On density of transitive algebras

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1. Introduction

In the following an *algebra* is a weakly closed algebra (containing the identity) of bounded linear operators on a Hilbert space $\mathfrak{H}$. An algebra $\mathscr{A}$ is *transitive* if the only (closed) subspaces of $\mathfrak{H}$ invariant under every operator in $\mathscr{A}$ are $\{0\}$ and $\mathfrak{H}$. If $\mathfrak{H}$ is finite-dimensional, then the only transitive algebra on $\mathfrak{H}$ is $\mathscr{B}(\mathfrak{H})$, the algebra of all operators on $\mathfrak{H}$, by BURNSIDE's Theorem. It is not known whether or not there are transitive algebras on $\mathfrak{H}$ other than $\mathscr{B}(\mathfrak{H})$ if $\mathfrak{H}$ is infinite-dimensional.

ARVESON [1] has shown that a transitive algebra which contains a maximal abelian von Neumann algebra is $\mathscr{B}(\mathfrak{H})$. In the same paper ARVESON shows that a transitive algebra which contains the unilateral shift of multiplicity one is $\mathscr{B}(\mathfrak{H})$. In this note we show that a transitive algebra containing a weighted shift of a certain type (a *Donoghue operator*, defined below) or a finite rank operator, must be $\mathscr{B}(\mathfrak{H})$. These results will follow from a more general result given below.

2. Sufficient conditions that a transitive algebra be $\mathscr{B}(\mathfrak{H})$

If A is an operator on $\mathfrak{H}$, let $A^{(n)}$ denote the operator $A\oplus\ldots\oplus A$ (n copies) on $\mathfrak{H}\oplus\ldots\oplus\mathfrak{H}$ (n copies). If $\mathscr{A}$ is an algebra on $\mathfrak{H}$ let $\mathscr{A}^{(n)}=\{A^{(n)}: A\in\mathscr{A}\}$. The following lemma is the main tool developed by ARVESON for studying transitive algebras. The proof of the lemma is implicitly contained in the proof of the main theorem of [1]; (for an alternate exposition of the proof see the proof of Theorem 1 of [5]).

Lemma. *Let $\mathscr{A}$ be a transitive algebra on $\mathfrak{H}$ having the following property: whenever, for any n, $\{T_i\}_{i=1}^n$ is a collection of (not necessarily bounded) operators with a common dense domain $\mathfrak{D}$ such that*

$$\{(x, T_1 x, \cdots, T_n x): x\in\mathfrak{D}\}$$

is an invariant subspace of $\mathscr{A}^{(n+1)}$, *then the* T_i *are each scalar multiples of the identity operator. Then* $\mathscr{A}=\mathscr{B}(\mathfrak{H})$.

This lemma can be strengthened as follows.

Theorem. *Let* $\mathscr{A}$ *be a transitive algebra on* $\mathfrak{H}$ *having the following property: whenever, for any* n, $\{T_i\}_{i=1}^n$ *is a collection of (not necessarily bounded) operators with a common dense domain* $\mathfrak{D}$ *such that*

$$\{(x, T_1 x, \cdots, T_n x) : x \in \mathfrak{D}\}$$

is an invariant subspace of $\mathscr{A}^{(n+1)}$, *then each* T_i *has an eigenvector (in* $\mathfrak{D}$*). Then* $\mathscr{A}=\mathscr{B}(\mathfrak{H})$.

Proof. Let $\mathfrak{M}=\{(x, T_1x, \ldots, T_nx): x\in\mathfrak{D}\}$ be an invariant subspace of $\mathscr{A}^{(n+1)}$. By the above lemma it sufficies to show that each T_i is a multiple of the identity. By hypothesis there is a unit vector x_1 in $\mathfrak{D}$ such that $T_1x_1=\lambda_1x_1$ for some λ_1. Let $\mathfrak{D}_1=\{x\in\mathfrak{D}: T_1x=\lambda_1x\}$. Then $\mathfrak{D}_1$ is an invariant linear manifold of $\mathscr{A}$, since if $x\in\mathfrak{D}_1$ and $A\in\mathscr{A}$, then $T_1Ax=AT_1x$ (by the invariance of $\mathfrak{M}$ under $\mathscr{A}^{(n+1)}$), and hence $T_1Ax=\lambda_1Ax$.

Now let $\mathfrak{M}_1=\{(x, \lambda_1x, T_2x, \ldots, T_nx): x\in\mathfrak{D}_1\}$. Then $\mathfrak{M}_1$ is closed and is an invariant subspace of $\mathscr{A}^{(n+1)}$ contained in $\mathfrak{M}$. Also $\mathfrak{D}_1$ is an invariant linear manifold of $\mathscr{A}$ and is therefore dense in $\mathfrak{H}$. Thus $T_2|\mathfrak{D}_1$ has an eigenvector by hypothesis. If λ_2 is the corresponding eigenvalue, let

$$\mathfrak{D}_2 = \{x\in\mathfrak{D}_1 : T_2x = \lambda_2x\},$$

and let $\mathfrak{M}_2=\{(x, \lambda_1x, \lambda_2x, T_3x, \ldots, T_nx): x\in\mathfrak{D}_2\}$. Then $\mathfrak{M}_2$ is an invariant subspace of $\mathscr{A}^{(n+1)}$.

We can continue this procedure and get a linear manifold $\mathfrak{D}_n\subset\mathfrak{D}$ and complex numbers $\lambda_1, \ldots, \lambda_n$ such that the subspace

$$\mathfrak{M}_n = \{(x, \lambda_1 x, \cdots, \lambda_n x) : x\in\mathfrak{D}_n\}$$

is an invariant subspace of $\mathscr{A}^{(n+1)}$ other than $\{0\}$. Then $\mathfrak{D}_n$ is an invariant linear manifold of $\mathscr{A}$ and therefore is dense in $\mathfrak{H}$. Also

$$\mathfrak{D}_n = \{x : (x, \lambda_1 x, \cdots, \lambda_n x)\in\mathfrak{M}\}$$

and is therefore closed. Hence $\mathfrak{D}_n=\mathfrak{H}$ and, since $\mathfrak{D}\supset\mathfrak{D}_n$, $\mathfrak{D}=\mathfrak{H}$ and $T_i=\lambda_iI$ on $\mathfrak{H}$.

Corollary 1. *Let* $\mathscr{A}$ *be a transitive algebra. If there is an operator* A *in* $\mathscr{A}$ *such that:*

(i) *every eigenspace of* A *is one dimensional, and*

(ii) *for each* n, *every non-zero invariant subspace of* $A^{(n)}$ *contains an eigenvector of* $A^{(n)}$,

then $\mathscr{A}=\mathscr{B}(\mathfrak{H})$.

Proof. We use the above theorem. Let

$$\mathfrak{M} = \{(x, T_1 x, \cdots, T_n x) : x \in \mathfrak{D}\}$$

be an invariant subspace of $\mathscr{A}^{(n+1)}$.

By hypothesis there is a vector x_0 in $\mathfrak{D}$ such that $(x_0, T_1 x_0, \ldots, T_n x_0)$ is an eigenvector of $A^{(n+1)}$. Then x_0 is an eigenvector of A; suppose that $Ax_0 = \lambda x_0$. Then for each i, $AT_i x_0 = \lambda T_i x_0$. The fact that the eigenspace of A is one-dimensional implies that for each i there is a λ_i such that $T_i x_0 = \lambda_i x_0$.

Corollary 2. *The only transitive algebra which contains a non-zero finite rank operator is* $\mathscr{B}(\mathfrak{H})$.

Proof. Let $\mathscr{A}$ be a transitive algebra containing a non-zero finite rank operator A. We use the above theorem again.

Let $\mathfrak{M} = \{(x, T_1 x, \ldots, T_n x) : x \in \mathfrak{D}\}$ be an invariant subspace of $\mathscr{A}^{(n+1)}$. We first show that the range of A is contained in $\mathfrak{D}$. For this let $y = Ax$. Since $\mathfrak{D}$ is dense in $\mathfrak{H}$, we can choose a sequence $\{x_k\} \subset \mathfrak{D}$ such that $x_k \to x$. Then Ax_k is in the range of A and in $\mathfrak{D}$ for each k. But the intersection of the range of A and $\mathfrak{D}$ is a finite-dimensional subspace and therefore $Ax = \lim_{k \to \infty} Ax_k$ is in $\mathfrak{D}$.

Now the fact that $AT_i = T_i A$ for each i implies that the range of A is invariant under each T_i. Hence each T_i has a finite-dimensional invariant subspace and therefore has an eigenvector.

3. Algebras containing Donoghue operators

The above has an interesting application to a special case. A *Donoghue operator* is an operator A such that there is an orthonormal basis $\{e_i\}_{i=0}^{\infty}$ for $\mathfrak{H}$ and a square-summable sequence $\{a_n\}_{n=1}^{\infty}$ of monotone decreasing positive numbers such that $Ae_0 = 0$ and $Ae_i = a_i e_{i-1}$ for $i > 0$. It is well known (see [3], for example) that the non-trivial invariant subspaces of a Donoghue operator are the subspaces $\mathfrak{M}_i = \bigvee_{j=0}^{i} \{e_j\}$ for non-negative integers i.

Corollary 3. *If $\mathscr{A}$ is a transitive algebra containing a Donoghue operator then* $\mathscr{A} = \mathscr{B}(\mathfrak{H})$.

Proof. Let A be the Donoghue operator in $\mathscr{A}$. To apply Corollary 1 we need only show that A satisfies conditions (i) and (ii) of the hypothesis. It is trivial to see that A satisfies (i); 0 is the only eigenvalue of A, and the corresponding eigenvectors are all multiples of e_0. The proof that A satisfies (ii) was given in [6]; it involves a computation which we outline below. (A stronger result, that every invariant subspace of $A^{(n)}$ is spanned by the finite-dimensional invariant subspaces that it contains, has been proven in [4].)

To see that A satisfies (ii), fix n and let $S=A^{(n)}$. Let $\mathfrak{M}$ be any invariant subspace of S other than $\{0\}$ and let $(x_1, \ldots, x_n)$ be a non-zero vector in $\mathfrak{M}$. Let $x_j=\sum_{i=0}^{\infty}\alpha_{i,j}e_i$ for $j=1, \ldots, n$. If the sequence $\{\alpha_{i,j}\}_{i=0}^{\infty}$ has only finitely many non-zero terms for every j, then the invariant subspace of S generated by $(x_1, \ldots, x_n)$ is finite-dimensional and thus contains an eigenvector. We therefore can assume that, for each N, the number

$$\alpha_N = \max\{|\alpha_{i,j}| : i \geqq N;\ j=1, \cdots, n\}$$

is greater than 0. Then for each N there is an $i(N)$ greater than or equal to N and a $j(N)$ such that

$$\alpha_N = |\alpha_{i(N), j(N)}|.$$

For each fixed N

$$\frac{1}{\alpha_{i(N),j(N)}a_{i(N)} \cdots a_1} S^{i(N)}(x_1, \cdots, x_n)$$

is equal to

$$\left(\frac{\alpha_{i(N),1}}{\alpha_{i(N),j(N)}}e_0, \frac{\alpha_{i(N),2}}{\alpha_{i(N),j(N)}}e_0, \cdots, \frac{\alpha_{i(N),n}}{\alpha_{i(N),j(N)}}e_0\right)+(h_{N,1}, \cdots, h_{N,n})$$

where

$$h_{N,j} = \sum_{i=i(N)+1}^{\infty} \frac{\alpha_{i,j}a_i \cdots a_{i-i(N)+1}}{\alpha_{i(N),j(N)}a_{i(N)} \cdots a_1} e_{i-i(N)}.$$

It is easily shown that, for each j, $h_{N,j}$ approaches 0 as N approaches infinity (cf. [3], p. 304). Some number j_0 between 1 and n must occur infinitely often as a value $j(N)$. Also for each fixed j the sequence $\left\{\frac{\alpha_{i(N),j}}{\alpha_{i(N),j(N)}}\right\}_{N=1}^{\infty}$ is contained in the unit disk, and hence has a subsequence converging to a number β_j. Choosing an appropriate subsequence of $\{N\}$ we see that the eigenvector $(\beta_1 e_0, \beta_2 e_0, \ldots, \beta_n e_0)$ of S (with $\beta_{j_0}=1$) lies in $\mathfrak{M}$.

The following corollary seems surprising.

Corollary 4. *If A is a Donoghue operator and B is any operator without point spectrum, then the algebra generated by A and B is $\mathscr{B}(\mathfrak{H})$.*

Proof. If B has no point spectrum, then A and B have no common invariant subspaces since every invariant subspace of A is finite-dimensional. Hence the algebra generated by A and B satisfies the hypotheses of Corollary 3.

4. Remarks

The results of section 3 remain valid if the definition of a Donoghue operator is extended to mean any unilateral weighted shift operator whose sequence of weights (the sequence $\{a_j\}$) is monotone decreasing in absolute value, non-zero, and p-summable for some $p>0$. The proof of Corollary 3 remains essentially the same for this case.

Another application of the theorem is to transitive algebras containing operators of the form $A\oplus(A+\lambda)$ where A is a Donoghue operator and λ is a non-zero complex number. Property (ii) for such an operator follows from the fact that the spectra of A and $A+\lambda$ are "sufficiently disjoint" to imply that every invariant subspace of $A\oplus(A+\lambda)$ is the direct sum of an invariant subspace of A and an invariant subspace of $A+\lambda$ (see [2]).

In view of Corollary 3 and ARVESON's result about the unilateral shift it seems likely that the following is true.

Conjecture. *A transitive algebra containing a weighted shift with non-zero weights is* $\mathcal{B}(\mathfrak{H})$.

Corollary 2 suggests that one might be able to show that a transitive algebra which contains a non-zero compact operator must be $\mathcal{B}(\mathfrak{H})$. Such a theorem would undoubtedly be extremely difficult to prove however. The corollaries of such a theorem would be striking: for example, one corollary would be the result that if A is a compact operator and B is any operator that commutes with A, then A and B have a common non-trivial invariant subspace. In particular this would prove that every operator which commutes with a compact operator has a non-trivial invariant subspace.

Note that ARVESON's lemma, the above Theorem, and Corollaries 1 and 2 remain valid if $\mathfrak{H}$ is a Banach space and the algebras considered are strongly closed. It can then be shown that Corollaries 3 and 4 hold for Donoghue operators on l^p for $1<p<\infty$.

References

[1] W. B. ARVESON, A density theorem for operator algebras, *Duke Math. J.*, **34** (1967), 635—647.

[2] T. CRIMMINS and P. ROSENTHAL, On the decomposition of invariant subspaces, *Bull. A. M. S.*, **73** (1967), 97—99.

[3] P. R. HALMOS, *A Hilbert space problem book* (Princeton, 1967).

[4] E. A. NORDGREN, Invariant subspaces of a direct sum of weighted shifts, *Pac. J. Math.*, **27** (1968), 589—598.

[5] H. RADJAVI and P. ROSENTHAL, On invariant subspaces and reflexive algebras (to appear in *Amer. J. Math.*).

[6] P. ROSENTHAL, *On lattices of invariant subspaces*, Dissertation, University of Michigan (1967).

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(Received Aug. 21, 1968)