On interpolation functions. III

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In several previous notes (see PEETRE [8], [9], [10]; see also GOULAOUIC [5]) we found various conditions, both necessary and sufficient, for a function to be an *interpolation function*, of given power p, $1 < p < \infty$ — a notion which has its origin in the work of FOIAŞ—LIONS [4]. In particular what concerns *non-exact* interpolation functions our results were almost complete, while as for *exact* interpolation functions the problem is, up to our knowledge, still essentially open (unless $p=2$, see DONOUGHUE [3]). This note is devoted to the observation that the methods of [8], [9], [10] are sufficiently powerful to settle the question not only in the limiting case $p=1$ (and, by a conveniently modified argument, the case $p=\infty$ too), which is fairly obvious (see [5]), but also in two additional cases of a quite different nature: 1° $0<p<1$, 2° $0<p<\infty$ and, in place of the field of real numbers R, a general local field F (e.g. the field of P-adic numbers Q_P, P being any (rational) prime number). In case 1° we thus have to leave the realm of Banach spaces and admit "quasi-Banach" spaces; in case 2° we encounter analogous vector spaces over the field F. The possibility of both types of extensions, when dealing with interpolation in general, was first realized by KRÉE [6]. In fact it is possible to treat both cases simultanously within the framework of what we call "ϱ-normed additive groups", with a given ϱ, $0<\varrho\leqq\infty$, and p ranging in the interval $0<p\leqq\varrho$. Clearly $\varrho=1$ in case 1° and $\varrho=\infty$ in case 2°. (It should be noted that there are also other parallels between the two cases. E.g. to DAY'S theorem [2] to the effect that (in general) $(L^p)'=0$ if $p<1$ (case 1°) there corresponds $(L^p)'=0$ if $p<\infty$ (case 2°): there is (in general) no integral for functions with values in F (see MONNA [7]).

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Let G be an additive (Abelian) group. By a ϱ-norm, where $0<\varrho\leqq\infty$, in G we mean a mapping $G\ni a\to\|a\|\in R_+$ such that

a) $\|a\|=0\Leftrightarrow a=0$,

b) $\|a+b\|\leqq(\|a\|^\varrho+\|b\|^\varrho)^{1/\varrho}$ (i.e. $\|a+b\|\leqq\max(\|a\|,\|b\|)$ if $\varrho=\infty$).

If $\varrho<\infty$ then $a\to\|a\|$ is a ϱ-norm if and only if $a\to\|a\|^\varrho$ is a 1-norm. Therefore there are really only two cases: 1° $\varrho=1$ and 2° $\varrho=\infty$. But it is, from the notational point of view, convenient not to pretend of this fact. An additive group G in which a ϱ-norm is singled out we call a ϱ-normed additive group. The principal example is of course when G is a vector space over a "ϱ-valued" field F. If $F=R$ with its usual valuation (absolute value) we must have $\varrho \leqq 1$ (unless $G=0$) but if F is a local field (say, the field of P-adic numbers Q_P) the case $\varrho=\infty$ of course can occur (see [7]). If π is an endomorphism of G (i.e. $\pi(a+b) = \pi(a)+\pi(b)$) we say that π is bounded with bound M if

(1) $$\|\pi a\| \leqq M\|a\|.$$

The additive group of bounded endomorphisms of G we denote by $\mathscr{B}(G)$.

Let X be a locally compact space provided with a positive measure μ, ζ a positive μ-measurable function on X, G a *complete* ϱ-normed additive group, $0<p<\infty$. Denote by $\mathscr{K}=\mathscr{K}(G)$ the space of bounded μ-measurable functions on X with values in G and compact support. If $a \in \mathscr{K}$ we set

(2) $$\|a\|_\zeta = \|a\|_{L^p_\zeta} = \Big[\int_X (\zeta(x)\|a(x)\|)^p \, d\mu\Big]^{1/p}.$$

This is clearly a ϱ_1-norm in $\mathscr{K}$, with $\varrho_1=\min(\varrho, p)$. The completion of $\mathscr{K}$ in this ϱ_1-norm we denote by $L^p_\zeta=L^p_\zeta(G)$. A great portion of the theory of L^p spaces with values in a Banach space E (over R), as developped e.g. in Bourbaki [1], chap. IV, can be carried over to the present case, L^p spaces with values in a complete ϱ-normed additive group G (and weight function ζ). But if $p<\varrho$, as we have already remarked, there is (in general) no integral (see [7]).

Now we come to our main definition. We say that a function $H=H(z_0, z_1)$, defined, continuous, and positive for $z_0>0$, $z_1>0$, is an *exact interpolation function*, of power p, with respect to G, if for any X, μ, ζ_0, ζ_1 it follows from $\pi \in \mathscr{B}(L^p_{\zeta_0}) \cap \mathscr{B}(L^p_{\zeta_1})$ that $\pi \in \mathscr{B}(L^p_\zeta)$, with $\zeta=H(\zeta_0, \zeta_1)$ and

(3) $$M \leqq \max(M_0, M_1)$$

for the three bounds M_0, M_1, M involved. We consider here only functions H which moreover are homogeneous of degree 1. We can thus write

$$H(z_0, z_1) = z_0 h(z_1/z_0)$$

where h is uniquely determined by H.

Our main result now reads:

Theorem. *Assume that H is an exact interpolation function of power p, with respect to a complete ϱ-normed additive group G satisfying the condition:*

(∗) *For every $\varepsilon>0$ there exists a positive number $\lambda<\varepsilon$ and an endomorphism χ of G such that $\|\chi(a)\|=\lambda\|a\|$.*

Then $\varphi(\sigma)=\big(h(\sigma^{1/p})\big)^p$ is concave. If $p \leqq \varrho$ this condition is also sufficient for H to be an exact interpolation function of power p, with respect to any G.

Remark. If G is a vector space over a field F one can take χ in $(*)$ to be multiplication with a suitable $c \in F$. E.g. if $F = Q_P$ we may take c to be a power of P.

Proof (necessity). As in [9], p. 170, we take X to be the set of $n+1$ points $x, x_1, \ldots, x_n$ and assume that μ to each of these points assigns the mass 1. Furthermore we take $\zeta_0 \equiv 1$ and $\zeta_1(x) = z$, $\zeta_1(x_i) = z_i$ $(i = 1, \ldots, n)$ where

$$z^p = \frac{1}{n}(z_1^p + \cdots + z_n^p). \tag{4}$$

For a given $\varepsilon > 0$ we choose λ and χ as in $(*)$ and take n to be the integer part of $1/\lambda^p$, i.e. $n \leqq 1/\lambda^p < n+1$ or

$$1 - \varepsilon^p < 1 - \lambda^p < n\lambda^p \leqq 1. \tag{5}$$

We define π by

$$\pi a(x) = 0, \quad \pi a(x_i) = \chi(a(x)) \quad (i = 1, \ldots, n).$$

For the three bounds of π we have then (using the condition on χ in $(*)$)

$$M_0 = \lambda n^{1/p}, \quad M_1 = \lambda \frac{1}{z}[z_1^p + \cdots + z_n^p]^{1/p}, \quad M = \lambda \frac{1}{h(z)}\left[(h(z_1))^p + \cdots + (h(z_n))^p\right]^{1/p}$$

or, in view of (4) and (5),

$$M_0 \leqq 1, \quad M_1 \leqq 1, \quad M > (1-\varepsilon^p)^{1/p} \frac{1}{h(z)} \left\{\frac{1}{n}\left[(h(z_1))^p + \cdots + (h(z_n))^p\right]\right\}^{1/p}.$$

From (3) it follows now

$$(1-\varepsilon^p)\frac{1}{n}\left[(h(z_1))^p + \cdots + (h(z_n))^p\right] < (h(z))^p$$

or if we set $\sigma_i = z_i^p$ $(i = 1, \ldots, n)$ and use (4) again

$$(1-\varepsilon^p)\frac{\varphi(\sigma_1) + \cdots + \varphi(\sigma_n)}{n} < \Phi\left(\frac{\sigma_1 + \cdots + \sigma_n}{n}\right).$$

Assume for simplicity that n is even, say $n = 2m$. Then we may take $\sigma_i = \sigma$ if $i = 1, \ldots, m$ and $\sigma_i = \tau$ if $i = m+1, \ldots, n$. It follows that

$$(1-\varepsilon^p)\frac{\Phi(\sigma) + \Phi(\tau)}{2} < \Phi\left(\frac{\sigma+\tau}{2}\right)$$

or, since $\varepsilon > 0$ was arbitrary,

$$\frac{\Phi(\sigma) + \Phi(\tau)}{2} \leqq \Phi\left(\frac{\sigma+\tau}{2}\right).$$

This proves the concavity of φ.

Proof (sufficiency). Let us set (see [8])

$$K_p(t, a) = \inf_{a = a_0 + a_1} \left(\|a_0\|_{\zeta_0}^p + t^p \|a_1\|_{\zeta_1}^p\right)^{1/p}$$

where $0<t<\infty$ and $a\in L^p_{\zeta_0}+L^p_{\zeta_1}$. It is readily seen, using (1), that

$$K_p(t, \pi a) \leqq \max(M_0, M_1) K_p(t, a).$$

Thus if we can find a representation of the form

$$\|a\|_\zeta = \Phi[K_p(t, a)] \tag{6}$$

with a functional $\Phi[\varphi]$ which is *monotone* and homogeneous of degree 1, we are through, because we then get

$$\|\pi a\|_\zeta = \Phi[K_p(t, \pi a)] \leqq \max(M_0, M_1)\Phi[K_p(t, a)] = \max(M_0, M_1)\|a\|_\zeta,$$

which leads to (3). By (2) we obtain

$$[K_p(t, a)]^p = \inf \int_X [(\zeta_0(x)\|a_0(x)\|)^p + (t\zeta_1(x)\|a_1(x)\|)^p]\, d\mu = \tag{7}$$
$$= \int_X \inf [(\zeta_0(x)\|a_0(x)\|)^p + (t\zeta_1(x)\|a_1(x)\|)^p]\, d\mu.$$

We claim that (if $p\leqq\varrho$)

$$\inf[(\zeta_0(x)\|a_0(x)\|)^p + (t\zeta_1(x)\|a_1(x)\|)^p] = [\min(\zeta_0(x), t\zeta_1(x))\|a(x)\|]^p. \tag{8}$$

Indeed we have, by the "ϱ-triangle inequality" and using the fact that $p\leqq\varrho$,

$$\min(\zeta_0(x), t\zeta_1(x))\|a(x)\| \leqq [(\zeta_0(x)\|a_0(x)\|)^\varrho + (t\zeta_1(x)\|a_1(x)\|)^\varrho]^{1/\varrho} \leqq$$
$$\leqq [(\zeta_0(x)\|a_0(x)\|)^p + (t\zeta_1(x)\|a_1(x)\|^p)]^{1/p}.$$

This leads to "$\geqq$" in (8). But by considering the special decomposition $a_0=a$, $a_1=0$ or $a_0=0$, $a_1=a$, depending on the value of t, we see that the corresponding lower bound is attained. Thus we get effectively "$=$" in (8). Inserting next (8) in (7) we arrive at the formula

$$K_p(t, a) = \|a\|_{\min(\zeta_0, t\zeta_1)}. \tag{9}$$

Now every concave function φ admits the representation (see [9])

$$\varphi(\sigma) = C_0 + C_1\sigma + \int_0^\infty \min(1, \tau\sigma)\, d\xi(\tau)$$

where C_0 and C_1 are positive constants and ξ is a positive measure on $(0, \infty)$. It follows that

$$(H(\zeta_0, \zeta_1))^p = C_0\zeta_0^p + C_1\zeta_1^p + \int_0^\infty (\min(\zeta_0, t\zeta_1))^p\, d\xi(t^p)$$

or, by (9), with $d\alpha(t) = d\xi(t^p)$,

$$\|a\|_\zeta = \left[C_0\|a\|^p_{\zeta_0} + C_1\|a\|^p_{\zeta_1} + \int_0^\infty (K_p(t, a))^p\, d\alpha(t)\right]^{1/p}. \tag{10}$$

Since, by (9),

$$\|a\|_{\zeta_0} = \lim_{t\to\infty} K_p(t, a), \quad \|a\|_{\zeta_1} = \lim_{t\to 0} \frac{1}{t} K_p(t, a),$$

(10) is a representation of the desired type (6).

Remark. In conclusion we remark that the above result probably also can be extended to the case when not only the weight function ζ but also p is a varied, à la STEIN—WEISS [12] (i.e. we have spaces $L^{p_0}_{\zeta_0}$ and $L^{p_1}_{\zeta_1}$ in place of just $L^p_{\zeta_0}$ and $L^p_{\zeta_1}$), by making use of the corresponding ideas in PEETRE [11].

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