Some results and problems in the theory of additive functions

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1. A function $f(n)$ of a positive integer is said to be restrictedly additive (or, simply, additive) if $(n_1, n_2)=1$ implies $f(n_1 n_2)=f(n_1)+f(n_2)$. If this equation is satisfied for any pair of integers n_1, n_2, then we say that $f(n)$ is completely (or totally) additive.

P. ERDŐS [1] has proved the following two assertions.

(A) *If $f(n)$ is restrictedly additive and monotonic then it is a constant multiple of* $\log n$.

(B) *If $f(n)$ is restrictedly additive and $f(n+1)-f(n)\to 0$ $(n\to\infty)$ then it is a constant multiple of* $\log n$.

New proofs of these assertions have been given by several authors (for the references see for example [2]). Using the ideas of BESICOVITCH to the proof of (B) (see his paper [2]) the author proved in [3] the following assertion (C), which contains (A) and (B) as special cases and which was previously stated without proof by P. ERDŐS in [5]. This assertion was proved by A. MÁTÉ [4], too.

(C) *If $f(n)$ is restrictedly additive and*

$$\liminf_{n\to\infty}\bigl(f(n+1)-f(n)\bigr) \geqq 0$$

then it is a constant multiple of $\log n$.

Later the author proved in [6] the following generalization of (C).

(D) *If $f(n)$ is restrictedly additive and* $\liminf \Delta^k f(n)\geqq 0$ *for some integer $k\geqq 1$ where $\Delta^k f(n)$ denotes the kth difference of $f(n)$, then $f(n)$ is a constant multiple of* $\log n$.

The following assertion, which was proved in [7], is a generalization of (A).

(E) *If $f(n)$ and $g(n)$ are restrictedly additive functions and the function $h(n)=$ $=\max\bigl(f(n), g(n)\bigr)$ is increasing, then the following assertions hold:*

1) *$h(n)=c\log n+r(n)$ and $r(n)\to 0$ as $n\to\infty$. Furthermore $r(n)=0$, when all prime divisors of n are greater than a certain constant.*

2) *If $f(n)\geqq g(n)$ for almost every n, then*

$$f(n)=c\log n \quad \text{and} \quad g(n)=c\log n+\varepsilon(n),$$

where $\varepsilon(p^\alpha)\leqq 0$ for sufficiently large prime numbers p.

Let $S=\{p_1, p_2, \dots\}$ be the set of irregular primes p_i such that $\varepsilon(p_i^{\alpha_i})>0$ for some α_i. If S contains at least two elements then $\varepsilon(p_i^\beta)\leqq 0$ for every $p_i\in S$ and for β sufficiently large.

3) *If the set of n's satisfying the condition $f(n)\geqq g(n)$ has positive lower density, smaller than one, then $h(n)=c\log n$ $(n=1, 2, \dots)$. Furthermore $f(p^\alpha)=g(p^\alpha)=c\log p^\alpha$ $(\alpha=1, 2, \dots)$, with the exception of at most one prime.*

2. In this paper we deal with similar questions.

Let $p, p_1, \dots, q, q_1, \dots$ denote prime numbers.

We say that the subset P of prime numbers is the *support* of the additive function $l(n)$, if $l(p^\alpha)=0$ for $\alpha=1, 2, \dots$, when $p\notin P$, and $l(p^\alpha)\neq 0$ for at least one α, when $p\in P$. We say that $l(n)$ is a function of *finite support* if P contains finitely many elements only.

Let K be a fixed natural number. Let $f(n)$ and $g(n)$ be restrictedly additive functions satisfying the condition

$$g(n+K)-f(n)\to 0 \qquad (n\to\infty). \tag{2.1}$$

We prove the following

Theorem 1. *Under the assumption* (2. 1) *we have*

$$f(n) = c\log n+l_1(n), \tag{2.2}$$

$$g(n) = c\log n+l_2(n), \tag{2.3}$$

where $l_1(n)$, $l_2(n)$ are functions of finite support. Their support can contain only the prime divisors of K.

Furthermore, if $2^\alpha\|K$, then

$$\begin{cases} l_1(2^\beta) = l_2(2^\beta) & (\beta=1, \cdots, \alpha-1); \\ l_1(2^\alpha) = l_2(2^{\alpha+j}), \quad l_2(2^\alpha) = l_1(2^{\alpha+j}) & (j=1, 2, \cdots), \end{cases} \tag{2.4}$$

and if $p^\alpha\|K$ and $p\geqq 3$, then

$$\begin{cases} l_1(p^\beta) = l_2(p^\beta) & (\beta=1, \cdots, \alpha-1); \\ l_1(p^\alpha) = l_2(p^\alpha) = l_1(p^{\alpha+j}) = l_2(p^{\alpha+j}) & (j=1, 2, \cdots). \end{cases} \tag{2.5}$$

From (2. 4) *and* (2. 5) *it follows immediately, that $l_2(n+K) = l_1(n)$ for $n\geqq 1$. Conversely, if $f(n)$ and $g(n)$ satisfy the conditions stated in* (2. 2)—(2.5), *then* (2. 1) *holds.*

Proof. Let $H(n)=f(n)-g(n)$. First we deduce from (2. 1) that $H(n)=0$ for all n coprime to K. We distinguish the cases of K being even or odd.

a) Let $2^\alpha \| K$, $\alpha \geqq 1$. From (2. 1) it follows that $g(2n+2K)-f(2n) \to H(2)$ as n tends to infinity over odd n's. By (2. 1),

$$g(2n+2K)=f(2n+K)+o(1),\ f(2n)=g(2n+K)+o(1),$$

and thus $-H(2n+K)\to H(2)$ as $n\to\infty$, $2\nmid n$, i.e.

$$H(4k+K+2)\to -H(2) \qquad (k\to\infty).$$

According to the cases: $K+2\equiv 0 \pmod 4$, and $K+2\equiv 2 \pmod 4$ we have

$$H(4k)\to -H(2) \qquad (k\to\infty), \tag{2. 6$_1$}$$

$$H(2k+1)\to -2H(2) \qquad (k\to\infty). \tag{2. 6$_2$}$$

Let m be an arbitrary odd integer and n an infinite sequence of odd integers coprime to K. From (2. 6)$_1$ we have

$$-H(2)=\lim_{n\to\infty} H(4mn)=H(m)+\lim_{n\to\infty} H(4n)=H(m)-H(2).$$

Similarly, from (2. 6)$_2$

$$-2H(2)=\lim_{n\to\infty} H(mn)=H(m)+\lim_{n\to\infty} H(n)=H(m)-2H(2).$$

Hence $H(m)=0$.

b) Let now K be odd. We distinguish the subcases: 1) $K\equiv 1 \pmod 4$ and 2) $K\equiv -1 \pmod 4$. In the case 1) let $n\equiv 1 \pmod 4$, and in the case 2) let $n\equiv -1 \pmod 4$. Using similar arguments as in a) we have

$$H(2n+K)\to -g(4)+g(2)+f(2)=C,$$

i.e. $H(8k+l)\to C$ as $k\to\infty$ for at least one l among 1, 3, 5, 7. Hence it follows that $H(m)=0$ for every m in the residue class $\equiv 1 \pmod 8$. Indeed, if $m\equiv 1 \pmod 8$, then choosing an infinite sequence n_j, $\equiv l \pmod 8$, such that $(n_j, K)=1$, then $n_j m\equiv l \pmod 8$ and

$$C=\lim_{mn_j\to\infty} H(mn_j)=H(m)+\lim_{n_j\to\infty} H(n_j)=H(m)+C.$$

Using the additivity of $H(n)$ we obtain that $C=0$.

Let now m_1, m_2 be coprime integers, $m_1 m_2\equiv 1 \pmod 8$. Then $H(m_1)=-H(m_2)$. Hence it follows that $H(m)$ is constant in every reduced residue class mod 8. But this is possible only if $H(m)=0$ for every odd m.

Now we prove that $H(2^\alpha)=0$ for $\alpha=1,2,\ldots$. Let n be an integer such that $(n(n+K),3)=1$. Then using (2. 1) and that $H(3)=0$ we have

$$o(1)=g(n+K)-f(n)=g(3n+3K)-f(3n)=[g(3n+3K)-f(3n+2K)]+$$
$$+[f(3n+2K)-f(3n+K)]+[f(3n+K)-f(3n)]=o(1)-H(3n+2K)-H(3n+K)$$

i.e.

$$H(3n+K)+H(3n+2K)\to 0.$$

Since $(n(n+K),3)=1$ and $2^\beta\|3n+K$ hold for infinitely many n, we have $H(2^\beta)=0$. Consequently, $H(n)=0$ for every n coprime to K.

We need the following

Lemma 1. *If*

(2. 7) $$f(n+K)-f(n)\to 0$$

as $n\to\infty$ over the n's coprime to K, then $f(n)=c\log n$ holds whenever $(n,K)=1$.

Proof. Firstly we deduce that $f(n)$ is totally additive in the set $(n,K)=1$, i.e. that

(2. 8) $$f(nm)=f(n)+f(m),$$

whenever $(nm,K)=1$.

For this purpose let p be a prime or a prime power, $p\nmid K$, and let ν be a large integer. Let $\varepsilon>0$ and l be so large, that

(2. 9) $$|f(n+K)-f(n)|<\varepsilon \quad \text{if} \quad n\geqq p^l.$$

Then

$$f(p^\nu)=f(p^\nu+Kp)+\theta_1\varepsilon p=f(p)+f(p^{\nu-1}+K)+\theta_1\varepsilon p=$$
$$=f(p)+f(p^{\nu-1}+Kp)+\theta_2\varepsilon p=\cdots=(\nu-l+1)f(p^{l-1}+K)+\nu\theta_{\nu-l}\varepsilon p$$
$$(|\theta_1|\leqq 1,\ \cdots,\ |\theta_{\nu-l}|\leqq 1).$$

Hence it follows immediately that

$$\lim_{\nu\to\infty}\frac{f(p^\nu)}{\nu}=f(p),\quad \text{i.e.}\quad \lim_{\nu\to\infty}\frac{f(p^\nu)}{\log p^\nu}=\frac{f(p)}{\log p}.$$

Applying this relation for $p=q^\mu$ and for $p=q$ we have

$$\frac{f(q^\mu)}{\log q^\mu}=\lim_{\nu\to\infty}\frac{f(q^{\mu\nu})}{\log q^{\mu\nu}}=\lim_{\nu\to\infty}\frac{f(q^\nu)}{\log q^\nu}=\frac{f(q)}{\log q};$$

hence $f(q^\mu)=\mu f(q)$ follows. Consequently (2. 8) holds.

Let now p be a prime. We take N large, $(N,K)=1$, and write it in the form

$$N=a_0p^\nu+a_1p^{\nu-1}+\cdots+a_\nu,\quad 0\leqq a_j<p\qquad (j=0,\cdots,\nu),\quad a_0\geqq 1.$$

Using the inequality (2. 9) we have

$$f(kN) = f(Ka_0p^\nu + \cdots + Ka_\nu) = f(Ka_0p^\nu + \cdots + Ka_{\nu-1}p) + \theta_1 \varepsilon a_\nu =$$

$$= f(p) + f(Ka_0p^{\nu-1} + \cdots + Ka_{\nu-1}) + \theta_2 \varepsilon p = \cdots =$$

$$= (\nu - l + 1) f(p) + f(Ka_0p^{l-1} + \cdots + Ka_{l-1}) + \theta_{\nu-l}\varepsilon p \quad (|\theta_1| \leqq 1, \cdots, |\theta_{\nu-l}| \leqq 1).$$

Writing

$$M = \max_{m \leqq K_p l} |f(m)|$$

we have

$$f(N) = (\nu - l + 1) f(p) - f(K) + \theta M + \theta \varepsilon \nu p \quad (|\theta| \leqq 1).$$

Observing that $p^\nu \leqq N < p^{\nu+1}$ we get

$$\lim_{N\to\infty} \frac{\nu}{\log N} = \frac{1}{\log p}.$$

Hence

$$\lim_{\substack{N\to\infty \\ (N,K)=1}} \frac{f(N)}{\log N} = \frac{f(p)}{\log p}.$$

Let now N, M be arbitrary integers such that $(N, K) = (M, K) = 1$. Since

$$\frac{f(N)}{\log N} = \lim_{k\to\infty} \frac{f(N^k)}{\log N^k} = \lim_{k\to\infty} \frac{f(M^k)}{\log M^k} = \frac{f(M)}{\log M},$$

$f(N)/\log N$ is constant if $(N, K) = 1$. This finishes the proof of Lemma 1.

By this we proved that under the condition (2. 1) the functions $f(n)$ and $g(n)$ have the form (2. 2), (2. 3).

Since $c \log (n+K) - c \log n \to 0$, we have $l_2(n+K) - l_1(n) \to 0$ $(n \to \infty)$. Hence we deduce the relations (2. 4), (2. 5).

Let $2^\alpha \| K$, $\beta \leqq \alpha - 1$. Since there exist infinitely many n satisfying the conditions $n = 2^\beta m$, $(m, K) = 1$, $(n+K, K) = 2^\beta$, we have $l_2(n+K) = l_2(2^\beta)$, $l_1(n) = l_1(2^\beta)$. Consequently $l_1(2^\beta) = l_2(2^\beta)$. Choosing n such that $2^{\alpha+j} \| n$ $(j \geqq 1)$ and $(n, 2^{-\alpha}K) = 1$, we have $2^\alpha \| n+K$ and $(n+K, 2^{-\alpha}K) = 1$. Hence $l_2(2^\alpha) = l_1(2^{\alpha+j})$ follows. Let $2^{\alpha+j} \| n$, $(n, 2^{-\alpha}K) = 1$. Then $2^\alpha \| n+K$ and $(n+K, 2^{-\alpha}K) = 1$. Consequently $l_1(n) = l_1(2^{\alpha+j})$, $l_2(n+K) = l_2(2^\alpha)$. Hence we obtain that $l_1(2^\alpha) = l_2(2^{\alpha+j})$ $(j \geqq 1)$. This completes the proof of (2. 4).

The proof of (2. 5) is similar and can be omitted.

From (2. 4) and (2. 5) it follows immediately, that $l_2(n+K) = l_1(n)$ for $n = 1, 2, \ldots$ Consequently the relations (2. 2)—(2. 5) are sufficient to guarantee the fulfilment of (2. 1).

Remarks. 1) It would be interesting to prove the more general assertion: *If $f_i(n)$ $(i=0, \ldots, k)$ are additive functions satisfying the condition*

$$\sum_{i=0}^{k} f_i(n+i) \to 0 \quad (n \to \infty),$$

then

$$f_i(n) = c_i \log n + l_i(n) \qquad (i=0, \ldots, k),$$

where $l_i(n)$ have finite support. I am unable to prove this for $k \geqq 2$.

2) It seems probable that the following generalization of the conjecture of P. ERDŐS holds: *If $f(n)$ and $g(n)$ are additive functions such that $g(n+1)-f(n)$ is bounded, then $g(n) = c \log n + v(n), f(n) = c \log n + u(n)$, and $u(n)$, $v(n)$ are bounded.*

3. Now we investigate the class of additive functions satisfying

(3. 1) $$f(2n+1)-f(n) \to C \qquad (C \text{ is a constant}).$$

Theorem 2. *If $f(n)$ is a completely additive function satisfying* (3. 1), *then $f(n)=c \log n$, $c=C/\log 2$.*

Proof. Without loss of generality we may suppose $C=0$. Then we need to show that $f(n)=0$ identically.

Let N be a large integer, which we represent in the dyadical form:

(3. 2) $$N = 2^{\nu_1}+2^{\nu_2}+\cdots+2^{\nu_k} \quad (\nu_1 > \nu_2 > \cdots > \nu_k).$$

Let $\alpha(N)$ denote the length of this representation, i.e. $\alpha(N)=k$.

Using (3. 1) with $C=0$ and the total additivity of $f(n)$ we have

(3. 3) $$f(2n+1)-f(2n) \to -f(2) \qquad (n \to \infty).$$

Hence we get

$$f(N) = f(2^{\nu_k})+f(2^{\nu_1-\nu_k}+\cdots+2^{\nu_{k-1}-\nu_k}+1) =$$
$$= \nu_k f(2)-f(2)+f(2^{\nu_1-\nu_k}+\cdots+2^{\nu_{k-1}-\nu_k})+o(1).$$

Repeating this process we obtain that

(3. 4) $$f(N) = \nu_1 f(2)-kf(2)+o(1)k \qquad (N \to \infty).$$

Since $2^{\nu_1} \leqq N < 2^{\nu_1+1}$, we have $\dfrac{\nu_1 \log 2}{\log N} \to 1$. Consequently, from (3. 4),

(3. 5) $$\frac{f(N)}{\log N} = \frac{f(2)}{\log 2} - f(2) \log 2 \cdot \frac{\alpha(N)}{\log N} + o(1).$$

Now we prove that $f(2)=0$. For this let $N_l = 2+2^3+\ldots+2^{2l+1}$. Then

$3N_l = 2+2^2+\ldots+2^{2l+2}$. Hence we obtain that $\alpha(3N_l)=2\alpha(N_l)$, $\alpha(N_l) = = (1+o(1))\frac{\log N_l}{2\log 2}$. By (3. 5) we have

$$f(3) = f(3N_l)-f(N_l) = -f(2)\log 2[\alpha(3N_l)-\alpha(N_l)]+o(\log N_l) =$$

$$= -f(2)\log 2\cdot\alpha(N_l)+o(\log N_l) = -\frac{f(2)}{2}(1+o(1))\log N_l.$$

Hence it follows immediately that $f(2)=0$.

Thus from (3. 5),

$$\lim_{N\to\infty}\frac{f(N)}{\log N} = 0. \tag{3. 6}$$

Using (3. 6) and the total additivity of $f(n)$ we have

$$\frac{f(N)}{\log N} = \lim_{k\to\infty}\frac{f(N^k)}{\log N^k} = 0,$$

and hence $f(N)=0$. This completes the proof of Theorem 2.

Remarks.

1) I am unable to prove Theorem 2 for restrictedly additive functions $f(n)$.

2) Similarly, I cannot decide whether from $g(2n+1)-f(n)\to C$ it follows or not that $f(n)$ and $g(n)$ are constant multiples of $\log n$.

3) It would be interesting to give all the solutions of the relation

$$f(An+B)-f(an+b)\to C \quad (n\to\infty)$$

in additive functions $f(n)$, for arbitrary integers A, B, a, b.

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