Some results concerning regressive functions

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A function f defined on a set of ordinals and having ordinals as values satisfying $f(\alpha)<\alpha$ will be called regressive. As a matter of fact we shall not require the validity of this inequality for "small" ordinals — what we mean by this latter phrase will be quite clear later. In this paper we are going to examine some questions concerning the existence of regressive functions subjected to some additional conditions such as we shall require f to be one-to-one or divergent (for the definition see below) and we shall restrict the domain of f or occasionally its range too. To this end a review of some definitions and theorems in the theory of stationary sets is necessary.

1. Preliminary definitions and theorems. Let p be a cardinal number the cofinality p^* of which is $>\aleph_0$. A function f defined for some ordinals preceding p and having ordinals $<p$ as values is said to be *regressive* if it satisfies the inequality $f(\alpha)<\alpha$ in its domain whenever $\alpha>\alpha_0$, where α_0 is an ordinal $<p$ depending on f. The function f is called *divergent* if the set

$$\{\alpha : f(\alpha)\leqq\mu\}$$

is not cofinal to p whichever the ordinal $\mu<p$ may be.

The set $S\subseteqq p$ is said to be *stationary* if there can not be defined any divergent regressive function on it. According to a celebrated result of W. NEUMER [1] this condition can be stated equivalently in a form that every *band* meets S. Here by band we mean any set cofinal to p which is closed in the topology induced by the natural ordering of p.

Considering the first form of the definition it is quite clear that the union of less than p^* non-stationary sets is not stationary, either. In the sequel, however, we shall need the following much stronger result, established by the first of the authors (see [2]):

Theorem 1. 1. *Let $\{S_\alpha\}_{\alpha<p^*}$ be a sequence of non-empty and non-stationary sets and suppose that the set of their initial elements, which are supposed to be distinct, is also non-stationary and cofinal to p. Then the union class $\bigcup_{\alpha<p^*} S_\alpha$ is not stationary either.*

2. If the regular cardinal number m is smaller than p^* then it is easily seen that the set of all ordinals preceding p and being cofinal to m is stationary. In fact, it is quite simple to verify that the mth element of each band belongs to this set. When, however, we consider the set S of all ordinals preceding p which have cofinalities greater than or equal to that of p, the situation is different. Indeed if we take the closure of a set of type p^* cofinal to p and containing only successor ordinals, then the band obtained will contain only ordinals of cofinalities smaller than p^*. This means that there exists a divergent regressive function defined on this set S. It can be shown, however, that there does not exist a divergent regressive function which maps ordinals into ordinals having greater cofinalities. More exactly the following theorem holds, where, of course, we tacitly suppose that p is a singular cardinal number.

Theorem 2. 1. *Let m be a regular cardinal number, $p^* \leqq m < p$. Let us denote by M the set of ordinals $<p$ which are cofinal to m and by N those cofinal to m or to some greater regular ordinals $<p$. Then there does not exist any divergent regressive function which maps M into N.*

For this theorem we first present what we think to be the most concise proof of it, and after that we outline another way leading to its proof, which, apart from the fact that it is in some respects more illuminating than the first proof, will employ some ideas useful later too.

Proof. Suppose, on the contrary, that there is a divergent regressive function f which maps M into N and let g be a divergent regressive function on N — such a function g *does* exist in view of the non-stationarity of N seen above. For any $\alpha \in N$ let $g'(\alpha)$ be the least element of M exceeding $g(\alpha)$. Then obviously we have $g'(\alpha) \leqq \alpha$. Now for any $\xi \in M$ we define the function $h(\xi)$ as $g'(f(\xi))$. This function is divergent, regressive, and maps M into itself; but the existence of such a function is a clear contradiction, since in view of the fact that the sets M and p have similar order types this would mean that there exists a divergent regressive function on the whole set p, which, however, is obviously stationary.

If we had required the existence of a divergent mapping f of M into N statisfying the regressivity condition $f(\alpha) < \alpha$ all over its domain, then it would have been easy to give a negative answer for this question. In fact if we consider the initial element of the set M we cannot find a smaller one in N. And yet, it might be of some interest that the essential point of the above theorem is the non-existence of such a mapping.

Indeed, on account of the non-stationarity of N we have the existence of a band B in the complement of N. Now, according to the fact that a band is always stationary, if we define a divergent regressive function f (in the sense used here generally) on the set M into the set N, it cannot always jump over the elements of B. More precisely said, if we define on B the function g by

$$g(\beta) = \min\{f(\alpha): \alpha > \beta \ \& \ \alpha \in M\},$$

then g will not be regressive. So if β is large enough and such that $g(\beta) \geqq \beta$, then the first element α of M exceeding β cannot have an image $\beta \leqq f(\alpha) < \alpha$ such that $f(\alpha) \in N$, which, however, is required for all "large" α.

3. If S is a non-stationary set, then by definition there can be defined a divergent regressive function f on S. If p is a regular cardinal number, the divergence of f is an obvious restriction on the cardinalities of the inverse images

$$\{\alpha : f(\alpha) = \mu\};$$

indeed it says that these cardinalities are all smaller than p. In what follows we are going to investigate the question whether this condition can be strengthened in some way. This problem for singular p has also a meaning even if it has no such inspiration as in the regular case — thus in the sequel we shall not exclude the singular case, either. However we will return once more to this point later (cf. Section 5).

Since the behaviour of the above stated problem is different for $p = \aleph_1$ and for p chosen greater, we shall treat the first case separately. A lemma will be useful to this end.

Lemma 3. 1. *If ϑ is an arbitrary ordinal and A is the set of all successor ordinals preceding it, then the unique one-to-one function f defined on A such that $f(\alpha) < \alpha$ for any $\alpha \in A$, is the one which transforms each element of A into its immediate predecessor.*

The proof of this lemma can be easily carried out by transfinite induction on ϑ so we do not go into further details here.

This lemma can be simply translated into an assertion about regressive functions. Here we are interested only in the case $p = \aleph_1$.

Lemma 3. 2. *If A denotes the set of all successor ordinals before $\aleph_1$, then the essentially unique one-to-one regressive function f defined on A takes over each element of A into its immediate predecessor.*

Here the word "essentially" refers to the fact, that there are several functions which meet the requirements of the lemma but they differ only in their values assumed for small arguments. As a matter of fact, when talking about regressive functions, we are in no way concerned which values they take for small arguments. E.g. in the above lemma we only require f to be essentially one-to-one — it is quite clear what we mean by this. As to the proof of the lemma it uses the same idea as occurred in the second proof of Theorem 1. 1.

Proof. Let us temporarily denote by B the set of all limit ordinals less than $\aleph_1$ and for any $\beta \in B$ let

$$g(\beta) = \min \{f(\alpha): \alpha > \beta \ \& \ \alpha \in A\}.$$

Since the set B is obviously stationary, the function g cannot be regressive. Choose a $\beta \in B$ such that $g(\beta) \geqq \beta$. If β is chosen large enough, then Lemma 3. 1 provides for the uniqueness of $f(\alpha)$ whenever $\alpha > \beta$.

Now we are ready to state our main result concerning the case $p = \aleph_1$:

Theorem 3. 3. *If S is a non-stationary subset of $\aleph_1$ then there is a regressive function f on S which assumes each of its values at most twice. This bound is the best possible one.*

Proof. In view of Lemma 3. 2 it is simple to verify the second assertion of the theorem. In fact if we choose the set S such that it contains all successor ordinals, then by the lemma we have that the values of f assumed on this part of S cover the whole set of the countable ordinals (i.e. the set of all ordinals less than $\aleph_1$). Thus the values assumed by f for limit ordinals will be assumed at least twice.

In order to prove the first assertion of the theorem it is sufficient to show that if S is a non-stationary set containing only limit ordinals, then there exists a one-to-one regressive function on it. This can be done as follows:

Let f be an arbitrary divergent regressive function on S, and consider its values assumed in any of the intervals $I_\tau = [\xi_\tau, \xi_{\tau+1})$ formed by two consecutive countable limit ordinals. According to the divergence of f it takes each ordinal at most countably many times as value; thus f assumes its value in I_τ at most countably many times. So we can modify the values of f assumed on the inverse image of I_τ so that they still remain in I_τ but are all different. If we carry out this step for all such intervals I_τ we obtain a one-to-one regressive function on S as required.

4. Now we are going to discuss the case $p > \aleph_1$. Since we admit also singular cardinal numbers as p here, we shall not be able in general to omit the adjective "divergent" from beside the expression "regressive function" as we sometimes did before. Nonetheless this question will be discussed later separately, and lastly we sall see that for other reasons this adjective can be omitted for singular p as well.

As said before, in the sequel p shall denote a cardinal number $> \aleph_1$ not cofinal to $\aleph_0$. This latter restriction on p is necessary since otherwise the theory of regressive functions becomes trivial and is of no interest for us in what follows.

The result we shall obtain says essentially that the cardinality conditions for the inverse images of regressive function cannot in general be strengthened. In order to show this in a precise form we first prove a lemma.

Lemma 4. 1. *For any regular cardinal number m satisfying* $\aleph_0 < m < p$ *there exists a non-stationary set such that any divergent regressive function defined on it assumes some large values at least m times.*

Here, of course, the word "large" expresses the fact that there are such values greater than any given ordinal less than p.

Proof. Let the band B be the closure of the set consisting of all ordinals $<p$ which are cofinal to m and let S be its complement; by definition S is not stationary. Now if f is any divergent regressive function on S then define the function g for $\beta \in B$ by

$$g(\beta) = \min \{f(\alpha): \alpha > \beta \ \& \ \alpha \in S\}.$$

Since a band is a fortiori stationary, the function g is not regressive. Consider a large $\beta \in B$ with $g(\beta) \geqq \beta$ and denote by β' its immediate successor in B. It follows that the ordinal type of the interval $[\beta, \beta')$ is m, a regular cardinal number greater than $\aleph_0$; thus there cannot be defined on it a divergent regressive mapping it into itself — considering these concepts "relativized" to the given interval. And this latter assertion says exactly that there exists some ordinal μ in the interval $[\beta, \beta')$ such that the set

$$\{\alpha \in S: \beta < \alpha < \beta' \ \& \ f(\alpha) = \mu\}$$

has power m. Thus the lemma is proved.

Of course it does not make any difference whether or not we require m to be regular in the above lemma, unless p is the successor of a singular cardinal. As we shall see in Section 6, the regularity of m in this latter case is essential. Thus neither can we, in general, omit the regularity condition imposed on m from the following theorem, which is an extension of the preceding lemma.

Theorem 4. 2. *Suppose* $p > \aleph_1$ *and is not cofinal to* $\aleph_0$. *Then p has a non-stationary subset S such that, whichever the regular cardinal* $m < p$ *may be, each divergent regressive function defined on S assumes some large values at least m times.*

Proof. If p is the immediate successor of a regular cardinal, then the assertion of the theorem coincides with that of the preceding lemma. Thus the remaining cases are:

a) p is the immediate successor of a singular cardinal m;

b) p is a limit cardinal.

The proof in case a) is most simple. Indeed, for any regular cardinal number $m < p$ let S_m be a non-stationary set satisfying the requirements of the preceding lemma and denote by S their union if m runs over all such values. Since the number of values taken by m is less than $p = p^*$, the set S is non-stationary and thus meets all the requirements of the theorem.

The case b) will be contained in Theorem 5. 1 below, which will be verified in a way independent of what we have done up to now. But, for the sake of the more ambitious readers, we point out that this case can be dealt with in a manner similar to case a), as follows.

As is well known, p can be represented as the sum of p^* smaller regular cardinal numbers:

$$p = \bigcup_{\alpha < p^*} m_\alpha.$$

As before, for each m_α we designate a non-stationary set S_α satisfying the requirements of the above lemma with $m = m_\alpha$, i.e. such that any divergent regressive function on it assumes some large values at least m_α times.

Now if we took the union of all such sets S_α, then the obtained set S would have the required properties of the theorem except that it might be stationary. In view of Theorem 1. 1, however, this latter case can be avoided by taking the union only of some appropriate upper sections of the sets S_α.

Now we shall indicate more exactly how this may be done.

Select an arbitrary non-stationary set Σ of power p^* which is cofinal to p and adjoin in turn its elements to the sets S_α. Omit those elements in S_α preceding the corresponding adjoined elements of Σ; then in view of Theorem 1. 1, the union of all sets obtained this way will be non-stationary, and — as seen just now — it satisfies the other requirements of the theorem, too.

5. As pointed out earlier, Theorem 4. 2 remains true, with a slight change, even if we do not require the divergence of the regressive functions mentioned there, i.e. we have the following:

Theorem 5. 1. *Suppose the cardinal p is greater than, and not cofinal to $\aleph_0$. Then there exists a non-stationary subset S of it such that, whichever the regular cardinal $m < p$ may be, each regressive function on S assumes some values at least m times.*

The price of the omission of the divergence condition imposed on f is that, unlike in the earlier cases, here we cannot expect f to assume large values at all. A natural substitute for this still remains true; namely, as easily seen from the proof below, f assumes some values at least m times even if we confine ourselves to large arguments in the domain of f.

Since in case p is regular, the theorem is trivially true for any non-divergent regressive function, the assertion for regular p is contained in Theorem 4. 2: thus we have to deal only with singular cardinals in place of p. Nevertheless the proof given here will apply for any limit cardinal so as to fulfil our earlier promise of giving an alternative proof of Theorem 4. 2 in this critical case.

Proof. If p is a limit cardinal, then the cardinal numbers preceding it form a band; select its complement as the non-stationary set S. Let f be an arbitrary regressive function on S and choose m to be a sufficiently large successor cardinal. Then f will be regressive on the lower section of S formed by its elements contained in m, too. This set, however, is stationary in m and hence the function f cannot be divergent here. This means that for some $\mu < m$ the set

$$\{\alpha : \alpha < m \ \& \ f(\alpha) = \mu\}$$

has power m, which implies the assertion of the theorem.

6. We mentioned earlier that Lemma 4. 1 fails for singular m. In fact, the following theorem is true.

Theorem 6. 1. *Let m be a singular cardinal number and let p be its immediate successor. If S is a non-stationary subset of p, then there exists a regressive function f on it which takes each value less than m times.*

Since p is obviously regular, such a function f must be divergent. The proof of the theorem might be compared to that of the first assertion in Theorem 3. 3.

Proof. Let M denote the closure of the set formed by all ordinals $<p$ having endings similar to m — i.e. each of which has some appropriate upper section of type m. Obviously, M is a band.

First we want to show that the assertion of the theorem is valid for the set $S = p - M$ which is now clearly non-stationary.

To this end let us consider any of the intervals $I_\tau = (\xi_\tau, \xi_{\tau+1})$ formed by two consecutive elements of M. According to the singularity of m, such an interval can be decomposed as the sum of a number less than m of its subsets each of which has cardinality $< m$. Now define the regressive function f_τ on I_τ as follows: transform every element of such a subset onto its initial element with the exception of this latter one the image of which will be ξ_τ. If we consider a function f_τ obtained this way on each of the intervals I_τ as a part of a function f defined on the whole set S, this latter function will meet all the requirements of the theorem.

Thus what now remains for us to verfy is that the assertion of the theorem holds for any non-stationary subset of M. For this aim we modify slightly the definition of the above intervals I_τ inasmuch as we adjoin to them their left endpoints, i.e. we put $I_\tau = [\xi_\tau, \xi_{\tau+1})$.

Let us now be given a non-stationary subset S of M and a divergent regressive function f defined on it. If I_τ is any of the considered intervals, then clearly its whole inverse image under f has power $\leqq m$; thus, in view of the singularity of m, we can modify the values in it assumed by f such that they still remain in I_τ but each of them will be taken less than m times. Since the disjoint intervals I_τ altogether cover the entire set p, by carrying out these modifications of the function f in all of the considered intervals, the obtained function will satisfy all the requirements of the theorem.

7. What now remains concerns our "standard" idea. It can be formulated in a lemma and why we did not do it before is that we thought that it was more simple to carry out the proof in each particular case. Nonetheless, for the sake of its own interest, we now bring it into the limelight:

Lemma 6. 1. *If B is a stationary set in p and f is a divergent regressive function defined on some subset of p, then f cannot jump over all the large elements of B, i.e. there are some large β in B such that for every α in the domain of f and exceeding β we have $f(\alpha) \geqq \beta$.*

We omit the proof.

References

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