On local properties of pseudo-differential operators

By VIOREL BARBU in Iaşi (Roumania)

Pseudo-differential operators have been developed by KOHN—NIRENBERG [1] and L. HÖRMANDER [3]. VOLEVIČ [4] considers a wider class of symbols which generalizes the differential operators of constant strength. Our aim is to complete the results of [4] by studying the Gevrey regularity of pseudo-differential operators.

1. Notations

We set $D_j = -i\,\partial/\partial x_j$, $\partial_j = \partial/\xi_j$ for $1 \leqq j \leqq n$, and for each n-tuple $\alpha = (\alpha_1, \ldots, \alpha_n)$ we set $D^\alpha = D_1^{\alpha_1} \ldots D_n^{\alpha_n}$, $\partial^\alpha = \partial_1^{\alpha_1} \ldots \partial_n^{\alpha_n}$, $x^\alpha = x_1^{\alpha_1} \ldots x_n^{\alpha_n}$, $\xi^\alpha = \xi_1^{\alpha_1} \ldots \xi_n^{\alpha_n}$, and $|\alpha| = \sum_1^n \alpha_j$. By S we denote the space C^∞ of complex valued functions $\varphi(x)$ such that $\sup_{x \in R^n} |x^\beta D^\alpha \varphi(x)| < \infty$ for all multi-indices α and β. For real s we introduce the norm

$$\|u\|_s = \left((2\pi)^{-n} \int |\hat{u}(\xi)|^2 (1+|\xi|^2)^s \, d\xi\right)^{1/2}, \tag{1}$$

where $\hat{u}$ is the Fourier transform of u. Let H^s the space obtained by the completion of S in this norm. We set

$$H^{-\infty} = \bigcup_{s=-\infty}^{\infty} H^s.$$

If K is any compact set of R^n, we shall use the notations

$$\|u, K\| = \left(\int_K |u(x)|^2 \, dx\right)^{1/2}, \quad \|u, K\|_\infty = \operatorname*{ess\,sup}_K |u(x)|.$$

A function $u(x) \in C^\infty$ defined on an open subset $\Omega \subset R^n$ is said to be hypoanalytic of class ϱ $(1 \leqq \varrho < \infty)$ if for any compact set $K \subset \Omega$ there exists a constant M such that for any multi-index α the inequality

$$\|D^\alpha u, K\|_\infty \leqq M^{|\alpha|+1} \Gamma(\varrho|\alpha|) \tag{1.2}$$

holds, where Γ is Euler's function. The Gevrey class $G^\varrho(\Omega)$ is the space of all functions of class ϱ on Ω. If $\varrho > 1$, $G_0^\varrho(\Omega)$ will denote the space $C_0^\infty(\Omega) \cap G^\varrho(\Omega)$.

2. Pseudo-differential operators. Pseudo-local properties

We consider pseudo-differential operators of the form

(2. 1) $$Au(x) = (2\pi)^{-n}\int e^{i(x,\xi)}a(x,\xi)\hat{u}(\xi)\,d\xi \qquad (u\in S)$$

or

(2. 1)′ $$(Au)\hat{}\,(\xi) = (2\pi)^{-n}\int \hat{a}'(\xi-\eta,\eta)\hat{u}(\eta)\,d\eta + a(\xi)\hat{u}(\xi) \qquad (u\in S),$$

with the symbol $a(x,\xi) = a(\xi)+a'(x,\xi)$, where $a'(x,\xi)$, as a function of x, vanishes at ∞. Here $\hat{a}'(\eta,\xi)$ denotes the Fourier transform of $a'(x,\xi)$ with respect to x.

Concerning the symbol $a(x,\xi)\in C^\infty(R^n\times R^n)$ we assume that there are positive constants m, M, C independent of α such that

(I) $$|\partial^\alpha a(\xi)| \leqq C^{|\alpha|}(1+|\xi|)^{m-|\alpha|/\varrho},$$

(II) $$\int |D^\beta\partial^\alpha a'(x,\xi)|\,dx \leqq M^{|\alpha|+|\beta|+1}\Gamma(\varrho|\beta|)(1+|\xi|)^{m-|\alpha|/\varrho}.$$

Theorem 1. *Let $a(x,\xi)$ be a C^∞ symbol satisfying* I, II *and let A be the associated pseudo-differential operator. Suppose that $u\in H^\infty\cap G^\varrho$. Then $Au\in G^\varrho(\Omega)$.*

Proof. We choose $\varphi\in C_0^\infty(\Omega)$ equal to 1 in a set $\bar{\Omega}'\subset\Omega$. We may suppose that $u\in H^\sigma$. We put

$$a_{\alpha,\beta}(x,\xi) = D^{\alpha-\beta}a(x,\xi)$$

and denote by $A_{\alpha,\beta}$ the operators of the form (2. 1) with associated symbols $a_{\alpha,\beta}(x,\xi)$. We have

(2. 2) $$\varphi D^\alpha Au = \sum_{|\beta|\leqq|\alpha|}\binom{\beta}{\alpha}A_{\alpha\beta}(\varphi D^\beta u) + \sum_{|\beta|\leqq|\alpha|}\binom{\beta}{\alpha}[A_{\alpha\beta};\varphi](D^\beta u)$$

where $[A_{\alpha\beta};\varphi]$ is the commutator of $A_{\alpha\beta}$ with φ.

The essential point in the proof is the estimation of $\|[A_{\alpha\beta};\varphi]D^\beta u\|$, where $\|\ \|$ denotes the L^2 norm. For $\alpha\neq\beta$ and $u, v\in S$ we have

(2. 3) $$([A_{\alpha\beta};\varphi](D^\beta u, v)) =$$
$$= (2\pi)^{-n}\iiint (D^\beta u)\hat{}\,(\xi)\hat{\varphi}(\tau-\xi)\big(\hat{a}'_{\alpha\beta}(\tau-\eta,\tau)-\hat{a}'_{\alpha\beta}(\tau-\eta,\xi)\big)\hat{v}(\eta)\,d\xi\,d\eta\,d\tau.$$

Using a finite series expansion for the difference in the integral, and substituting this expression in (2. 3), we find

(2. 4) $$([A_{\alpha\beta};\varphi](D^\beta u), v) =$$
$$= \sum_{|\gamma|\leqq N}(1/\gamma!)\iint (D^\beta u D^\gamma\varphi)\hat{}\,(\tau)\hat{v}(\eta)(D^{\alpha-\beta}\partial^\gamma a')\hat{}\,(\tau-\eta,\eta)\,d\eta\,d\tau +$$
$$+(2\pi)^{-n}\iiint (D^\beta u)\hat{}\,(\xi)\hat{\varphi}(\tau-\xi)R_N(\tau-\eta,\xi)\hat{v}(\eta)\,d\xi\,d\eta\,d\tau,$$

where R_N denotes the remainder: $(1/N!)^N(\xi-\tau)(\partial^N a')^\wedge(\tau-\eta,\theta)$. From II it follows by partial integration that for any multi-indices α, γ we have

$$(2.5)\quad |D^\alpha\partial^\gamma \hat{a}'(\tau-\eta,\tau)| \leqq M^{|\alpha|+|\gamma|+1}\Gamma(\varrho|\alpha|+N_1)(1+|\tau-\eta|)^{-N_1}(1+|\tau|)^{m-|\gamma|/\varrho}$$

for every non-negative N_1. From (2. 4) it follows

$$(2.6)\quad |([A_{\alpha\beta};\varphi]D^\beta u, v)| \leqq M^{|\alpha-\beta|+N+1}\Gamma(\varrho|\alpha-\beta|)\cdot$$
$$\cdot\sum_{0\leqq|\gamma|\leqq N}(1/\gamma!)\|D^\beta u D^\gamma\varphi\|_{m-|\gamma|/\varrho}\|v\| + \int K(\xi,\eta)(1+|\xi|)^{\sigma-|\beta|}|D^\beta\hat{u}(\xi)|\,|\hat{v}(\eta)|\,d\xi\,d\eta$$

with

$$K(\xi,\eta) = (2\pi)^{-n}\sum_{|\gamma|=N}(1/\gamma!)\cdot$$
$$\cdot\int\frac{(1+|\xi|)^{|\beta|-\sigma}((1-\Delta)^p\varphi)^\wedge(\xi-\tau)\cdot(\xi-\tau)^\gamma((1-\Delta)^q D^{\alpha-\beta}\partial^\gamma a')^\wedge(\tau-\eta)}{(1+|\xi-\tau|^2)^p(1+|\tau-\eta|^2)^q}\,d\tau,$$

where p, q are non-negative numbers. Using the inequality

$$\frac{1+|\xi|}{2(1+|\xi-\eta|)} \leqq 1+|\theta| \leqq (1+|\xi|)(1+|\xi-\tau|)$$

we get

$$(2.7)\quad |K(\xi,\eta)| \leqq$$
$$\leqq (1/N!)M^{|\alpha-\beta|+1}\Gamma(\varrho|\alpha-\beta|)(1+|\xi|)^{|\beta|-\sigma-N/\varrho+m}(1+|\eta|)^{-n-1}\|\varphi\|_{N(1+1/\varrho)+p},$$

for p, q sufficiently large. If we choose N so that

$$\varrho(|\beta|-\sigma+m+n+1) < N \leqq \varrho(|\beta|-\sigma+m+n+1)+1$$

we obtain

$$(2.8)\quad |K(\xi,\eta)| \leqq (1/N!)M^{|\alpha-\beta|+1}\Gamma(\varrho|\alpha-\beta|)(1+|\xi|)^{-n-1}(1+|\eta|)^{-n-1}.$$

To prove Theorem 1 we first suppose that $\varrho>1$. Thus we may choose $\varphi\in G_0^d$ so that $1<d<2\varrho/(\varrho+1)$. Applying Schur's lemma (see HÖRMANDER [3]) from (2. 6) and (2. 8) we get

$$(2.9)\quad \|[A_{\alpha\beta},\varphi]D^\beta u\| \leqq$$
$$\leqq M^{|\alpha-\beta|+1}\Gamma(\varrho|\alpha-\beta|)\sum_{|\gamma|\leqq N}1/\gamma!\,\|D^\beta u D^\gamma\varphi\|_{m-|\gamma|/\varrho} + M^{|\alpha|+1}\Gamma(\varrho|\alpha|)\|u\|_\sigma.$$

with N defined above and for $u, v\in S$. Similarly it follows

$$(2.10)\quad \|A_{\alpha\beta}(\varphi D^\beta u)\| \leqq M_1^{|\alpha-\beta|+1}\Gamma(\varrho|\alpha-\beta|)\|\varphi D^\beta u\|_m.$$

Let now $u\in H^\sigma\cap C^\infty(\Omega)$. Combining (2. 2), (2. 9) and (2. 10) we get

$$(2.11)\quad \|\varphi D^\alpha Au\| \leqq \sum_{|\beta|\leqq|\alpha|}M^{|\alpha-\beta|+1}\Gamma(\varrho|\alpha-\beta|)\sum_{|\gamma|\leqq N}1/\gamma!\,\|D^\beta u D^\gamma\varphi\|_{m-|\gamma|/\varrho} +$$
$$+M^{|\alpha|+1}\Gamma(\varrho|\alpha|)\|u\|_\sigma.$$

Let ψ be a G_0^d-function with its support in Ω and equal to 1 on the support of φ. Clearly $D^\gamma\varphi D^\beta(u\psi) = D^\gamma\varphi D^\beta u$. Using the inequality (see HÖRMANDER [2])

$$\|D^\beta u D^\gamma\varphi\|_{m-|\gamma|/\varrho} \leqq C\|D^\beta(u\psi)\|_{m-|\gamma|/\varrho}\|D^\gamma\varphi\|_{|m-|\gamma|/\varrho|}$$

and the fact that $u\in G^\varrho(\Omega)$ from (2. 11) we deduce

$$\|\varphi D^\alpha Au\| \leqq M^{|\alpha|+1}\Gamma(\varrho|\alpha|). \tag{2. 12}$$

Hence

$$\|D^\alpha Au, \bar{\Omega}'\|_\infty \leqq M_1^{|\alpha|+1}\Gamma(\varrho|\alpha|). \tag{2. 13}$$

Since Ω' is an arbitrary open subset of Ω, this completes the proof. Now we suppose $\varrho=1$. We choose a sequence $\{\varphi_k\}\in C_0^\infty(\Omega)$ such that $\varphi_k=1$ on $\bar{\Omega}'$ and

$$|D^\alpha\varphi_k(x)| \leqq C^{|\alpha|+1}k^{|\alpha|} \quad \text{for} \quad |\alpha|\leqq k. \tag{2. 14}$$

Taking $\varphi=\varphi_{2N}$ in (2. 6), from (2. 14) it follows

$$\|[A_{\alpha\beta}, \varphi_{2N}]D^\beta u\| \leqq M^{|\alpha-\beta|+1}\Gamma(|\alpha-\beta|)\sum_{|\gamma|\leqq N} 1/\gamma!\|D^\beta u D^\gamma\varphi_{2N}\|_{m-|\gamma|} + \tag{2. 15}$$
$$+M^{|\alpha|}\Gamma(|\alpha|)\|u\|_\sigma,$$

where $|\beta|-\sigma+m+n+1<N\leqq|\beta|-\sigma+m+n+2$. As above we obtain

$$\|\varphi_{2N}D^\alpha Au\| \leqq \sum_{|\beta|\leqq|\alpha|} M^{|\alpha-\beta|+1}\Gamma(|\alpha-\beta|)\sum_{|\gamma|\leqq N}\|D^\beta u D^\gamma\varphi_{2N}\|_{m-N} + \tag{2. 16}$$
$$+M^{|\alpha|+1}\Gamma(|\alpha|)\|u\|_\sigma.$$

Let $\psi_{2N}\in C_0^\infty(\Omega)$ equal to 1 in the support of φ_{2N} such that

$$|D^\alpha\psi_{2N}(x)| \leqq C^{|\alpha|+1}2N^{|\alpha|} \quad \text{for} \quad |\alpha|\leqq 2N.$$

Then it follows

$$\|\varphi D^\alpha Au\| \leqq \tag{2. 17}$$
$$\leqq \sum_{|\beta|\leqq|\alpha|} M^{|\alpha-\beta|+1}\Gamma(|\alpha-\beta|)\sum_{|\gamma|\leqq N}(1/\gamma!)\|D^\beta(u\psi_{2N})\|_{m-|\gamma|}\|D^\gamma\varphi_{2N}\|_{|m-|\gamma||} +$$
$$+M^{|\alpha|+1}\Gamma(|\alpha|)\|u\|_\sigma.$$

This implies that

$$\|D^\alpha Au, \bar{\Omega}'\|_\infty \leqq M^{|\alpha|+1}|\alpha|! \tag{2. 18}$$

Hence the proof of Theorem 1 is completed.

Remark. Let $K\in D'(R^n\times R^n)$ be a distribution defined by

$$K(F) = (2\pi)^{-n}\int e^{i(x,\xi)}a(x,\xi)F(x,\xi)\,dx\,d\xi \quad \text{for} \quad F\in C_0^\infty(R^n\times R^n), \tag{2. 19}$$

where $\hat{F}(x, \xi) = \int e^{-i(x,\xi)} F(x, y)dy$. Obviously the distribution K is the kernel of a pseudo-differential operator A, i.e.

$$(Au, v) = K(u \otimes v) \quad (u, v \in S).$$

It is easily seen that under assumptions I, II the kernel K is ϱ-hypoanalytic in the domain $\{(x, y) \in R^n \times R^n;\ x \neq y\}$.

3. Hypoelliptic pseudo-differential operators

Let $a(x, \xi)$ be the symbol considered above. Assume there are non-negative constants M_1, N_1 independent of α, β such that

(III) $$|a(\xi) - a(\eta)| \leqq M(1 + |\xi - \eta|)^{N_1}(1 + |\xi|)^{m-\sigma},$$

(IV) $$\int |D^\alpha a'(x, \xi) - D^\alpha a'(x, \eta)|\, dx \leqq M_1^{|\alpha|+1}\Gamma(\varrho|\alpha|)(1 + |\xi - \eta|)^N(1 + |\eta|)^{m-\sigma}$$

with a real $\sigma \geqq 2$, and

(V) $$|a(\xi)|,\quad |a'(x, \xi)| \geqq C(1 + |\xi|)^m$$

for $|\xi|$ sufficiently large.

Theorem 2. *Let $a(x, \xi)$ be a C^∞ symbol which satisfies the assumptions* I—V, *and let Ω be an open subset of R^n. Then $u \in H^{-\infty}$ and $Au \in G^\varrho(\Omega)$ imply $u \in G^\varrho(\Omega)$* $(\varrho \geqq 2)$.

Suppose that $|a(x, \xi)|$, $|a(\xi)| \geqq C(1 + |\xi|)^m$ for $|\xi| \geqq R$. In the following $\chi(\xi)$ will denote a C^∞ non-negative function which is equal to 1 for $|\xi| \geqq R + 1$ and vanishes for $|\xi| < R$. Consider the symbol $e(x, \xi) = \chi(\xi)/a(x, \xi)$ and denote by E and G the pseudo-differential operators with the associated symbols $e(x, \xi)$ and $\chi(\xi)$. Setting $T = EA - G$ and $T_1 = I - G$ we decompose u as

(3. 1) $$u = EAu - Tu + T_1 u.$$

Lemma 1. *For any real s there exists a non-negative constant C_s such that*

(3. 2) $$\|Eu\|_s \leqq C_s \|u\|_{s-m} \quad \text{for} \quad u \in S,$$

(3. 3) $$\|Tu\|_s \leqq C_s \|u\|_{s-\sigma} \quad \text{for} \quad u \in S,$$

and

(3. 4) $$\|[A_{\alpha\beta}, \varphi]u\|_s \leqq C_s \|u\|_{s+m-2} \quad \text{for} \quad u \in S, \quad \text{where} \quad \varphi \in C_0^\infty(R^n).$$

Proof. To prove (3. 2) we remark that the symbol $e(x, \xi)$ satisfies conditions (I), (II) with m replaced by $-m$. The estimates (3. 3) and (3. 4) follow in a similar way as (2. 9).

Proof of Theorem 1. Under our conditions the operator A is hypo-elliptic (see VOLEVIČ [4]); hence we may assume that $u \in C^\infty(\Omega) \cap H^t$. Since the statement of the theorem is local it is sufficient to prove that every point in Ω has an open neighborhod ω in which $u \in G^\varrho$. In the following we denote by ω_ε the set of all points of ω at a distance $>\varepsilon$ from $C\omega$. Let $\varphi, \psi \in C_0^\infty(\omega)$ be fixed functions such that $\operatorname{supp} \varphi \subset \omega_{2\varepsilon}$, $\operatorname{supp} \psi \subset \omega_\varepsilon$, and $\varphi = 1$ in $\omega_{3\varepsilon}$, $\varphi = 1$ in $\omega_{2\varepsilon}$. For $u_1 = u\psi$ we get from (2. 2) and (3. 1):

$$\varphi D^\alpha u_1 = E\varphi D^\alpha A u_1 - \sum_{|\beta|<|\alpha|} \binom{\beta}{\alpha} EA_{\alpha\beta}(\varphi D^\beta u_1) - T(\varphi D^\alpha u_1) + \tag{3. 5}$$

$$+ \sum_{|\beta| \leqq |\alpha|} \binom{\beta}{\alpha} E[A_{\alpha\beta}, \varphi] D^\beta u_1 + T_1(\varphi D^\alpha u_1).$$

We remark that

$$\varphi D^\alpha A u_1 = \varphi D^\alpha A u - \varphi D^\alpha A(1-\psi)u. \tag{3. 6}$$

Since $(1-\psi)u = 0$ on $\omega_{2\varepsilon}$ it follows from Theorem 1 that $A(1-\psi)u \in G^\varrho(\omega_{2\varepsilon})$. This implies that

$$\|\varphi D^\alpha A u_1\| \leqq M^{|\alpha|+1} \Gamma(\varrho|\alpha|). \tag{3. 7}$$

From (2. 10), (3. 2), (3. 3), and (3. 4) we obtain that

$$\|EA_{\alpha\beta}(\varphi D^\beta u_1)\| \leqq M^{|\alpha-\beta|+1} \Gamma(\varrho|\alpha-\beta|) \|\varphi D^\beta u\|, \tag{3. 8}$$

$$\|T(\varphi D^\alpha u_1)\| \leqq C \|D^{\alpha-2}(u\psi)\| \, \|\varphi\|_2, \quad \text{and} \tag{3. 9}$$

$$\|E[A_{\alpha\beta}, \varphi] D^\beta u_1\| \leqq M^{|\alpha-\beta|+1} \Gamma(\varrho|\alpha-\beta|) \|D^\beta u_1\|_{-2}. \tag{3. 10}$$

Applying Leibniz's formula we may write

$$\|T(\varphi D^\alpha u_1)\| \leqq C \sum_{|\beta|<|\alpha|-1} \|D^\beta u; \omega_\varepsilon\| \, \|D^{\alpha-\beta}\psi\| \binom{\beta}{\alpha}, \tag{3. 11}$$

$$\|E[A_{\alpha\beta}, \varphi] D^\beta u_1\| \leqq M^{|\alpha-\beta|+1} \Gamma(\varrho|\alpha-\beta|) \sum_{|\gamma| \leqq |\beta|} \|D^{\beta-\gamma} u; \omega_\varepsilon\| \, \|D^\gamma \psi\| \binom{\beta}{\alpha}. \tag{3. 12}$$

Denote by $\alpha(\xi)$ the function $\chi(\xi) - 1$. It is easy to see that $\alpha(\xi) \in C_0^\infty(R^n)$. Obviously

$$T_1(\varphi D^\alpha u_1)(x) = (2\pi)^{-n} \int e^{i(x,\xi)} \alpha(\xi) (\varphi D^\alpha u)^\wedge(\xi) \, d\xi.$$

Applying Parseval's formula we obtain

$$\|T_1(\varphi D^\alpha u_1)\| \leqq M^{|\alpha|+1} \|\varphi u\| + \sum_{|\beta|<|\alpha|-1} \binom{\beta}{\alpha} \|D^\beta u; \omega_\varepsilon\| \, \|D^{\alpha-\beta}\varphi\|. \tag{3. 13}$$

Combining (3. 5), (3. 8), (3. 11), (3. 12), and (3. 13) we get

(3. 14)
$$\|D^\alpha u;\omega_{3\varepsilon}\| \leqq \sum_{|\beta|<|\alpha|} M^{|\alpha-\beta|+1}\Gamma(\varrho|\alpha-\beta|)\|D^\beta u;\omega_{2\varepsilon}\| +$$
$$+\sum_{|\beta|<|\alpha|-1}\binom{\beta}{\alpha}\|D^\beta u;\omega_\varepsilon\|(\|D^{\alpha-\beta}\psi\|+\|D^{\alpha-\beta}\varphi\|)+$$
$$+\sum_{|\beta|<|\alpha|-1} M^{|\alpha-\beta|+1}\Gamma(\varrho|\alpha-\beta|)\sum_{|\gamma|\leqq|\beta|}\binom{\gamma}{\beta}\|D^{\beta-\gamma}u;\omega_\varepsilon\|\,\|D^\gamma\varphi\|.$$

We choose two sequences $\varphi_k\in C_0^\infty(\omega_{(k-1)\varepsilon})$, $\psi_k\in C_0^\infty(\omega_{(k-2)\varepsilon})$ such that $\psi_k(x)=1$ in $\omega_{(k-1)\varepsilon}$, $\varphi_k(x)=1$ in $\omega_{k\varepsilon}$, and

$$\|D^\alpha\varphi_k\|_\infty \leqq C^{k+1}k^{|\alpha|}\varepsilon^{-|\alpha|},\quad \|D^\alpha\psi_k\|_\infty \leqq C^{k+1}k^{|\alpha|}\varepsilon^{-|\alpha|}$$

for $|\alpha|\leqq k$. If in (3. 13) we take $\varphi=\varphi_k$ and $\psi=\psi_k$ from (3. 14) it follows

(3. 15)
$$\|D^\alpha u;\omega_{|\alpha|\varepsilon}\| \leqq \sum_{|\beta|<|\alpha|} M^{|\alpha-\beta|+1}\Gamma(\varrho|\alpha-\beta|)\|D^\beta u;\omega_{(|\alpha|-1)\varepsilon}\| +$$
$$+C\sum_{|\beta|<|\alpha|-1}\binom{\beta}{\alpha}\|D^\beta u;\omega_{(|\alpha|-2)\varepsilon}\|\,|\alpha|^{|\alpha-\beta|}\varepsilon^{-|\alpha-\beta|}+$$
$$+\sum_{|\beta|<|\alpha|} M^{|\alpha-\beta|+1}\Gamma(\varrho|\alpha-\beta|)\|D^\beta u;\omega_{(|\alpha|-2)\varepsilon}\|\sum_{|\gamma|\leqq|\beta|}|\gamma|^{-\varrho|\gamma|}\varepsilon^{-|\gamma|}|\alpha|^{-|\gamma|}.$$

Let δ be a non-negative constant, sufficiently small. If we take ε such that $|\alpha|\varepsilon\leqq\delta$, then from (3. 15) we obtain

$$\|D^\alpha u;\omega_{|\alpha|\varepsilon}\| \leqq \sum_{|\beta|<|\alpha|} M^{|\alpha-\beta|+1}\Gamma(\varrho|\alpha-\beta|)\|D^\beta u;\omega_{(|\alpha|-1)\varepsilon}\| +$$
$$+\sum_{|\beta|<|\alpha|-1} M^{|\alpha-\beta|+1}\Gamma(\varrho|\alpha-\beta|)\|D^\beta u;\omega_{(|\alpha|-2)\varepsilon}\|.$$

By recurrence with respect to $|\alpha|$ we get

$$\|D^\alpha u;\omega_{|\alpha|\varepsilon}\| \leqq M^{|\alpha|+1}\Gamma(\varrho|\alpha|).$$

Hence

$$\|D^\alpha u;\omega_\delta\| \leqq M^{|\alpha|+1}\Gamma(\varrho|\alpha|).$$

Since δ is arbitrary this implies that $u\in G^\varrho(\omega)$, and the proof of Theorem 2 is complete.

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FACULTY OF MATH. AND MECH.
UNIVERSITY OF IASHY (ROUMANIA)

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