A}$ is $\mathcal{G}$-finite and $\mathcal{K}(T, \mathcal{G})$ equals the strong closure of $co\{\alpha_g(T) | g \in \mathcal{G}\}$ then $\mathcal{K}(T, \mathcal{G}) \cap \mathfrak{A}^{\mathcal{G}}$ contains a unique point, where $\mathfrak{A}^{\mathcal{G}}$ is the von Neumann algebra of elements fixed by all α_g. This is then used to define the faithful normal map $T \to T^{\mathcal{G}}$ [3, p. 240].

Proposition 1. *Let $\mathfrak{A}$ be a $\mathcal{G}$-finite and suppose $\mathfrak{m}$ is an ultraweakly closed ideal in $\mathfrak{A}$ invariant under the $\{\alpha_g\}$. Then $\mathfrak{m} \cap \mathfrak{A}^{\mathcal{G}} = \{T^{\mathcal{G}} | \ T \in \mathfrak{m}\}$. If $\mathfrak{A}^{\mathcal{G}} \supseteqq \mathfrak{Z}$, every two-sided ultraweakly closed ideal is invariant.*

Proof. Let $\mathfrak{m}$ be u.w. closed and suppose $\mathfrak{m}$ is a left ideal. $\mathfrak{m} = \mathfrak{A}E$ [2, p. 45] with E a unique projection in $\mathfrak{A}$. Since $\alpha_g(\mathfrak{m}) = \mathfrak{m}$ we have $\alpha_g(E) = E$ by uniqueness. Let $T \in \mathfrak{m} \cap \mathfrak{A}^{\mathcal{G}}$ then $T = T^{\mathcal{G}}$ [3, p. 241]. Conversely suppose $T \in \mathfrak{m}$ then $T = SE$ so $T^{\mathcal{G}} = (SE)^{\mathcal{G}} = S^{\mathcal{G}} E \in \mathfrak{m}$. Moreover $(T^{\mathcal{G}})^{\mathcal{G}} = T^{\mathcal{G}}$ so $T^{\mathcal{G}} \in \mathfrak{m} \cap \mathfrak{A}^{\mathcal{G}}$ [3, p. 240]. The last statement follows from the previous remarks and the fact that any such ideal looks like $\mathfrak{A}z$ with $z \in \mathfrak{Z}$ [2, p. 45].

Remark 1. If we suppose $\mathfrak{A}^{\mathcal{G}} \supseteqq \mathfrak{Z}$ then if $\mathfrak{m} = \mathfrak{A}z$ is an ultraweakly closed two-sided ideal in $\mathfrak{A}$ then clearly $\mathfrak{A}^{\mathcal{G}} z$ ($= \mathfrak{m} \cap \mathfrak{A}^{\mathcal{G}}$) is one in $\mathfrak{A}^{\mathcal{G}}$. If $\mathfrak{n}$ is an ultraweakly closed two-sided ideal in $\mathfrak{A}^{\mathcal{G}}$ there is (if at all) at most one ultraweakly closed two-sided ideal in $\mathfrak{A}$ giving rise to $\mathfrak{n}$ in this manner. This is the case for if $\mathfrak{A}^{\mathcal{G}} z_1 = \mathfrak{A}^{\mathcal{G}} z_2$ then since $I \in \mathfrak{A}^{\mathcal{G}}$, $z_1 \leqq z_2$ and $z_2 \geqq z_1$, so $z_1 = z_2$. If we make the additional hypothesis that $\mathfrak{Z}$ is the center of $\mathfrak{A}^{\mathcal{G}}$, then the correspondence becomes complete and clearly preserves maximality.

The appropriate replacement for characters seems to be $\mathcal{G}$-clustering states, where

Definition. Let ϱ be an invariant state. ϱ is said to be $\mathcal{G}$-clustering [5, p. 18] if $\varrho(AB^{\mathcal{G}}) = \varrho(A)\varrho(B)$.

Proposition 2. *Let $\mathfrak{A}$ be $\mathcal{G}$-finite and suppose ϱ is a normal $\mathcal{G}$-clustering state. Then the support* [2, p. 61] *of ϱ is a minimal projection in $\mathfrak{A}^{\mathcal{G}}$, lying in the center of $\mathfrak{A}^{\mathcal{G}}$. Conversely to every minimal projection lying in the center of $\mathfrak{A}^{\mathcal{G}}$ there corresponds a unique normal $\mathcal{G}$-clustering state on $\mathfrak{A}$.*

Proof. Since ϱ is invariant we have that E_ϱ, the support of ϱ, belongs to $\mathfrak{A}^{\mathcal{G}}$ (this is noted in [3]). The map $A \to A^{\mathcal{G}}$ takes $\mathfrak{A}$ onto $\mathfrak{A}^{\mathcal{G}}$ thus ϱ restricted to $\mathfrak{A}^{\mathcal{G}}$ is a normal multiplicative state (for all normal invariant states ψ we have $\psi(A^{\mathcal{G}}) = \psi(A)$ [3, p. 240]). It is now clear that E_ϱ is also the support of ϱ restricted to $\mathfrak{A}^{\mathcal{G}}$ and thus by a result of PLYMEN [4] is minimal in $\mathfrak{A}^{\mathcal{G}}$ and lies in the center of $\mathfrak{A}^{\mathcal{G}}$.

Conversely suppose E belongs to the center of $\mathfrak{A}^{\mathcal{G}}$ and is minimal in $\mathfrak{A}^{\mathcal{G}}$. Then [4] there exists a unique, normal, multiplicative state $\bar{\varrho}$ on $\mathfrak{A}^{\mathcal{G}}$ whose support is E. We then define $\varrho(A) = \bar{\varrho}(A^{\mathcal{G}})$. ϱ is normal by the normality of $A \to A^{\mathcal{G}}$. Further $\varrho(\alpha_g(A)) = \bar{\varrho}([\alpha_g(A)]^{\mathcal{G}}) = \bar{\varrho}(A^{\mathcal{G}}) = \varrho(A)$ [3, p. 240], thus ϱ is invariant. For $A, B \in \mathfrak{A}$ we have $\varrho(AB^{\mathcal{G}}) = \bar{\varrho}((AB^{\mathcal{G}})^{\mathcal{G}}) = \bar{\varrho}(A^{\mathcal{G}} B^{\mathcal{G}}) = \bar{\varrho}(A^{\mathcal{G}})\bar{\varrho}(B^{\mathcal{G}}) = \varrho(A)\varrho(B)$ i.e. ϱ is $\mathcal{G}$-clustering on $\mathfrak{A}$. The uniqueness follows from the fact that the state $\bar{\varrho}$ is uniquely determined by E and the fact that a normal invariant state is uniquely determined by its values on $\mathfrak{A}^{\mathcal{G}}$ [3, p. 242].

Under appropriate conditions we obtain the analogue of [Proposition 5. 2, p. 277].

Proposition 3. *Suppose $\mathfrak{A}$ is $\mathcal{G}$-finite and $\mathfrak{Z}$ is the center of $\mathfrak{A}^{\mathcal{G}}$. Then there exists a one-to-one correspondence between maximal two-sided ultraweakly closed ideals in $\mathfrak{A}$ and normal $\mathcal{G}$-clustering states on $\mathfrak{A}$.*

Proof. By Remark 1 it suffices to exhibit a correspondence with ideals in $\mathfrak{A}^{\mathcal{G}}$.

We consider the kernel of $\varrho|\mathfrak{A}^{\mathcal{G}}$. By the $\mathcal{G}$-clustering and a result of PLYMEN, this equals $\mathfrak{A}^{\mathcal{G}}(I - E_\varrho)$ which is a two sided ultraweakly closed ideal in $\mathfrak{A}^{\mathcal{G}}$. Suppose there exists $\mathfrak{m}$ with $\mathfrak{A}^{\mathcal{G}} \supset \mathfrak{m} \supset \mathfrak{A}^{\mathcal{G}}(I - E_\varrho)$, $\mathfrak{m}$ two sided u.w. closed ideal in $\mathfrak{A}^{\mathcal{G}}$. Since $\mathfrak{Z}$ is the center of $\mathfrak{A}^{\mathcal{G}}$, $\mathfrak{m} = \mathfrak{A}^{\mathcal{G}} z$ with $z \in \mathfrak{Z}$. Thus $I > z > I - E_\varrho$ and $O < I - z < E_\varrho$ which contradicts the minimality of E_ϱ. Thus $\mathfrak{A}^{\mathcal{G}}(I - E_\varrho)$ is maximal.

2. We now discuss a result of STØRMER [5] and obtain a more explicit ideal correspondence.

Definition. Let $\mathfrak{A}$ be a von Neumann algebra and $\mathfrak{B}$ a von Neumann subalgebra of $\mathfrak{A}$. Then a positive linear map Φ of $\mathfrak{A}$ onto $\mathfrak{B}$ is called an expectation if $\Phi(I)=I$ and $\Phi(BA)=B\Phi(A)$ for $B\in\mathfrak{B}$ and $A\in\mathfrak{A}$.

In [5] STØRMER constructs an expectation onto a subalgebra of $\mathfrak{Z}$ under the condition that the algebra is acted upon by a large group of automorphisms given by unitaries.

Definition. Let $\mathscr{U}$ be a group of unitaries giving rise to automorphisms of $\mathfrak{A}$, a C^*-algebra. Then $\mathscr{U}$ is said to be a large group of automorphisms if $co(UAU^{-1}: U\in\mathscr{U})\cap\mathfrak{A}' \neq \emptyset$ for $A\in\mathfrak{A}$ (the closure is in the strong topology).

We show that one can obtain a normal invariant expectation onto the same subalgebra of $\mathfrak{Z}$ without this assumption. We do not however obtain the full strength of STØRMER's results.

Theorem 1. *Let $\mathfrak{A}$ be a von Neumann algebra acted upon by a group of automorphisms $\{\alpha_g\}$. Set $\mathfrak{B}=\mathfrak{A}^g\cap\mathfrak{Z}$ and suppose there exists a normal state, ϱ, invariant under the $\{\alpha_g\}$ which is faithful on $\mathfrak{B}$. Then there exists an expectation, Φ taking $\mathfrak{A}$ onto $\mathfrak{B}$ such that*

(i) $\varrho(B\Phi(X))=\varrho(BX)$ $B\in\mathfrak{B}$ *and* $X\in\mathfrak{A}$,

(ii) $\Phi(\alpha_g(A))=\Phi(A)$,

(iii) Φ *is normal,*

(iv) *if* $\mathfrak{m}$ *is an ultraweakly closed two-sided invariant ideal and* $X\in\mathfrak{m}$, *then* $\Phi(X)\in\mathfrak{m}$.

Proof. The existence of an expectation with property (i) is a special case of a result of DE KORVIN [1]. One first realizes $\mathfrak{B}$ as a Hilbert algebra with inner product $(A, B)=\varrho(B^*A)$. Then one defines $\sigma(B)=\varrho(BX)$ for $X\in\mathfrak{A}^+$, $B\in\mathfrak{B}$. Riesz' lemma and a standard Hilbert algebra argument yield the desired result.

From (i)

$$\varrho\, B\Phi(\alpha_g(X))=\varrho(B\alpha_g(X))=\varrho(\alpha_g(BX))=\varrho(BX)=\varrho(B\Phi(X)) \qquad B\in\mathfrak{B}.$$

Thus $\Phi(\alpha_g(X))=\Phi(X)$ since ϱ is faithful on $\mathfrak{B}$.

Normality follows as in [5, p. 10] since the map Φ is positive.

Now let $\mathfrak{m}$ be an ultraweakly closed two-sided ideal in $\mathfrak{A}$. Then $\mathfrak{m}=\mathfrak{A}z$. If $\mathfrak{m}$ is invariant then $\alpha_g(\mathfrak{m})=\mathfrak{m}$. By the uniqueness of z, $\alpha_g(z)=z$ for all g and $z\in\mathfrak{B}$. We must show that if $X\in\mathfrak{m}$ then $\Phi(X)\in\mathfrak{m}$ or equivalently $z\Phi(X)=\Phi(X)$. But for $X\in\mathfrak{m}$

$$\varrho(Bz\Phi(X))=\varrho(BzX)=\varrho(BX)=\varrho(B\Phi(X)).$$

An appropriate choice of B gives $z\Phi(X)=\Phi(X)$.

The expectation Φ' that STØRMER constructs has the nice property that it preserves normal invariant states in that if ψ is any such $\psi\circ\Phi'=\psi$. While this is not necessarily true for the above expectation, nevertheless we have

Corollary. *Let $\mathfrak{A}$ be as in the theorem. If ψ is an invariant, multiplicative, normal state on $\mathfrak{A}$ then $\psi\circ\Phi=\psi$.*

Proof. Since ψ is invariant so is $\ker\psi$ (the kernel of ψ), which PLYMEN has shown [4] is an ultraweakly closed two-sided ideal. By (iv) of the theorem $\Phi(\ker\psi)\subseteqq$ $\subseteqq\ker\psi$ so $\ker\psi\circ\Phi\supseteqq\ker\psi$. Thus $\psi\circ\Phi=\lambda\psi$. However $\Phi(I)=I$ so $\lambda=1$ and $\psi\circ\Phi=\psi$.

In this case we can, following [2, p. 273], obtain a characterization of the ideal $\mathfrak{m}$ corresponding to an ideal $\mathfrak{n}$ in $\mathfrak{B}$.

Proposition 4. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be as in Theorem 1. Let $\mathfrak{n}$ be a two-sided ultraweakly closed ideal in $\mathfrak{B}$. Let $\mathfrak{m}=\{T\in\mathfrak{A}|\Phi(T_1TT_2)\in\mathfrak{n}$ for $T_1, T_2\in\mathfrak{A}\}$. Then $\mathfrak{m}$ is the largest two-sided ideal of $\mathfrak{A}$ that $\mathfrak{m}\cap\mathfrak{B}\subseteqq\mathfrak{n}$, $\mathfrak{m}$ is invariant and ultraweakly closed.* $\mathfrak{m}\cap\mathfrak{B}=\mathfrak{n}$.

Proof. Linearity and ultraweak continuity [2, p. 56] of Φ imply that $\mathfrak{m}$ is a two-sided ultraweakly closed ideal. Suppose now that $T\in\mathfrak{m}\cap\mathfrak{B}$. Then $\Phi(T)=T\in\mathfrak{n}$ so $\mathfrak{m}\cap\mathfrak{B}\subseteqq\mathfrak{n}$. If $T\in\mathfrak{n}$ then for $T_1, T_2\in\mathfrak{A}$ we have $\Phi(T_1TT_2)=\Phi(TT_1T_2)=$ $=T\Phi(T_1T_2)\in\mathfrak{n}$, i.e. $T\in\mathfrak{m}\cap\mathfrak{B}$. — $\mathfrak{m}$ is invariant for if $T\in\mathfrak{m}$ then by (ii) of Theorem 1

$$\Phi(T_1\alpha_g(T)T_2)=\Phi(\alpha_g(T_1'TT_2'))=\Phi(T_1'TT_2')\in\mathfrak{n}.$$

Suppose $\mathfrak{m}'$ is another ultraweakly closed two-sided invariant ideal in $\mathfrak{A}$ with $\mathfrak{m}'\cap\mathfrak{B}\subseteqq\mathfrak{n}$. By (iv) of Theorem 1 we have $\mathfrak{m}'\cap\mathfrak{B}=\{\Phi(T)|\,T\in\mathfrak{m}'\}$, i.e. $T\in\mathfrak{m}'$ gives $\Phi(T)\in\mathfrak{n}$. Since $\mathfrak{m}'$ is an ideal $\Phi(T_1TT_2)\in\mathfrak{n}$, i.e. $\mathfrak{m}'\subseteqq\mathfrak{m}$.

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