On invertible elements in compact semigroups

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The paper [6] investigates the structure of abstract semigroups containing invertible elements. The purpose of this paper is to study the structure of compact semigroups containing invertible elements. Throughout this paper S will denote a Hausdorff compact semigroup.

In the first place we set down some notions and statements.

An element a of S is called *totally maximal,* if $SaS=S$. An element a of S is called *left (right, two-sided) invertible* if $Sa=S$ $(aS=S, aSa=S)$.

Further, $\mathcal{K}$ is the set of elements of a semigroup S which are neither left nor right invertible, $\mathcal{L}$ is the set of elements of S which are left invertible but not right invertible, $\mathcal{R}$ is the set of elements of S which are right invertible but not left invertible, and finally $\mathcal{G}$ is the set of elements of S which are both left and right invertible. From [5] it is known that each of the sets $\mathcal{K}, \mathcal{L}, \mathcal{R}, \mathcal{G}$ is a subsemigroup of the semigroup S.

Denote by L^* the maximal proper left ideal of a semigroup S, which contains every proper left ideal of S. The maximal proper right ideal R^* and the maximal proper two-sided ideal M^* are defined similarly.

All ideals in this paper are considered in the algebraic sense.

We admit in our considerations that the ideals L^*, R^*, M^* possibly are void sets.

Lemma 1. [6] *If in a semigroup S there exists at least one left invertible element, then S contains the unique maximal proper left ideal L^* and the complement of this ideal is the set of left invertible elements of S; hence $S=L^*\cup\mathcal{L}\cup\mathcal{G}$.*

Remark 1. A similar statement holds if S contains at least one right invertible element.

Lemma 2. [2] *Let S be a compact semigroup. If the ideal $L^*(R^*)$ exists in S, then M^* and $M^*=L^*$ $(M^*=R^*)$ also exist in S.*

From [6] it is known that every left invertible or right invertible element of a semigroup S is totally maximal. The converse statement does not hold.

Let $\mathcal{M}$ denote the set of all totally maximal elements of S which are neither left nor right invertible. Then, evidently $\mathcal{M} \subseteqq \mathcal{K}$.

Corollary. *Let S be a compact semigroup, which contains invertible elements. Then every totally maximal element is either left invertible or right invertible.*

Theorem 1a. *Let S be a compact semigroup. Then $P = \mathcal{L} \cup \mathcal{G}$ is a closed subset of the compact semigroup S.*

Proof. Since $P = \{a \in S \colon Sa = S\}$, there exists for an arbitrary $b \in S$ an $x \in S$ such that $xa = b$. In order to prove that P is closed it is sufficient to show that for an arbitrary $a \in \bar{P}$ ($\bar{P}$ means the closure of P) and for an arbitrary $b \in S$ there exists an $x \in S$ such that the relation $xa = b$ holds, since in this way it will be proved that $\bar{P} = P$. Let us assume that this is not true. This means that for some $b \in S$ the equation $xa = b$ has no solution in S and, therefore $xa \neq b$ for every $x \in S$. Since S is a Hausdorff space, it follows from the continuity of multiplication that there exist neighbourhoods $o(x) \in O(x)$, $o_x(a) \in O(a)$ and $o_x(b) \in O(b)$ such that

$$o_x(b) \cap [o(x), o_x(a)] = \emptyset,$$

where $O(x)$, $O(a)$ and $O(b)$ are complete systems of neighbourhoods of the elements x, a, b. Let us consider a system of neighbourhoods $\{o(x)\}, x \in S$. It is evident that $S = \bigcup_{x \in S} o(x)$. Since S is a compact semigroup, there exists such a finite system $o(x_1), o(x_2), \ldots, o(x_n)$, which also covers S. For $i = 1, 2, \ldots, n$ we have

$$o_{xi}(b) \cap [o(x_i) \cdot o_{xi}(a)] = \emptyset.$$

Evidently, there exist such neighbourhoods $o(b) \in O(b)$, $o(a) \in O(a)$, that $o(b) \subset \bigcap_{i=1}^{n} o_{xi}(b)$ and $o(a) \subset \bigcap_{i=1}^{n} o_{xi}(a)$. But then we have

$$o(b) \cap [o(x_i) \cdot o(a)] = \emptyset,$$

for $i = 1, 2, \ldots, n$. Since $S = \bigcup_{i=1}^{n} o(x_i)$, it follows from the preceeding relations:

$$(*) \qquad o(b) \cap [S \cdot o(a)] = \emptyset.$$

We show that the last relation is not correct. Since $a \in \bar{P}$, it follows that in every neighbourhood of a there exists at least one element of P. Therefore there exists such an element $\xi \in P$ that $\xi \in o(a)$ and, since $\xi \in P$, there exists an $\eta \in S$ such that $b = \eta \cdot \xi$. But $b \in o(b)$, so $b = \eta \cdot \xi \in S \cdot o(a)$, and this contradicts relation $(*)$.

Analogously, one proves the following

Theorem 1b. *Let S be a compact semigroup. Then $Q = \mathcal{R} \cup \mathcal{G}$ is a closed subset of S.*

Corollary. *P and Q are compact subsets of S.*

We say that a semigroup S is left simple (right simple, simple) if S contains no proper left (right, two-sided) ideal of S, distinct of S and the void set.

Theorem 2a. *Let S be a compact semigroup. Then* $P=\mathcal{L}\cup\mathcal{G}$ *is a left simple compact subsemigroup of S.*

Proof. Let $a, b\in P$. Then $Sa=S$, $Sb=S$, $S(ab)=(Sa)b=Sb=S$. This means that $ab\in P$ and P is a subsemigroup of S. Assume that $L'\subset S-L^*$ is a proper left ideal of $S-L^*$. Then Lemma 2 implies: $S(L'\cup L^*)=SL'\cup SL^*=(S-L^*)L'\cup \cup L^*L'\cup SL^*\subset L'\cup L^*$ and this is a contradiction with the assumption that L^* contains every proper left ideal of S.

The proof of the following statement is analogous.

Theorem 2b. *Let S be a compact semigroup. Then* $Q=\mathcal{R}\cup\mathcal{G}$ *is a right simple compact subsemigroup of S.*

Let e be an idempotent of a compact semigroup S. We say that an element $a\in S$ belongs to the idempotent e, if e is the unique idempotent of the closure $\bar{A}$ of the semigroup $A=\{a, a^2, \ldots\}$.

Let us denote by K_α the set of all elements of a semigroup S, which belongs to the idempotent e_α; we shall call it a K-class. From [1] it is known that any compact semigroup S can be written as the union of disjoint K-classes.

We say that a group G_α is a maximal group belonging to the idempotent e_α, if G_α contains e_α and if there exists no group $G'\neq G_\alpha$ such that $G_\alpha\subset G'\subset K_\alpha$.

Theorem 3a. *Either of the subsemigroups* L^*, P *of a compact semigroup S is the union of some K-classes of S.*

Proof. To prove our statement it is sufficient to show that no K-class can have a non-void intersection with both subsemigroups.

Let us assume that $K_\alpha\cap L^*\neq\emptyset$ and $K_\alpha\cap P\neq\emptyset$ holds for some K-class K_α. Let $a\in K_\alpha\cap L^*$. $a\in K_\alpha$ means that a belongs to the idempotent e_α. But also $a\in L^*$. Let us consider the principal left ideal generated by a: $(a)_L=a\cup Sa$. $(a)_L\subset L^*$ and $(a)_L$ is a closed subsemigroup of a compact semigroup S, therefore $(a)_L$ is also a compact subsemigroup. So we have $\bar{A}\subseteqq(a)_L$ where $A=\{a, a^2, \ldots\}$.The subsemigroup $\bar{A}$ contains the unique idempotent and since $a\in K_\alpha$ this idempotent must be e_α. We have obtained that $e_\alpha\in\bar{A}\subset(a)_L\subset L^*$. Let now $b\in K_\alpha\cap P$. The element b also belongs to the idempotent e_α. But since $b\in P$ and P is a compact subsemigroup of the semigroup S, we have $e_\alpha\in\bar{B}\subset P$ for $B=\{b, b^2, \ldots\}$. We have obtained that $e_\alpha\in L^*$ and at the same time $e_\alpha\in P$. But this is impossible, since $L^*\cap P=\emptyset$.

Analogously, we can prove

Theorem 3b. *Either of the subsemigroups R^*, Q of a compact semigroup S is the union of some K-classes of S.*

Remark 2. In [3] it is proved that a left (right) simple semigroup having at least one idempotent is a disjoint union of algebraically isomorphic groups G_α. But P is a left simple subsemigroup, Q is a right simple subsemigroup. Moreover, either of P, Q is a compact subsemigroup, so they contain at least one idempotent. From this we have

Theorem 4. *Either of the subsemigroups P, Q of a compact semigroup S is a disjoint union of topologically isomorphic maximal groups G_α.*

Proof. The statement that $P(Q)$ is a disjoint union of maximal algebraically isomorphic groups G_α follows from Remark 2 and Theorems 3a and 3b. It is only necessary to show that these groups are isomorphic also topologically. It is known from [3] that if $e_\alpha \in P$ then $G_\alpha = e_\alpha P$ and for $x \in G_\alpha$

$$x \to e_\beta x \tag{1}$$

is an isomorphic mapping of the group G_α onto G_β and for $y \in G_\beta$

$$y \to e_\alpha y \tag{2}$$

is the inverse mapping.

But according to the assumption, the multiplication in S is continuous. This means that the transformation (1) is a continuous transformation of a topological group G_α onto the topological group G_β, and the inverse transformation (2) is a continuous transformation of G_β onto G_α. Hence, G_α and G_β are topologically isomorphic.

Corollary. *If in a compact semigroup S, $\mathcal{L} \neq \emptyset$ ($\mathcal{R} \neq \emptyset$) then the semigroup $\mathcal{L}$ ($\mathcal{R}$) contains at least one idempotent.*

Theorem 5a. *If a compact semigroup S contains at least one left invertible element, then S contains at least one right unit.*

Proof. From Theorem 2a. we know that P contains at least one idempotent. Let $e_1 \in P$. Then, evidently, $Se_1 = S$. Let $x \in S$ be an arbitrary element. $Se_1 = S$ implies $ye_1 = x$ for some $y \in S$. Hence, $xe_1 = (ye_1)e_1 = ye_1^2 = ye_1 = x$. This means that e_1 is a right unit of S.

Analogously we can prove

Theorem 5b. *If a compact semigroup S contains at least one right invertible element, then S contains at least one left unit.*

Theorem 6. *Let S be a compact semigroup. Then only one of $\mathcal{L}$, $\mathcal{R}$, and $\mathcal{G}$ can be non-empty.*

Proof. Let $\mathscr{L}\neq\emptyset$, $\mathscr{G}\neq\emptyset$. From [5] we know that $\mathscr{G}$ is a subgroup of S and its unit is the unit of S, and Corollary of Theorem 4 implies that $\mathscr{L}$ contains at least one idempotent which is a right unit of S, this is a contradiction.

Corollary 1. *If S is a compact semigroup, then only the following cases are possible:* 1) $S=\mathscr{K}$, 2) $S=\mathscr{L}$, 3) $S=\mathscr{R}$, 4) $S=\mathscr{G}$, 5 $S=\mathscr{K}\cup\mathscr{L}$, 6) $S=\mathscr{K}\cup\mathscr{R}$, *and* 7) $S=\mathscr{K}\cup\mathscr{G}$.

Theorems 1a, 1b, and 6 imply

Corollary 2. *The subsemigroups $\mathscr{L}$, $\mathscr{R}$, and $\mathscr{G}$ of a compact semigroup S are compact subsemigroups.*

Theorems 2a, 2b, 4 and 6 imply

Corollary 3. *The subsemigroup $\mathscr{L}$ ($\mathscr{R}$) is left simple (right simple) and both are disjoint unions of topologically isomorphic maximal groups G_α.*

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