Note on power series with positive coefficients

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We prove the following

Theorem. *Let $\lambda(t)$ be a positive, non-increasing, integrable function on the interval* $0 < t \leqq 1$ *such that* $\lambda\left(\frac{1}{n+1}\right) = O\left(\lambda\left(\frac{1}{n}\right)\right)$, *and let $F(x)$ be defined on the interval* $0 \leqq x < 1$ *by the series* $\sum_{k=0}^{\infty} a_k x^k$ *with* $a_k \geqq 0$. *Further let* $1 \leqq p \leqq \infty$. *Then we have*

$$\left\{\int_0^1 \lambda(1-x)(F(x))^p\,dx\right\}^{1/p} < \infty \tag{1}$$

if and only if

$$\left\{\sum_{n=1}^{\infty} \lambda\left(\frac{1}{n}\right) n^{-2}\left(\sum_{k=0}^{n} a_k\right)^p\right\}^{1/p} < \infty. \tag{2}$$

If $\lambda(t) = t^{-r}$ $(r < 1)$, this theorem reduces to a theorem of Rais Shan Khan [3], which in its turn includes a theorem of Askey ([1], $r=0$) and a theorem of Heywood ([2], $p=1$).

The proof is similar to that of the mentioned theorems.

Proof. We may assume $1 \leqq p < \infty$, since under the assumptions of the theorem both (1) and (2) mean for $p = \infty$ that the series $\sum_{k=0}^{\infty} a_k$ converges. First we show that (1) implies (2). Set $y = 1 - x$ and $A_n = \sum_{k=0}^{n} a_k$. Since $\left(1 - \frac{1}{n}\right)^n$ is an increasing sequence, we have for $\frac{1}{n+1} \leqq y \leqq \frac{1}{n}$ $(n \geqq 2)$:

$$F(1-y) \geqq \sum_{k=0}^{n} a_k (1-y)^k \geqq \sum_{k=0}^{n} a_k \left(1 - \frac{1}{n}\right)^k \geqq \left(1 - \frac{1}{n}\right)^n \sum_{k=0}^{n} a_k \geqq \frac{1}{4} A_n.$$

Using this we obtain for $m \geqq 2$:

$$\sum_{n=1}^{m} \lambda\left(\frac{1}{n}\right) n^{-2} A_n^p \leqq 2 \sum_{n=1}^{m} \int_{\frac{1}{n+1}}^{\frac{1}{n}} \lambda(y)\, dy\, A_n^p \leqq 2\left(\int_{\frac{1}{2}}^{1} \lambda(y) A_1^p\, dy + \sum_{n=2}^{m} \int_{\frac{1}{n+1}}^{\frac{1}{n}} \lambda(y) A_n^p\, dy\right) \leqq$$

$$\leqq O(1) + 8 \sum_{n=2}^{m} \int_{\frac{1}{n+1}}^{\frac{1}{n}} \lambda(y)(F(1-y))^p\, dy \leqq O(1) + 8 \int_0^1 \lambda(1-x)(F(x))^p\, dx.$$

This proves that (2) follows from (1).

The proof of the inverse statement is a bit longer. We have for $m \geqq 1$

$$\int_0^{1-\frac{1}{m+1}} \lambda(1-x)(F(x))^p\, dx = \sum_{n=1}^{m} \int_{\frac{1}{n+1}}^{\frac{1}{n}} \lambda(y)(F(1-y))^p\, dy = \tag{3}$$

$$= \sum_{n=1}^{m} \int_{\frac{1}{n+1}}^{\frac{1}{n}} \lambda(y)\left(\sum_{k=0}^{\infty} a_k (1-y)^k\right)^p dy \leqq \sum_{n=1}^{m} \int_{\frac{1}{n+1}}^{\frac{1}{n}} \lambda(y)\left(\sum_{k=0}^{\infty} a_k \left(1-\frac{1}{n+1}\right)^k\right)^p dy \leqq$$

$$\leqq O(1) \sum_{n=1}^{m} \lambda\left(\frac{1}{n}\right) n^{-2} \left(\sum_{k=0}^{\infty} a_k \left(1-\frac{1}{n+1}\right)^k\right)^p.$$

Since $\frac{1}{2} \geqq \left(1-\frac{1}{n+1}\right)^n \geqq \left(1-\frac{1}{n+2}\right)^{n+1}$ for $n = 1, 2, \ldots,$ we have

$$\sum_{k=0}^{\infty} a_k \left(1-\frac{1}{n+1}\right)^k \leqq \sum_{j=0}^{\infty} \sum_{k=nj}^{n(j+1)} a_k \left(1-\frac{1}{n+1}\right)^k \leqq \tag{4}$$

$$\leqq \sum_{j=0}^{\infty} \left(1-\frac{1}{n+1}\right)^{nj} \sum_{k=nj}^{n(j+1)} a_k \leqq 2 \sum_{i=1}^{\infty} 2^{-i} A_{ni}.$$

If $p > 1$, then we have

$$\sum_{i=1}^{\infty} 2^{-i} A_{ni} \leqq \left(\sum_{i=1}^{\infty} 2^{-\frac{ip'}{2}}\right)^{\frac{1}{p'}} \left(\sum_{i=1}^{\infty} 2^{-\frac{ip}{2}} A_{ni}^p\right)^{\frac{1}{p}}. \tag{5}$$

Hence and from (3), (4), and (5) we deduce for $m \geqq 1$ and $1 \leqq p < \infty$, that

$$\int_0^{1-\frac{1}{m+1}} \lambda(1-x)(F(x))^p\,dx \leqq O(1) \sum_{n=1}^{m} \lambda\left(\frac{1}{n}\right) n^{-2} \sum_{i=1}^{\infty} 2^{-\frac{ip}{2}} A_{ni}^p \leqq$$

$$\leqq O(1) \sum_{i=1}^{\infty} 2^{-\frac{ip}{2}} \sum_{n=1}^{m} \lambda\left(\frac{1}{n}\right) n^{-2} A_{ni}^p \leqq$$

$$\leqq O(1) \sum_{i=1}^{\infty} 2^{-\frac{ip}{2}} i^2 \sum_{n=1}^{m} \lambda\left(\frac{1}{ni}\right) (ni)^{-2} A_{ni}^p \leqq O(1) \sum_{n=1}^{\infty} \lambda\left(\frac{1}{n}\right) n^{-2} A_n^p.$$

Thus (2) implies (1), and this completes the proof.

References

[1] R. Askey, L^p behavior of power series with positive coefficients, *Proc. Amer. Math. Soc.*, **19** (1968), 303—305.

[2] P. Heywood, Integrability theorems for power series and Laplace transforms. I, *J. London Math. Soc.*, **30** (1955), 302—310.

[3] Rais Shah Khan, On power series with positive coefficients, *Acta Sci. Math.*, **30** (1969), 255—257.

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