A generalization of the Rees theorem in semigroups

By GÉRARD LALLEMENT and MARIO PETRICH in University Park (Penn., U.S.A.)

1. Introduction and summary

The Rees theorem asserts that a semigroup S is completely 0-simple if and only if S is isomorphic to a regular Rees matrix semigroup $\mathcal{M}^{\circ}(G; I, \Lambda; P)$ over a group with zero G^0 (3. 5, [1]; see also the original paper of REES [5]). As with the Rees matrix semigroups over a group with zero, we can construct a semigroup $\mathcal{M}^0(D; I, \Lambda; P)$ starting with any semigroup D instead of a group. A natural way of generalizing the Rees theorem consists on solving the following problem: to give an abstract characterization of semigroups $\mathcal{M}^0(D; I, \Lambda; P)$, where D is taken in a class of semigroups containing the class of groups. The purpose of this paper is to give several solutions of this problem, with some restrictions on P, using the notion of a 0-matrix decomposition of a semigroup [3].

We say that a semigroup S has a 0-matrix decomposition if S has a zero 0 and there exists a congruence ϱ on S such that (a) 0 is a ϱ-class and (b) S/P is a rectangular 0-band (i.e., a completely 0-simple semigroup with trivial subgroups). In such a case, $ABA \subseteq A$ or 0 for all ϱ-classes A, B. If all the ϱ-classes which are subsemigroups of S belong to a class $\mathcal{T}$ of semigroups, we say that S is a 0-matrix of semigroups of type $\mathcal{T}$. In case S has no zero, obvious modifications of the preceding definitions yield the concepts of a matrix decomposition and a matrix of semigroups of type $\mathcal{T}$. Using this terminology and separating the cases with and without zero, we obtain the following weakened versions of the Rees theorem:

(i) A semigroup S is a matrix of groups if and only if $S \cong \mathcal{M}(G; I, \Lambda; P)$, where G is a group (Theorem 12, [4]).

(ii) A semigroup S is a 0-matrix of groups such that the classes of the corresponding congruence ϱ satisfy $ABA = A$ or 0, if and only if $S \cong \mathcal{M}^{\circ}(G; I, \Lambda; P)$, where G is a group (4. 5 [3]).

In view of this situation, for the case without zero, we introduce the class of composable semigroups (2. 1), which, e.g., contains the class of bisimple semigroups with identity (2. 3). For the case with zero, we introduce a special kind of matrix decomposition, the Rees 0-composition (3. 3). Our main results are:

(1) A semigroup S is a matrix of composable semigroups if and only if $S \cong \mathcal{M}(D; I, \Lambda; P)$, where D is a composable semigroup and P is a $\Lambda \times I$-matrix over G, the group of units of D (3. 10).

(2) A semigroup S is a Rees 0-composition if and only if $S \cong \mathcal{M}^0(D; I, \Lambda; P)$ where D is a semigroup with identity and P is a regular $\Lambda \times I$-matrix over G^0, G being the group of units of D (3. 4).

(3) We give an abstract characterization of $\mathcal{M}^0(D; I, \Lambda; P)$ when D is a bisimple inverse semigroup with identity, P is a regular $\Lambda \times I$-matrix over G, and G is the group of units of D (5. 1). This characterization uses properties of the partially ordered set of idempotents and the fact that principal right [left] ideals form a semilattice under intersection.

In section 2, we study right [left] composable semigroups. Using the notion of an r-composition of semigroups (2. 4), introduced by YOSHIDA [8], we show in 2. 5 that any r-composition of right composable semigroups is isomorphic to $D \times R$, where D is right composable and R is a right zero semigroup. Our main results (1) and (2) are established in section 3. We also prove that the class of composable semigroups is the largest class $\mathscr{C}$ with the property that every matrix of semigroups of type $\mathscr{C}$ is a Rees composition (3. 9). Section 4 is devoted to 0-restricted homomorphisms of semigroups $\mathcal{M}^0(D; I, \Lambda; P)$ discussed above; they can be described in essentially the same manner as those of a Rees matrix semigroup over a group with zero (4. 1, 4.2). We also characterize Rees matrix semigroups which can be expressed as products of some special semigroups (4. 3). The abstract characterization described in (3) above is given in section 5. It is of interest to note that 5. 1 makes it possible to construct certain 0-bisimple regular semigroups from bisimple inverse semigroups with identity. The characterization given in 5. 1 simplifies if we assume that the idempotents form a subsemigroup. We obtain, e.g., the structure of any bisimple semigroup whose idempotents form a subsemigroup isomorphic to the Cartesian product of a semilattice with identity and a rectangular band (5. 6). In Section 6, we conclude by giving an example of a bisimple regular semigroup whose set of idempotents does not satisfy the conditions of the version of 5. 1 without zero.

Recently STEINFELD [7] gave an abstract characterization of matrix semigroups $\mathcal{M}^0(D; I, \Lambda; P)$ which are locally regular (i.e. the entries of P are not necessarily taken in G^0, where G is the group of units of D, but certain entries of P have invertibility properties). Our results concern the instance in which the entries of P are in G^0 and widely supplement those of Steinfeld in this case.

Except for the concepts defined in the paper, we follow the notation and terminology of CLIFFORD and PRESTON [1]. In section 3 and 5, we use a number of concepts introduced and results proved in [3]; however, the knowledge of [3] is not indispensable. In order to avoid repetition, instead of "S is a semigroup with identity [zero]" we write $S=S^1$ $[S=S^0]$. If $S=S^1$ $[S=S^0]$, then 1 [0] denotes the identity [zero] of S unless stated otherwise.

2. Composable semigroups

Definition 2. 1. A semigroup $S=S^1$ is called *right* [*left*] *composable* if for any $a\in S$, $axa=xa$ $[axa=ax]$ for all $x\in S$ implies $a=1$. A semigroup is called *composable* if it is both right and left composable.

The reason for this terminology as well as the importance of such semigroups will become clear later (2. 5). We consider now some properties and examples of these semigroups.

Proposition 2. 2. *A semigroup S is* [*right*] *composable if and only if $S=S^1$ and the identity transformation on S is the only inner* [*right*] *translation of S which is also a homomorphism.*

Proof. The bracketted part follows directly from the equivalence of the statements: (i) ϱ_a is a homomorphism, (ii) $(xa)(ya)=(xy)a$ for all $x, y\in S$, (iii) $aya=ya$ for all $y\in S$, when $S=S^1$.

Proposition 2. 3. *Any bisimple semigroup $S=S^1$ is composable.*

Proof. Let $a\in S$ and suppose that $axa=xa$ for all $x\in S$. Since S is bisimple, there is $z\in S$ such that $a\mathcal{L}z$ and $z\mathcal{R}1$; $a\mathcal{L}z$ implies $za=z$ since $a^2=a$. If z' is an inverse of z, then $z\mathcal{L}z'z$, which implies $z'z=z'za$. Since $axa=xa$ for all $x\in S$, we obtain $z'z=z'za=z'aza=z'az$. On the other hand, $z\mathcal{R}1$ implies $zz'=1$, which together with $z'z=z'az$ yields

$$1=zz'=z(z'z)z'=z(z'az)z'=(zz')a(zz')=1a1=a.$$

Hence S is right composable; analogously S is also left composable.

Example 1.*) Let S be a left group which is not a group. Then S^1 is right composable. Since every idempotent e of S is a right identity of S, we have $exe=ex$ for all $x\in S^1$; hence S^1 is not left composable.

Example 2. Let $S=S^1$ and let S have a minimal two-sided completely simple ideal K which is neither a left nor a right group. Further suppose that 1 is the only idempotent of S not contained in K. Then S is composable. For if $axa=xa$

*) The referee points out that the right composable semigroups are precisely those semigroups with identity containing no proper left ideals with identity (the verification is left to the reader). Hence any left simple semigroup with identity, or a semigroup S^1 where S is a left simple semigroup without identity, is right composable.

for every $x \in S$ and some $a \in S$, then $a^2 = a$ and thus either $a = 1$ or $a \in K$. The latter possibility is excluded since $axa = xa$ for every $x \in K$ implies $a\mathcal{R}x$ for every $x \in K$, which in turn implies that K is a right group, contradicting the hypothesis. Thus S is right composable; by symmetry S is also left composable.

Example 3. Let $S = S^1$ be the union of groups such that no $\mathcal{D}$-class of S different from the $\mathcal{D}$-class containing the identity is a left or a right group. Similar reasoning as in the previous example shows that S is composable.

Definition 2.4 (cf. [8]). A semigroup S is said to be an *r-composition* [*l-composition*] *of semigroups* $\{D_\lambda\}_{\lambda \in \Lambda}$ if $S = \bigcup_{\lambda \in \Lambda} D_\lambda$, $D_\lambda \cap D_\mu = \Box$ if $\lambda \neq \mu$, and each D_λ is a left [right] ideal of S.

Note that if S is an r-composition of semigroups D_λ, the equivalence relation induced on S is a congruence ϱ such that S/ϱ is a right zero semigroup, and conversely, every such congruence induces an r-composition of S. Furthermore, for a given family of pairwise disjoint semigroups, there may exist no r-composition (see [8]). The importance of the class of right composable semigroups stems from the next two theorems.

Theorem 2.5. *Let S be an r-composition of right composable semigroups D_λ, $\lambda \in \Lambda$, with identities 1_λ. Then the set $R_\Lambda = \{1_\lambda | \lambda \in \Lambda\}$ is a right zero semigroup, all D_λ are isomorphic, and $S \cong D_1 \times R_\Lambda$, where D_1 is any of the semigroups D_λ.*

Proof. For any $\lambda, \mu \in \Lambda$ and $x \in D_\mu$, we get $x1_\lambda \in D_\lambda$ so that $x1_\lambda = 1_\lambda x 1_\lambda$; since also $x = 1_\mu x$, we obtain $1_\lambda 1_\mu x 1_\lambda 1_\mu = x 1_\lambda 1_\mu$ for every $x \in D_\mu$. Since $1_\lambda 1_\mu \in D_\mu$ and D_μ is right composable, it follows that $1_\lambda 1_\mu = 1_\mu$. This proves that R_Λ is a right zero semigroup. Fix any index, say $1 \in \Lambda$, and define φ by $x\varphi = (x1_1, 1_\lambda)$ if $x \in D_\lambda$. A straightforward calculation shows that φ is an isomorphism of S onto $D_1 \times R_\Lambda$. (This is a special case of Theorem 14, [4].) It is now clear that all D_λ are isomorphic.

Consider the following conditions on a class $\mathscr{C}$ of semigroups:

(A) Every semigroup in $\mathscr{C}$ has an identity.

(B) $\mathscr{C}$ is closed under isomorphisms.

(C) If a semigroup S is an r-composition of semigroups C_λ in $\mathscr{C}$, $\lambda \in \Lambda$, then $R_\Lambda = \{1_\lambda | 1_\lambda$ is the identity of $C_\lambda, \lambda \in \Lambda\}$ is a subsemigroup of S (and thus, by the proof of 2.5, $S \cong C_1 \times R_\Lambda$, where C_1 is any of the semigroups C_λ and R_Λ is a right zero semigroup).

Theorem 2.6. *Let $\mathscr{C}$ be a class of semigroups satisfying* (A), (B), (C). *Then every semigroup in $\mathscr{C}$ is right composable.*

Proof. Let $C \in \mathscr{C}$ and suppose that $exe = xe$ for some $e \in C$ and all $x \in C$. Let α be an isomorphism of C onto a semigroup D disjoint from C. In $S = C \cup D$

define multiplication as follows:

$$x*y=\begin{cases} xy & \text{if} \quad x,y\in C \quad \text{or} \quad x,y\in D,\\ [(xe)\alpha]y & \text{if} \quad x\in C, y\in D,\\ (x\alpha^{-1})ey & \text{if} \quad x\in D, y\in C \end{cases}$$

(multiplication in C and D is denoted by juxtaposition). A simple calculation shows that this multiplication is associative. Hence S is an r-composition of C and D. By (B), $D\in\mathscr{C}$ and thus by (C), the identities 1_C and 1_D of C and D, respectively, form a right zero semigroup. Hence

$$e\alpha=[(1_Ce)\alpha]1_D=1_C*1_D=1_D,$$

which implies that $e=1_C$. Consequently C is right composable.

Corollary 2.7. *The class of right composable semigroups is the largest class of semigroups satisfying* (A), (B), (C).

3. The main theorem

Recall that a *rectangular* 0-*band* is a regular Rees matrix semigroup over a one element group, and that a congruence ϱ on a semigroup S is called an I-matrix congruence if S/ϱ is a rectangular 0-band and I is the complete inverse image of 0. The classes of ϱ which are complete inverse images of nonzero idempotents in S/ϱ are called *nonzero classes* of ϱ, the others are *zero classes*. We are interested here solely in the case when S has a zero and $I=0$; in such a case, ϱ is called a 0-*matrix congruence* on S. These concepts were introduced and studied in [3] (see particularly section 1).

Definition 3.1. Let $\mathscr{C}$ be a class of semigroups. A semigroup S is said to be a 0-*matrix of semigroups of type* $\mathscr{C}$ if $S=S^0$ and there is a 0-matrix congruence $\mathfrak{M}$ on S whose nonzero classes are in $\mathscr{C}$.

Proposition 3.2. *If $S=S^0$ is a semigroup having a* 0-*matrix congruence $\mathfrak{M}$ all of whose nonzero classes have an identity, then $\mathfrak{M}$ is the finest* 0-*matrix congruence on S.*

Proof. Let $\mathfrak{M}$ be as in the statement of the proposition, and $\Phi(0)$ be the finest 0-matrix congruence on S (2.6, [3]). If A is a nonzero class of $\mathfrak{M}$, then $\alpha=\Phi(0)|_A$ is a matrix congruence (i.e., A/α is a rectangular band), and since A has an identity, α must be the universal relation. Hence A is a class of $\Phi(0)$. Conversely, if B is a nonzero class of $\Phi(0)$, it must be contained in a nonzero class A of $\mathfrak{M}$ and thus $B=A$, i.e., B is a class of $\mathfrak{M}$. It follows that $\mathfrak{M}$ and $\Phi(0)$ have the same nonzero classes which by 2.2, [2], implies $\mathfrak{M}=\Phi(0)$.

Definition 3. 3. A semigroup S is called a *Rees 0-composition* if $S=S^0$ and there is a 0-matrix congruence $\mathfrak{M}$ on S whose classes, denoted by $\Sigma_{i\lambda}$ $(i\in I, \lambda\in\Lambda)$, satisfy the condition

(D) for every $\mathfrak{M}$-class $\Sigma_{i\lambda}$, there exists an element $x_{i\lambda}\in\Sigma_{i\lambda}$ with the property that for every $j\in I$, $\mu\in\Lambda$:

$$x_{i\lambda}\Sigma_{j\mu}=\Sigma_{i\mu} \text{ or } 0 \text{ and } \Sigma_{j\mu}x_{i\lambda}=\Sigma_{j\lambda} \text{ or } 0.$$

Remarks. i) More precisely, we should speak of a „Rees 0-composition relative to $\mathfrak{M}$"; however, in 3. 5 we will prove that every nonzero class of $\mathfrak{M}$ has an identity, which by 3. 2 will imply uniqueness of $\mathfrak{M}$.

ii) Note that $\Sigma_{i\lambda}\Sigma_{j\mu}\neq 0$ if and only if $\Sigma_{j\lambda}$ is a nonzero class (p. 80, [3]) so that by (D), $x_{i\lambda}\Sigma_{j\mu}=\Sigma_{i\mu}$ if and only if $\Sigma_{j\lambda}$ is a nonzero class; analogously for $\Sigma_{j\mu}x_{i\lambda}$.

iii) A 0-matrix of semigroups of some type $\mathscr{C}$ need not be a Rees 0-composition; e.g., a 0-matrix of groups is in general an ideal extension of a completely 0-simple semigroup.

Definition 3. 4. Let $D=D^1$ be a semigroup with the group of units G (i.e., G is the $\mathscr{H}$-class of 1), and let P be a regular $\Lambda\times I$-matrix over G^0 (i.e., in each row and each column of P there is at least one nonzero entry). By $\mathscr{M}^0(D; I, \Lambda; P)$ denote the set of all elements $(a; i, \lambda)$, with $a\in D^0$ (D with zero adjoined even if D already has a zero), $i\in I$, $\lambda\in\Lambda$ (the elements $(0; i, \lambda)$ are identified with a single element 0, the zero of $\mathscr{M}^0(D; I, \Lambda; P)$) together with the multiplication

$$(a; i, \lambda)(b; j, \mu)=(ap_{\lambda j}b; i, \mu).$$

Then $\mathscr{M}^0(D; I, \Lambda; P)$ is a semigroup which we call the *Rees matrix semigroup* (over D^0). The congruence $\mathfrak{M}$ defined by $(a; i, \lambda)\mathfrak{M}(b; j, \mu)\Leftrightarrow i=j$, $\lambda=\mu$, and $0\mathfrak{M}0$ is called the *associated congruence.*

If D is a group, $D=G$ and our terminology and notation agree with that used in [1] except that we consider only a regular sandwich matrix P. We are now ready to state our main result.

Theorem 3. 4. *A semigroup S is a Rees 0-composition if and only if S is isomorphic to a Rees matrix semigroup $\mathscr{M}^0(D; I, \Lambda; P)$, where $D=D^1$.*

Proof. Sufficiency. Let $Q=\mathscr{M}^0(D; I, \Lambda; P)$ where $D=D^1$. Note first that the associated congruence $\mathfrak{M}$ on Q is a 0-matrix congruence; its classes different from 0 are the sets $\Sigma_{i\lambda}=\{(a; i, \lambda)\,|\,a\in D\}$, $i\in I$, $\lambda\in\Lambda$. Let $x_{i\lambda}=(1; i, \lambda)$; since $\mathfrak{M}$ is a 0-matrix congruence, $x_{i\lambda}\Sigma_{j\mu}\subseteqq\Sigma_{i\mu}\cup 0$ for any $j\in I$, $\mu\in\Lambda$. If $x_{i\lambda}\Sigma_{j\mu}\neq 0$, then $x_{i\lambda}\Sigma_{j\mu}\subseteqq\Sigma_{i\mu}$ and $p_{\lambda j}\neq 0$. Consequently, for any $(y; i, \mu)\in\Sigma_{i\mu}$, we obtain

$$(y; i, \mu)=(1; i, \lambda)(p_{\lambda j}^{-1}y; j, \mu)\in x_{i\lambda}\Sigma_{j\mu},$$

whence $x_{i\lambda}\Sigma_{j\mu}=\Sigma_{i\mu}$. The other half of condition (D) is established similarly.

Necessity. The proof is broken into several lemmas in which S is a Rees 0-composition, and $\Sigma_{i\lambda}$ are the classes of the congruence induced.

Lemma 3. 5. *Every nonzero class* $\Sigma_{i\lambda}$ *has an identity (denoted by* $1_{i\lambda}$*).*

Proof. Let $\Sigma_{i\lambda}$ be a nonzero class; then $x_{i\lambda}\Sigma_{i\lambda}=\Sigma_{i\lambda}x_{i\lambda}=\Sigma_{i\lambda}$ (by (D)). There is $t\in\Sigma_{i\lambda}$ such that $x_{i\lambda}=x_{i\lambda}t$ and for every $y\in\Sigma_{i\lambda}$, $y=ux_{i\lambda}$ for some $u\in\Sigma_{i\lambda}$. Hence $yt=ux_{i\lambda}t=ux_{i\lambda}=y$, i.e., t is a right identity of $\Sigma_{i\lambda}$. Dually, $\Sigma_{i\lambda}$ also has a left identity which implies that $1_{i\lambda}=t$ is the identity of $\Sigma_{i\lambda}$.

Lemma 3. 6. *If* $y\in\Sigma_{i\lambda}$, *then* $y=1_{i\mu}y=y1_{j\lambda}$ *whenever* $\Sigma_{i\mu}$ *and* $\Sigma_{j\lambda}$ *are nonzero classes.*

Proof. Since $\Sigma_{i\mu}$ is a nonzero class, $x_{i\mu}\Sigma_{i\lambda}=\Sigma_{i\lambda}$ by (D). For any $y\in\Sigma_{i\lambda}$, we obtain $y=x_{i\mu}u$ for some $u\in\Sigma_{i\lambda}$, so that $1_{i\mu}y=1_{i\mu}x_{i\mu}u=x_{i\mu}u=y$. The equality $y=y1_{j\lambda}$ is established analogously.

As a consequence of 3. 6, we have

$$1_{i\mu}1_{i\delta}=1_{i\delta},\ 1_{i\lambda}1_{j\lambda}=1_{i\lambda}$$

provided that $\Sigma_{i\mu}$, $\Sigma_{i\delta}$, $\Sigma_{i\lambda}$, and $\Sigma_{j\lambda}$ are nonzero classes. We will use this without express mention.

Lemma 3. 7. *Let*

$$S_1=\{x\in S|x\mathscr{R}1_{i\nu},\ x\mathscr{L}1_{k\lambda}\ \text{for some}\ i,k\in I,\ \nu,\lambda\in\Lambda\}\cup 0;$$

then S_1 *is a completely* 0*-simple subsemigroup of* S, *and* S_1 *intersects every class of* $\mathfrak{M}$.

Proof. Let $x\in\Sigma_{i\lambda}\cap S_1$ and $y\in\Sigma_{j\mu}\cap S_1$. If $xy=0$, then $xy\in S_1$. Suppose $xy\neq 0$. We have $x\mathscr{R}1_{i\nu}$, $x\mathscr{L}1_{k\lambda}$, $y\mathscr{R}1_{j\delta}$, $y\mathscr{L}1_{m\mu}$ for some $k,m\in I$, $\nu,\delta\in\Lambda$. Consequently

$$x1_{j\delta}1_{j\lambda}=x1_{j\lambda}=x1_{k\lambda}1_{j\lambda}=x1_{k\lambda}=x.$$

So we have $x1_{j\delta}\mathscr{R}x$; thus $x\mathscr{R}1_{i\nu}$ implies $x1_{j\delta}\mathscr{R}1_{i\nu}$. Since $\mathscr{R}$ is a left congruence, $y\mathscr{R}1_{j\delta}$ implies $xy\mathscr{R}x1_{j\delta}$, and hence $xy\mathscr{R}1_{i\nu}$. One shows similarly that $xy\mathscr{L}1_{m\mu}$, which proves that $xy\in S_1$. Thus S_1 is a subsemigroup of S.

Let $\Sigma_{i\lambda}$ be any class and $\Sigma_{i\mu}$, $\Sigma_{j\lambda}$ be nonzero classes. Then by (D), $x_{i\lambda}\Sigma_{j\mu}=\Sigma_{i\mu}$ whence $x_{i\lambda}t=1_{i\mu}$ for some $t\in\Sigma_{j\mu}$; this together with $x_{i\lambda}=1_{i\mu}x_{i\lambda}$ (3. 6) implies $x_{i\lambda}\mathscr{R}1_{i\mu}$. Dually, we obtain $x_{i\lambda}\mathscr{L}1_{j\lambda}$, and thus $x_{i\lambda}\in\Sigma_{i\lambda}\cap S_1$, which proves the last statement of the lemma. Further, if $\Sigma_{i\lambda}$ is a nonzero class, then $\Sigma_{i\lambda}\cap S_1=G_{i\lambda}$, the group of units of $\Sigma_{i\lambda}$. For obviously $\Sigma_{i\lambda}\cap S_1\supseteqq G_{i\lambda}$, while the opposite inclusion holds since $x\in\Sigma_{i\lambda}\cap S_1$ implies $x\mathscr{R}1_{i\mu}$, $x\mathscr{L}1_{j\lambda}$ for some $j\in I$, $\mu\in\Lambda$; this together with $1_{i\lambda}\mathscr{R}1_{i\mu}$, $1_{i\lambda}\mathscr{L}1_{j\lambda}$ implies $x\mathscr{H}1_{i\lambda}$. It then follows that the restriction of $\mathfrak{M}$ to

S_1 is a 0-matrix congruence whose nonzero classes are groups. By 3. 6, every element of S_1 has a left (and a right) identity, and thus 4. 1 and 4. 5 of [3] imply that S_1 is completely 0-simple.

Let $H_{i\lambda}=\Sigma_{i\lambda}\cap S_1$ and choose any nonzero class Σ_{11}; then H_{11} is the group of units of Σ_{11}. For each $i\in I, \lambda\in\Lambda$, select $r_i\in H_{i1}$ and $q_\lambda\in H_{1\lambda}$ and define P as the $\Lambda\times I$-matrix $P=(p_{\lambda i})$ over H_{11}^0 by

$$p_{\lambda i}=\begin{cases} q_\lambda r_i & \text{if } q_\lambda r_i\in H_{11},\\ 0 & \text{otherwise.}\end{cases}$$

Lemma 3. 8. *Every nonzero element of S is uniquely representable in the form $r_i a q_\lambda$ with $a\in\Sigma_{11}$, $i\in I$, $\lambda\in\Lambda$ and the mapping Φ defined by $(a; i, \lambda)\Phi=r_i a q_\lambda$, $0\Phi=0$, is an isomorphism of $\mathcal{M}^0(\Sigma_{11}; I, \Lambda; P)$ onto S.*

Proof. For $\lambda\in\Lambda$, there exists $i\in I$ such that $\Sigma_{i\lambda}$ is a nonzero class. Hence q_λ has a unique inverse q'_λ in $R_{1_{i\lambda}}\cap L_{1_{11}}$ since $H_{i\lambda}$ is a group and S_1 is completely 0-simple. Thus $1_{11}q_\lambda=q_\lambda$ and $q_\lambda q'_\lambda=1_{11}$. Now $\mathfrak{M}=\mathfrak{R}\cap\mathfrak{L}$, where $\mathfrak{R}[\mathfrak{L}]$ is a 0-left [0-right] zero equivalence on S (1. 7 and 1. 10, [3]). Let C_i, $i\in I$, and Γ_λ, $\lambda\in\Lambda$, denote the $\mathfrak{R}$ and $\mathfrak{L}$ classes of S, respectively, different from 0. For every $x\in\Gamma_1$, by 3. 6, we obtain $xq_\lambda q'_\lambda=x1_{11}=x$, and analogously, for every $y\in\Gamma_\lambda$, $yq'_\lambda q_\lambda=y$. The mappings $x\to xq_\lambda (x\in\Gamma_1)$ and $y\to yq'_\lambda$ $(y\in\Gamma_\lambda)$ are mutually inverse C_i-class preserving one-one mappings of Γ_1 onto Γ_λ and of Γ_λ onto Γ_1, respectively. Using r_i and r'_i, one similarly establishes one-one Γ_λ-class preserving correspondences between C_1 and C_i. It follows that the mappings $x\to r_i x q_\lambda$ $(x\in\Sigma_{11})$ and $y\to r'_i y q'_\lambda$ $(y\in\Sigma_{i\lambda})$ are one-one inverse mappings. Since every nonzero element of S belongs to some $\Sigma_{i\lambda}$, this proves the first part of the lemma and also that Φ is one-one and onto. The proof that Φ is a homomorphism is the same as for the corresponding part of the Rees theorem in [1], pages 93 and 94.

This completes the proof of 3. 4.

Recall that a *matrix congruence* ϱ on a semigroup S is a congruence such that S/ϱ is a rectangular band (see, e.g., [4]). If we adjoin a zero to S and extend ϱ to S^0 by letting $0\varrho 0$, we get a 0-matrix congruence. Definitions 3. 1 and 3. 3 then carry over to this case if we then remove the zero. We thus obtain a *matrix of semigroups of type* $\mathscr{C}$ and a *Rees composition* $\mathcal{M}(D; I, \Lambda; P)$. The next theorem shows that for the case of a matrix of semigroups, the class of composable semigroups is the best in a certain sense.

Theorem 3. 9. *Let $\mathscr{C}$ be a class of semigroups closed under isomorphisms. Then every semigroup in $\mathscr{C}$ has an identity and every matrix of semigroups of type $\mathscr{C}$ is a Rees composition if and only if $\mathscr{C}$ is contained in the class of composable semigroups.*

Proof. Necessity. Let S be an r-composition of semigroups C_λ in $\mathscr{C}$, $\lambda \in \Lambda'$. By hypothesis and 3. 5, $S \cong \mathscr{M}(D; I, \Lambda; P)$ with $D = D^1$. Since every C_λ and D have identities, 3. 2 implies that, by identifying S with $\mathscr{M}(D; I, \Lambda; P)$, the congruences induced by the r-composition and by the Rees composition coincide. Hence we may set $I = \{1\}$, $\Lambda = \Lambda'$. If 1_λ is the identity of C_λ, we have $1_\lambda = (p_{\lambda 1}^{-1}; 1, \lambda)$; it follows that the set $R_\Lambda = \{1_\lambda | \lambda \in \Lambda\}$ is a subsemigroup of S. We have proved that $\mathscr{C}$ satisfies condition (C) (preceding 2. 6); since $\mathscr{C}$ satisfies (A) and (B) by hypothesis, 2. 6 implies that every semigroup in $\mathscr{C}$ is right composable. A dual proof shows that every semigroup in $\mathscr{C}$ is also left composable.

Sufficiency. Let S be a matrix of composable semigroups $\Sigma_{i\lambda}$ with identity $1_{i\lambda}$, $i \in I, \lambda \in \Lambda$. To establish condition (D) in this case, it suffices to show that $1_{i\lambda}\Sigma_{j\mu} = \Sigma_{i\mu}$ and $\Sigma_{j\mu}1_{i\lambda} = \Sigma_{j\lambda}$ for all $i, j \in I$, $\lambda, \mu \in \Lambda$. The set $C_i = \bigcup_{\lambda \in \Lambda} \Sigma_{i\lambda}$ is an r-composition of semigroups $\Sigma_{i\lambda}$ which are (right) composable; 2. 5 then implies $1_{i\lambda}1_{i\mu} = 1_{i\mu}$; dually, we have $1_{i\lambda}1_{j\lambda} = 1_{i\lambda}$. Hence

$$1_{i\mu} = 1_{i\lambda}1_{i\mu} = (1_{i\lambda}1_{j\lambda})1_{i\mu} = 1_{i\lambda}(1_{j\lambda}1_{i\mu}) \in 1_{i\lambda}\Sigma_{j\mu},$$

whence for all $x \in \Sigma_{i\mu}$,

$$x = 1_{i\mu}x \in 1_{i\lambda}\Sigma_{j\mu}x \subseteq 1_{i\lambda}\Sigma_{j\mu}.$$

Consequently $\Sigma_{i\mu} \subseteq 1_{i\lambda}\Sigma_{j\mu}$; the opposite inclusion holds since $\Sigma_{i\lambda}\Sigma_{j\mu} \subseteq \Sigma_{i\mu}$. Thus $1_{i\lambda}\Sigma_{j\mu} = \Sigma_{i\lambda}$; the equality $\Sigma_{j\mu}1_{i\lambda} = \Sigma_{j\lambda}$ is proved symmetrically. Therefore S is a Rees composition.

Corollary 3. 10. *A semigroup S is a matrix of composable semigroups if and only if $S \cong \mathscr{M}(D; I, \Lambda; P)$, where D is composable.*

It appears to be much more difficult to obtain a characterization of a 0-matrix of semigroups of type $\mathscr{T}$ without additional restrictions. The next theorem, which generalizes 4. 5, [3], points in this direction.

Theorem 3. 11. *Let S be a 0-matrix of bisimple semigroups with identity. Then the following conditions on S are equivalent:*

a) *S is regular.*

b) *S is 0-bisimple.*

c) *S is a Rees 0-composition.*

In such a case, $S \cong \mathscr{M}^0(D; I, \Lambda; P)$, where $D = D^1$ is bisimple.

Proof. Denote the classes of the 0-matrix congruence (see 3. 2) by $\Sigma_{i\lambda}$, $i \in I$, $\lambda \in \Lambda$, and if $\Sigma_{i\lambda}$ is a nonzero class, let $1_{i\lambda}$ denote its identity. Recall the notation $C_i = \bigcup_{\lambda \in \Lambda} \Sigma_{i\lambda}$, $\Gamma_\lambda = \bigcup_{i \in I} \Sigma_{i\lambda}$.

a)⇒b). If $x \in \Sigma_{i\lambda}$, then by regularity of S, $x = xyx$ for some $y \in \Sigma_{j\mu}$. It follows that $e = yx$ is an idempotent of $\Sigma_{j\lambda}$ and $x \mathscr{L} e$. Since $\Sigma_{j\lambda}$ is then a bisimple semigroup, we have $e\mathscr{D}1_{j\lambda}$, and thus $x\mathscr{D}1_{j\lambda}$. If $\Sigma_{k\lambda}$ is any nonzero class, 2. 3 and 2. 5 imply $1_{j\lambda}1_{k\lambda}=1_{j\lambda}$ and $1_{k\lambda}1_{j\lambda}=1_{k\lambda}$, i.e., $1_{j\lambda}\mathscr{R}1_{k\lambda}$. Thus $x\mathscr{D}1_{k\lambda}$ which shows that any two elements of Γ_λ are $\mathscr{D}$-equivalent. By symmetry we obtain that any two elements of C_i are also $\mathscr{D}$-equivalent. Since these statements hold for any i, λ it follows that S is 0-bisimple.

b)⇒c). Consider any $\Sigma_{i\lambda}$ and any nonzero classes $\Sigma_{i\nu}$ and $\Sigma_{k\lambda}$. Since S is 0-bisimple, there exists $x\in S$ such that $1_{i\nu}\mathscr{R}x$ and $x\mathscr{L}1_{k\lambda}$. It follows that $x\in\Sigma_{i\lambda}\cap S_1$. Let $x_{i\lambda}$ be any element of $\Sigma_{i\lambda}\cap S_1$ and suppose that $x_{i\lambda}\Sigma_{j\mu}\neq 0$; then $x_{i\lambda}\Sigma_{j\mu}\subseteq\Sigma_{i\mu}$. Let $y\in\Sigma_{i\mu}$. Since S is 0-bisimple and contains nonzero idempotents, S is regular and thus $y=ey$ for some idempotent $e\in\Sigma_{i\theta}$. Hence $\Sigma_{i\theta}$ is a nonzero class and thus $x_{i\lambda}\mathscr{R}1_{i\theta}$, which implies $1_{i\theta}=x_{i\lambda}z$ for some z, and $x_{i\lambda}=1_{i\theta}x_{i\lambda}$. By symmetry, we have $x_{i\lambda}=x_{i\lambda}1_{n\lambda}$ for some $n\in I$, which together with $1_{j\lambda}\mathscr{L}1_{n\lambda}$ implies $x_{i\lambda}=x_{i\lambda}1_{j\lambda}$. Consequently

$$y = ey = 1_{i\theta}(ey) = 1_{i\theta}y = (x_{i\lambda}z)y = (x_{i\lambda}1_{j\lambda})zy = x_{i\lambda}(1_{j\lambda}zy)\in x_{i\lambda}\Sigma_{j\mu}.$$

Therefore $\Sigma_{i\mu}\subseteq x_{i\lambda}\Sigma_{j\mu}$ and the equality holds. The proof of $\Sigma_{j\mu}x_{i\lambda}=\Sigma_{j\lambda}$, if $\Sigma_{i\mu}$ is a nonzero class, is dual. Therefore (D) holds and S is a Rees 0-composition. ∎

c)⇒a). By 3. 4, $S\cong\mathscr{M}^0(D; I, \Lambda; P)$ with $D=D^1$, and by the uniqueness of induced congruences (3. 2), D is bisimple. Item a) then follows by a straightforward computation in $\mathscr{M}^0(D; I, \Lambda; P)$ using regularity of D.

4. Homomorphisms of Rees matrix semigroups

A homomorphism φ of a semigroup $S=S^0$ into a semigroup $T=T^0$ is said to be *0-restricted* if $a\varphi=0\Leftrightarrow a=0$. A homomorphic image of a Rees matrix semigroup need not be a Rees matrix semigroup; however, if φ is a 0-restricted homomorphism of a Rees matrix semigroup S onto S^*, then S^* is also a Rees matrix semigroup. The next theorem describes all 0-restricted homomorphisms of a Rees matrix semigroup into another; it generalizes a result of Munn (3. 11, [1]). Recall that for a semigroup D, D^0 denotes the semigroup obtained by adjoining a zero to D (even if D already has a zero).

Theorem 4. 1. *Let* $S=\mathscr{M}^0(D; I, \Lambda; P)$, $S^*=\mathscr{M}^0(D^*; I^*, \Lambda^*; P^*)$, *where D and D^* are semigroups with identities* 1 *and* 1^*, *respectively. Let ω be a 0-restricted homomorphism of D^0 into $(D^*)^0$. Let $i\to u_i$ be a mapping of I into the $\mathscr{R}$-class of* 1ω, *$\lambda\to v_\lambda$ be a mapping of Λ into the $\mathscr{L}$-class of* 1ω, *and Φ, ψ be mappings of I into I^* and Λ into Λ^*, respectively, such that*

$$p_{\lambda i}\omega = v_\lambda p^*_{\lambda\psi, i\Phi}u_i \tag{1}$$

for all $i \in I$, $\lambda \in \Lambda$. *For each element* $(a; i, \lambda) \in S$, *define*

(2) $$(a; i, \lambda)\theta = [u_i(a\omega)v_\lambda; i\Phi, \lambda\psi].$$

Then θ *is a* 0*-restricted homomorphism of* S *into* S^*. *Conversely, every* 0*-restricted homomorphism of* S *into* S^* *can be obtained in this fashion.*

Proof. In the direct part, the proof that θ is a homomorphism is the same as in 3. 11, [1], and is omitted. It is clear that θ is 0-restricted.

For the converse, the proof of 3. 11, [1], is modified as follows. The mappings Φ and ψ are defined as there (substituting $\mathscr{R}$ and $\mathscr{L}$-classes by $\mathfrak{R}$ and $\mathfrak{L}$-classes, respectively; see the proof of 3. 8). We select a nonzero class Σ_{11} of the associated congruence $\mathfrak{M}$ of S, and denote its identity by 1_{11}. Then $1_{11}\theta$ is a nonzero idempotent so that the class of $\mathfrak{M}^*$ (the associated congruence of S^*) is nonzero, whence $p^*_{1\psi, 1\Phi} \neq 0$. The equation

(3) $$(p_{11}^{-1}x; 1, 1)\theta = [p^{*-1}_{1\psi, 1\Phi}(x\omega); 1\Phi, 1\psi]$$

defines a homomorphism of D into D^*. For every $i \in I$, define u_i by

(4) $$(1; i, 1)\theta = [u_i; i\Phi, 1\psi]$$

and for every $\lambda \in \Lambda$, define v_λ by

(5) $$(p_{11}^{-1}; 1, \lambda)\theta = [p^{*-1}_{1\psi, 1\Phi}v_\lambda; 1\Phi, \lambda\psi].$$

Since $(1; i, 1)\mathscr{L}(p_{11}^{-1}; 1, 1)$, by (3) and (4), we obtain

$$[u_i; i\Phi, 1\psi]\mathscr{L}[p^{*-1}_{1\psi, 1\Phi}(1\omega); 1\Phi, 1\psi],$$

which implies $u_i\mathscr{L}1\omega$. Similarly $(p_{11}^{-1}; 1, 1)\mathscr{R}(p_{11}^{-1}; 1, \lambda)$ implies, by (3) and (5),

$$[p^{*-1}_{1\psi, 1\Phi}v_\lambda; 1\Phi, \lambda\psi]\mathscr{R}[p^{*-1}_{1\psi, 1\Phi}(1\omega); 1\Phi, 1\psi],$$

which implies $p^{*-1}_{1\psi, 1\Phi}v_\lambda\mathscr{R}p^{*-1}_{1\psi, 1\Phi}(1\omega)$ whence $v_\lambda\mathscr{R}1\omega$. Writing $(a; i, \lambda) \in S$ in the form

$$(1; i, 1)(p_{11}^{-1}a; 1, 1)(p_{11}^{-1}; 1, \lambda)$$

and applying θ, we obtain (2). From (2), we have

$$(1; i, \lambda)^2\theta = [u_i(p_{\lambda i}\omega)v_\lambda; i\Phi, \lambda\psi],$$

$$[(1; i, \lambda)\theta]^2 = [u_i(1\omega)v_\lambda p^*_{\lambda\psi, i\Phi}u_i(1\omega)v_\lambda; 1\Phi, \lambda\psi]$$

and thus

(6) $$u_i(p_{\lambda i}\omega)v_\lambda = u_i(1\omega)v_\lambda p^*_{\lambda\psi, i\Phi}u_i(1\omega)v_\lambda.$$

Since $u_i\mathscr{L}1\omega$ and $v_\lambda\mathscr{R}1\omega$, we have $u_i(1\omega) = u_i$, $u_i'u_i = 1\omega$, $(1\omega)v_\lambda = v_\lambda$, $v_\lambda v_\lambda' = 1\omega$ for some $u_i', v_\lambda' \in D$. Taking into account $u_i(1\omega) = u_i$, $(1\omega)v_\lambda = v_\lambda$ and multiplying (6) on the left by u_i' and on the right by v_λ', we obtain (1).

To state the next corollary, using the notation of 4. 1, we define a left invertible

$I^* \times I$-matrix U over $(D^*)^0$ as a matrix which has exactly one nonzero entry in each row and in each column, this entry being in the $\mathcal{L}$-class of 1^*. A right invertible $\Lambda \times \Lambda^*$-matrix V is defined dually. The proof of the following corollary is essentially the same as the proof of 3. 12, [1].

Corollary 4. 2. *Two Rees matrix semigroups* $\mathcal{M}^0(D; I, \Lambda; P)$ *and* $\mathcal{M}^0(D^*; I^*, \Lambda^*; P^*)$ *are isomorphic if and only if there exists an isomorphism* ω *of* D^0 *onto* $(D^*)^0$, *a left invertible* $I^* \times I$-*matrix* U *over* $(D^*)^0$ *and a right invertible* $\Lambda \times \Lambda^*$-*matrix* V *over* $(D^*)^0$ *such that* $P\omega = VP^*U$.

We now consider the special cases of Rees matrix semigroups which can be conveniently expressed as products of certain semigroups. Let A and B be semigroups, where B has a zero 0. By $A \times^0 B$ denote the Rees quotient $A \times B / A \times 0$ ($A \times B$ is the Cartesian product of A and B). Let P be a $\Lambda \times I$-matrix over a group with zero G^0. We say that P satisfies condition (M) if every nonzero product of the form

$$p_{\lambda_1 i_1}^{-1} p_{\lambda_1 i_2} p_{\lambda_2 i_2}^{-1} p_{\lambda_2 i_3} \cdots p_{\lambda_{n-1} i_{n-1}}^{-1} p_{\lambda_{n-1} i_n} p_{\lambda_n i_n}^{-1} p_{\lambda_n i_1}$$

is equal to 1, the identity of G (p. 97, [3]). Recall the definition of S_1 (3. 7).

Theorem 4. 3. *Let* $S = \mathcal{M}^0(D; I, \Lambda; P)$ *and let* G *be the group of units of* $D = D^1$. *Let* $\bar{P}$ *be the* $\Lambda \times I$-*matrix with entries*

$$\bar{p}_{\lambda i} = \begin{cases} 1 & \text{if } p_{\lambda i} \neq 0 \\ 0 & \text{if } p_{\lambda i} = 0. \end{cases}$$

Let $B = \mathcal{M}^0(1; I, \Lambda; \bar{P})$, *where* 1 *denotes a one element group. Then* $S_1 = \mathcal{M}^0(G; I, \Lambda; P)$ *and the following statements are equivalent:*

a) $S \cong D \times^0 B$; b) $S_1 \cong G \times^0 B$; c) P *satisfies* (M).

Proof. The first statement follows easily from the proof of 3. 7; b) and c) are equivalent by 4. 13, [3] ((a)$\Leftrightarrow$(e)). Since $S_1 = \mathcal{M}^0(G; I, \Lambda; P)$, it follows easily that a) implies b). Suppose that c) holds. By 4. 13 and 4.10 of [3], there exists a subsemigroup F of S_1 intersecting every $\mathcal{H}$-class $H_{i\lambda}$ of S_1 in exactly one element; denote it by $e_{i\lambda}$. If $H_{i\lambda}$ is a group, $e_{i\lambda}$ is an idempotent and thus $e_{i\lambda} = (p_{\lambda i}^{-1}; i, \lambda)$. If for $(x; i, \lambda) \in S$, $(x; i, \lambda)e_{j\lambda} \neq 0$, then $p_{\lambda j} \neq 0$ and thus $e_{j\lambda} = (p_{\lambda j}^{-1}; j, \lambda)$. Consequently $(x; i, \lambda)e_{j\lambda} = (x; i, \lambda)$. Symmetrically, if $e_{i\mu}(x; i, \lambda) \neq 0$, then $e_{i\mu}(x; i, \lambda) = (x; i, \lambda)$. Applying 4. 8, [3], we obtain a).

Corollary 4. 4. *Let* $S = \mathcal{M}^0(D; I, \Lambda; P)$. *If the group of units of* D *is trivial, then* $S \cong D \times^0 B$, *where* B *is as in* 4. 3.

Proof. If the group of units of D is trivial, 4. 3 implies that $S_1 \cong B$, whence $S \cong D \times^0 B$ again by 4. 3.

5. Rees matrix semigroups over a bisimple inverse semigroup with identity

The principal object of this section is to give an abstract characterization of such a semigroup using certain properties of its set of idempotents. From this we then derive simple characterizations of several classes of semigroups. The set E_S of idempotents of a semigroup S is now considered as a partially ordered set under the usual order $e \leqq f \Leftrightarrow e = ef = fe$. If we write $E_S \cong C$, where C is a semigroup, it means that E_S is a subsemigroup of S and is isomorphic to C.

Theorem 5. 1. *Let S be a 0-bisimple semigroup. Then $S \cong \mathcal{M}^0(D; I, \Lambda; P)$, where $D = D^1$ is a bisimple inverse semigroup if and only if S satisfies:*

a) *for all $a, b, c \in S$, $abc = 0 \Rightarrow ab = 0$ or $bc = 0$;*
b) *there exist order isomorphisms φ and ψ of E_S onto E_A,*

where $A = T \times {}^0 B$, $T = T^1$ is a semilattice, B is a rectangular 0-band, such that for all $e, f \in E_S$,

i) $ef = f \Leftrightarrow (e\varphi)(f\varphi) = f\varphi$,
ii) $ef = e \Leftrightarrow (e\psi)(f\psi) = e\psi$,
iii) *if $e\varphi = (x, a)$ and $e\psi = (y, b)$, then $a = b$;*
c) *for all $e, f \in E_S$,*
i) $eS \cap fS \neq 0 \Rightarrow eS \cap fS = efS$,
ii) $Se \cap Sf \neq 0 \Rightarrow Se \cap Sf = Sef$.
In such a case, $T \cong E_D$, $B \cong \mathcal{M}^0(1; I, \Lambda; \bar{P})$, where $\bar{P}$ is as in 4. 3.

Proof. Necessity. For convenience we identify S with $\mathcal{M}^0(D; I, \Lambda; P)$. Item a) follows from the fact that the associated congruence $\mathfrak{M}$ is a 0-matrix congruence (1. 6, [3]). Let $T = E_D$, $B = \mathcal{M}^0(1; I, \Lambda; \bar{P})$, and $A = T \times {}^0 B$. It is easy to see that

(1) $$E_S = \{(x; i, \lambda) | p_{\lambda i} \neq 0, \ x p_{\lambda i} x = x\} \cup 0.$$

On E_S define the mappings φ and ψ by:

$$(x; i, \lambda)\varphi = \big(x p_{\lambda i}, (1; i, \lambda)\big) \quad \text{if} \quad x \neq 0, \quad \text{and} \quad 0\varphi = 0,$$

$$(x; i, \lambda)\psi = \big(p_{\lambda i} x, (1; i, \lambda)\big) \quad \text{if} \quad x \neq 0, \quad \text{and} \quad 0\psi = 0.$$

Note that

(2) $$E_A = \{(e, (1; i, \lambda)) | e \in T, p_{\lambda i} \neq 0\} \cup 0.$$

Using (1) and (2), it is straightforward to verify that φ and ψ satisfy all the conditions in b). We prove only that c) i) holds; c) ii) is treated analogously. Thus let $e = (x; i, \lambda)$, $f = (y; j, \mu)$ be idempotents of S such that $eS \cap fS \neq 0$. Then

$$(x; i, \lambda)(z; k, \nu) = (y; j, \mu)(w; m, \delta) \neq 0$$

for some $(z; k, v)$, $(w; m, \delta) \in S$ and hence $i=j$. Since $e, f \in E_S$, (1) yields $p_{\lambda i} \neq 0$, $p_{\mu i} = p_{\mu j} \neq 0$, which by commutativity of idempotents in D implies

$$x p_{\lambda i} y = (x p_{\lambda i})(y p_{\mu i}) p_{\mu i}^{-1} = (y p_{\mu i})(x p_{\lambda i}) p_{\mu i}^{-1}.$$

Consequently

$$ef = (x; i, \lambda)(y; i, \mu) = (x p_{\lambda i} y; i, \mu) = (y; i, \mu)(x p_{\lambda i} p_{\mu i}^{-1}; i, \mu)$$

which implies $ef \in eS \cap fS$. Conversely, if $x \in eS \cap fS$, then $x = ex = fx = efx \in efS$. Therefore $eS \cap fS = efS$.

Sufficiency. We will freely use the terminology and results of [3]. First note that S is regular (0-bisimple containing a nonzero idempotent). Since S is 0-bisimple, for $a \neq 0$, $b \neq 0$, there is $c \in S$ such that $aS = cS$, $Sc = Sb$. It follows $aSb = cSb = cSc \neq 0$ and 0 is a prime ideal of S, which together with a) implies that 0 is a matrix ideal of S (p. 74, [3]). By 1. 6, [3], S has a 0-matrix congruence; let $\mathfrak{M}$ be the finest such. Then $\mathfrak{M} = \sigma \cap \tau$, where $a \sigma b \Leftrightarrow a = b = 0$ or there exist $a_1, a_2, \cdots, a_n \in S$ such that

$$aS \setminus 0 \,|\, a_1 S \setminus 0 \,|\, \cdots \,|\, a_n S \setminus 0 \,|\, bS \setminus 0 \tag{3}$$

($|$ means "intersects") and τ is defined symmetrically using left ideals (2. 6, [3]). We will show that each nonzero class of $\mathfrak{M}$ is a bisimple inverse semigroup with identity.

If $B \cong \mathcal{M}^0(1; I, \Lambda; Q)$, then $A \cong \mathcal{M}^0(T; I, \Lambda; Q)$. We identify A with $\mathcal{M}^0(T; I, \Lambda; Q)$.

Suppose that for $e, f \in E_S$, $eS \cap fS \neq 0$. By c) i), $eS \cap fS = efS$ which implies $f(ef) = ef = e(ef)$. Thus $ef \in E_S$ so that by b) i), we obtain

$$(f\varphi)[(ef)\varphi] = (ef)\varphi = (e\varphi)[(ef)\varphi] \neq 0.$$

Hence if $e\varphi = (t; i, \lambda)$ and $f\varphi = (u; j, \mu)$, then $i = j$. Now, if $e, f \in E_S$, $e \neq 0$, $e \sigma f$, then by (3)

$$eS \setminus 0 | e_1 S \setminus 0 | \cdots | e_n S \setminus 0 | fS \setminus 0$$

for some $e_1, e_2, \cdots, e_n \in E_S$ since $a_i S = e_i S$ for some $e_i \in E_S$ by regularity of S. Letting again $e\varphi = (t; i, \lambda)$ and $f\varphi = (u; j, \mu)$, the preceding observation implies $i = j$. Dually, if $e \tau f$, $e\psi = (v; i, \lambda)$, $f\psi = (w; j, \mu)$, then $\lambda = \mu$.

Conversely, if $e\varphi = (t; i, \lambda)$ and $f\varphi = (u; i, \mu)$, then $(e\varphi)[(e\varphi)(f\varphi)] = (e\varphi)(f\varphi)$ and $(f\varphi)[(e\varphi)(f\varphi)] = (e\varphi)(f\varphi)$, which by b) i) implies

$$e[(e\varphi)(f\varphi)]\varphi^{-1} = [(e\varphi)(f\varphi)]\varphi^{-1} = f(e\varphi)(f\varphi)]\varphi^{-1},$$

i.e., $eS \cap fS \neq 0$. Dually $e\psi = (v; i, \lambda)$, $f\psi = (w; j, \lambda)$ implies $Se \cap Sf \neq 0$. Consequently

$$e \sigma f \Leftrightarrow e\varphi = (t; i, \lambda) \text{ and } f\varphi = (u; i, \mu), \tag{4}$$

$$e \tau f \Leftrightarrow e\psi = (v; i, \lambda) \text{ and } f\psi = (w; j, \lambda). \tag{5}$$

Using (4) and (5) we now show that the classes of $\mathfrak{M}$ different from 0 can be indexed by the set $\Lambda\times I$. Let $C\neq 0$ be a σ-class; then C contains an $\mathcal{R}$-class, and since S is regular, C also contains an idempotent e. If $e\varphi=(t;i,\lambda)$, write $C=C_i$. By (4), the index i is independent of the choice of the idempotent in C. If $C_i=C_j$, then clearly $i=j$. Further, for any $i\in I$, there is an idempotent $(t;i,\lambda)\in A$ for some $\lambda\in\Lambda$. Since φ is onto, there is $e\in E_S$ such that $e\varphi=(t;i,\lambda)$. But then the σ-class containing e has index i. We have proved that I can be used as an index set for the σ-classes distinct from 0. Similarly, the set of τ-classes Γ distinct from 0 can be indexed by Λ. Consequently, the $\mathfrak{M}$-classes distinct from 0 can be written as $\Sigma_{i\lambda}=C_i\cap\Gamma_\lambda$ with $i\in I$, $\lambda\in\Lambda$.

If $\Sigma_{i\lambda}$ is a nonzero $\mathfrak{M}$-class and $a\in\Sigma_{i\lambda}$, then $a^2\in\Sigma_{i\lambda}$. For b an inverse of a^2, we obtain $e=aba\in E_S$, whence $e\varphi=(t;i,\lambda)$ is an idempotent of A and $q_{\lambda i}\neq 0$. Conversely, if $q_{\lambda i}\neq 0$, for any $t\in T$, $(t;i,\lambda)\in E_A$ and thus $(t;i,\lambda)\varphi^{-1}=e\in C_i$. Hence $e\varphi=(t;i,\lambda)$ and by b) iii), $e\psi=(u;i,\lambda)$ so that $e\in\Gamma_\lambda$. Thus $e\in\Sigma_{i\lambda}$ which is then a nonzero $\mathfrak{M}$-class.

For the remainder of the proof let $\Sigma_{i\lambda}$ be a nonzero $\mathfrak{M}$-class. Let $a\in\Sigma_{i\lambda}$; then a has an inverse $a'\in\Sigma_{j\mu}$ for some $j\in I$, $\mu\in\Lambda$, and aa', $a'a$ are idempotents. Since $aa'\in\Sigma_{i\mu}$ and $a'a\in\Sigma_{j\lambda}$, we have $(aa')\varphi=(m;i,\mu)$ and $(a'a)\varphi=(n;j,\lambda)$ for some $m,n\in T$. In A, $(n;i,\lambda)$ and $(m;i,\lambda)$ are nonzero idempotents. Letting $f=(n;i,\lambda)\psi^{-1}$ and $g=(m;i,\lambda)\varphi^{-1}$, we obtain

$$[(a'a)\psi](f\psi)=(n;j,\lambda)(n;i,\lambda)=(n;j,\lambda)=(a'a)\psi,$$

which by b) ii) implies $(a'a)f=a'a$, so that $af=a$. Analogously, using b) i), we derive $ga=a$. Thus $a=aa'a=(af)a'(ga)=a(fa'g)a$, where $fa'g\in\Sigma_{i\lambda}$. Therefore $\Sigma_{i\lambda}$ is regular.

If $a,b\in\Sigma_{i\lambda}$, there exists $c\in S$ such that $aS=cS$, $Sb=Sc$, S being 0-bisimple. Clearly $c\in\Sigma_{i\lambda}$. Letting a', b', c' be any inverses in $\Sigma_{i\lambda}$ of a, b, c, respectively, we obtain $a=cc'a$, $c=aa'c$ which proves $a\Sigma_{i\lambda}=c\Sigma_{i\lambda}$, and $c=cb'b$, $b=bc'c$ which proves $\Sigma_{i\lambda}c=\Sigma_{i\lambda}b$. Hence $\Sigma_{i\lambda}$ is bisimple.

We show next that the idempotents of $\Sigma_{i\lambda}$ commute. Thus let $e,f\in E_S\cap\Sigma_{i\lambda}$. Since $e,f\in\Sigma_{i\lambda}$, we have $e\sigma f$, $e\tau f$, which by (4) and (5) yields $e\varphi=(t;i,\lambda)$, $f\varphi=(u;i,\mu)$, $e\psi=(v;j,\nu)$, $f\psi=(w;k,\nu)$ for some $t,u,v,w\in T$, $i,j,k\in I$, $\lambda;\mu,\nu\in\Lambda$. By b) iii), $i=j=k$, $\lambda=\mu=\nu$. Thus $e\varphi$, $f\varphi$, $e\psi$, $f\psi$ commute. Let $z\in\Sigma_{i\lambda}$ be an inverse of ef; for $g=fze$, we have $g\in E_S\cap\Sigma_{i\lambda}$ and $ge=g$. It follows that $eg\in E_S$ and $g(eg)=g$, which by b) ii) implies $(g\psi)[(eg)\psi]=g\psi$. Similarly $(eg)g=eg$ implies $[(eg)\psi](g\psi)=(eg)\psi$. Since eg, $g\in\Sigma_{i\lambda}$, $(eg)\psi$ and $g\psi$ commute and thus $(eg)\psi=g\psi$. Consequently $eg=g$, i.e., $efze=fze$. Hence $ef=efzef=fzef$ so that $fef=ef$ and $ef\in E_S$. By symmetry, we conclude that $efe=fe$, whence $fe\in E_S$. Further, $fef=(fe)(ef)=ef$ implies $[(fe)\varphi][(ef)\varphi]=(ef)\varphi$ and $efe=(ef)(fe)=fe$ implies $[(ef)\varphi][(fe)\varphi]=(fe)\varphi$ by b) i). Since $(fe)\varphi$ and $(ef)\varphi$ commute, we obtain $(ef)\varphi=(fe)\varphi$, whence $ef=fe$

From the above, we also see that $(1; i, \lambda)\varphi^{-1}$ is a left identity of $E_S \cap \Sigma_{i\lambda}$. Hence for $a \in \Sigma_{i\lambda}$ and its inverse $a^{-1} \in \Sigma_{i\lambda}$ (unique), we obtain

$$(1; i, \lambda)\varphi^{-1} a = [(1; i, \lambda)\varphi^{-1}]aa^{-1}a = a'a = a.$$

Analogously $(1; i, \lambda)\psi^{-1}$ is a right identity of $\Sigma_{i\lambda}$. Therefore $\Sigma_{i\lambda}$ has an identity.

We have proved that every nonzero $\mathfrak{M}$-class is a bisimple inverse semigroup with identity. By 3. 11, S is a Rees 0-composition, and since every nonzero $\mathfrak{M}$-class has an identity, by 3. 2, $\mathfrak{M}$ must be the congruence associated to S. Therefore $S \cong \mathcal{M}^0(D; I, \Lambda; P)$, where $D = D^1$ is a bisimple inverse semigroup. For $q_{\lambda i} \neq 0$, $\varphi | E_S \cap \Sigma_{i\lambda}$ is a semigroup isomorphism of $E_S \cap \Sigma_{i\lambda}$ onto $T_{i\lambda} = \{(t; i, \lambda) | t \in T\}$. Thus

$$E_D \cong E_S \cap \Sigma_{i\lambda} \cong T_{i\lambda} \cong T.$$

It was shown above that $q_{i\lambda} \neq 0 \Leftrightarrow \Sigma_{i\lambda}$ is a nonzero $\mathfrak{M}$-class. Hence $Q = \bar{P}$. This completes the proof.

Remark 5. 2. In the last part of the proof of necessity, we have in fact shown that $eS \cap fS \subseteqq efS$ always holds. A simple computation then shows that $eS \cap fS = efS$ in c) i) can be substituted by any one of the following expressions: (a) $efS \subseteqq fS$, (b) $ef \in fS$, (c) $ef = fef$.

In order to express conveniently the corollaries of 5. 1, we now introduce the notion of a sum of rectangular bands.

Proposition 5. 3. *Let $C = C^0$ be a semigroup, $|C| > 1$; then the following conditions are equivalent:*

a) *C is a band and for all $a, b, c \in C$, $ab \neq 0$, $bc \neq 0 \Rightarrow abc = ac \neq 0$;*

b) *C is an orthogonal sum of semigroups B_α^0, where B_α are pairwise disjoint rectangular bands* (5. 11, [3]; *i.e., C is the union of its subsemigroups B_α and* 0, *and $B_\alpha B_\beta = 0$ if $\alpha \neq \beta$*);

c) *$C \cong E_B$ for some rectangular* 0-*band B.*

Such a semigroup C will be called a *sum of rectangular bands.*

Proof. a)⇒b). For $a \neq 0$, $b \neq 0$, let: $a\tau b \Leftrightarrow ab \neq 0$. Then τ is an equivalence relation whose classes B_α are rectangular bands, and C is evidently an orthogonal sum of B_α^0.

b)⇒c). We may put $B_\alpha = \mathcal{M}(1; I_\alpha, \Lambda_\alpha; P_\alpha)$, where the sets I_α (respectively Λ_α) are pairwise disjoint. Let $I = \bigcup_\alpha I_\alpha$, $\Lambda = \bigcup_\alpha \Lambda_\alpha$; let $Q = (q_{\lambda i})$ be the $\Lambda \times I$-matrix defined by: for $i \in I_\alpha$, $\lambda \in \Lambda_\beta$

$$q_{\lambda i} = \begin{cases} 1 & \text{if } \alpha = \beta, \\ 0 & \text{if } \alpha \neq \beta, \end{cases}$$

and set $B = \mathcal{M}^0(1; I, \Lambda; Q)$. Clearly B is a rectangular 0-band and $C \cong E_B$.

c)⇒a). Let B be a rectangular 0-band and suppose that E_B is a subsemigroup of B. Letting $C=E_B$ and using the hypothesis that E_B is a semigroup, we see without difficulty that C satisfies a).

Definition 5. 4. Let P be a $\Lambda\times I$-matrix over a group with zero G^0 and identity 1. P is said to satisfy condition (N) if for all $i, j\in I$, $\lambda, \mu\in\Lambda$,

$$p_{\lambda i}\neq 0, \quad p_{\lambda j}\neq 0, \quad p_{\mu j}\neq 0 \Rightarrow p_{\lambda i}^{-1}p_{\lambda j}p_{\mu j}^{-1}p_{\mu i} = 1.$$

The statement of 5. 1 simplifies considerably if we suppose that the idempotents of S form a subsemigroup, or that S has no zero, or that the associated congruence is a Brandt congruence (3. 2, [3]). Also, using 4. 3, we obtain certain other characterizations of these semigroups; 5. 1 thus has the following corollaries.

Corollary 5. 5. *The following statements are equivalent for any semigroup S:*

a) *S is 0-bisimple and $E_S\cong T\times^0 C$, where $T=T^1$ is a semilattice and C is a sum of rectangular bands;*

b) *$S\cong\mathcal{M}^0(D; I, \Lambda; P)$, where $D=D^1$ is a bisimple inverse semigroup and E_S is a semigroup;*

c) *$S\cong\mathcal{M}^0(D; I, \Lambda; P)$, where D is as in* b) *and P satisfies (N);*

d) *$S\cong D\times^0 B$, where D is as in* b) *and B is a rectangular* 0*-band whose idempotents form a subsemigroup.*

Proof. a)⇒b). Let B as in the proof of 5. 3, b)⇒c), and let $A=T\times^0 B$. Then $E_B\cong C$ so that $E_S\cong T\times^0 C\cong T\times^0 E_B\cong E_A$. Since B is a rectangular 0-band, B satisfies 5. 1 a). It follows that in turn A, E_A, E_S, S satisfy 5. 1 a); the last implication holds since S is regular. If θ is a semigroup isomorphism of E_S onto E_A, then letting $\varphi=\psi=\theta$, all conditions in 5. 1 b) are trivially satisfied. Let $e, f\in E_S$ and suppose $eS\cap fS\neq 0$. Then $ex=fy\neq 0$ for some $x, y\in S$. Let x' be an inverse of x and w be an inverse of yx'. Using the fact that idempotents of S form a subsemigroup, we obtain

$$f(yx'w)=e(xx')w=e(xx')e(xx')w=e(xx')f(yx'w)\neq 0,$$

which implies $eE_S\cap fE_S\neq 0$. We identify E_S with $T\times^0 C$ and write $e=(a, u)$, $f=(b, v)$, so that $(a, u)(s, t)=(b, v)(p, q)\neq 0$ for some $(s, t), (p, q)\in T\times^0 C$. It follows that $as=bp$, $ut=vq\neq 0$. Since C is a sum of rectangular bands, we have $u, t, v, q\in B_\alpha$ for some rectangular band B_α. From $ut=vq=uvq$ it then follows $uv=u(vqv)=(uvqv)=vqv\in vC$. In T, trivially $ab\in bT$, so that $ef\in fE_S\subseteq fS$. Thus 5. 2 (b) holds and hence also 5. 1 c) i); c) ii) is verified analogously. By 5. 1, b) holds.

b)⇒c). If $p_{\lambda i}\neq 0$, $p_{\lambda j}\neq 0$, $p_{\mu j}\neq 0$, then

$$(p_{\lambda i}^{-1}; i, \lambda)(p_{\mu j}^{-1}, j, \mu) = (p_{\mu i}^{-1}; i, \mu)$$

since the product on the left is a nonzero idempotent. Hence $p_{\lambda i}^{-1}p_{\lambda j}p_{\mu j}^{-1}=p_{\mu i}^{-1}$ and (N) holds.

c)⇒d). It is easy to see that (N) implies (M). Thus by 4. 3, $S\cong D\times{}^0B$, $S_1\cong G\times{}^0B$, where D, B, G are as in 4. 3. Moreover, $S_1=\mathcal{M}^0(G; I, \Lambda; P)$, where P satisfies (N). From the proof of b)⇒c), it follows directly that E_{S_1} is a semigroup and since $S_1\cong G\times{}^0B$, E_B also is a semigroup.

d)⇒a). We identify S with $D\times{}^0B$. Since D is bisimple and B is 0-bisimple, it follows easily that S is 0-bisimple. Evidently $E_S=E_D\times{}^0E_B$, where E_D is a semilattice with identity and E_B is a sum of rectangular bands by 5. 3.

Corollary 5. 6. *The following statements are equivalent for any semigroup S:*

a) *S is bisimple and $E_S\cong T\times B$, where $T=T^1$ is a semilattice and B is a rectangular band;*

b) *$S\cong\mathcal{M}(D; I, \Lambda; P)$, where $D=D^1$ is a bisimple inverse semigroup and E_S is a semigroup;*

c) *$S\cong\mathcal{M}(D; I, \Lambda; P)$, where D is as in* b) *and P satisfies (N);*

d) *$S\cong D\times B$, where D is as in* b) *and B is a rectangular band.*

Recall that an inverse rectangular 0-band is called a Brandt 0-band (3. 2, [3]). A semigroup K which is a sum of rectangular bands each of which contains only one element is characterized by the fact that K has 0 and at least one more element, and for any $a, b\in K$:

$$ab=\begin{cases} a & \text{if } a=b; \\ 0 & \text{if } a\neq b; \end{cases}$$

call such a semigroup a *Kronecker semigroup.*

Corollary 5. 7. *The following statements are equivalent for any semigroup S:*

a) *S is 0-bisimple and $E_S\cong T\times{}^0K$, where $T=T^1$ is a semilattice and K is a Kronecker semigroup;*

b) *$S\cong\mathcal{M}^0(D; I, \Lambda; P)$, where $D=D^1$ is a bisimple inverse semigroup and E_S is a semilattice;*

c) *$S\cong\mathcal{M}^0(D; I, I; \Delta)$; where D is as in* b) *and Δ is the $I\times I$-unit matrix;*

d) *$S\cong D\times{}^0B$, where D is as in* b) *and B is a Brandt 0-band.*

The proof of 5. 6 and 5. 7 follows easily from 5. 5 and is omitted. Note that further characterizations of semigroups appearing in these corollaries can be given using the results of the previous section, i.e., using the notions of a Rees 0-composition and of a matrix of semigroups.

6. Example and conclusions

The following example shows that a bisimple regular semigroup need not be a matrix of bisimple inverse semigroups. Let S be the semigroup generated by a and b subject to the relations $a=aba$, $b=bab=ab^2$. The elements of S can be written in an array:

$$\underbrace{\begin{array}{cccc} a & a^2 & \cdots & a^m \cdots \\ ba & ba^2 & \cdots & ba^m \cdots \\ \vdots & \vdots & & \vdots \\ b^n a & b^n a^2 & \cdots & b^n a^m \cdots \\ \vdots & \vdots & & \vdots \end{array}}_{L_1} \qquad \underbrace{\begin{array}{cccc} ab & a^2 b & \cdots & a^m b \cdots \\ b & ba^2 b & \cdots & ba^m b \cdots \\ \vdots & \vdots & & \vdots \\ b^n & b^n a^2 b & \cdots & b^n a^m b \cdots \\ \vdots & \vdots & & \vdots \end{array}}_{L_2}$$

The $\mathscr{R}$-classes constitute the rows and the $\mathscr{L}$-classes the columns of this array. Hence S is bisimple and regular. E_S consists of two descending chains

$$ba > b^2a^2 > \cdots > b^m a^m > \cdots, \qquad ab > ba^2 b > \ldots > b^m a^{m+1} b \cdots > \cdots,$$

no two elements belonging to different chains are comparable, and E_S is a subsemigroup of S. Both L_1 and L_2 are left ideals and the partition induced is the maximal matrix decomposition of S ([4]). Since L_1 is not regular, S is not a matrix of inverse semigroups. L_1 is the subsemigroup of the bicyclic semigroup generated by $p_1=a$, $p_2=b$ obtained by omitting the $\mathscr{L}$-class of the identity; L_2 is the bicyclic semigroup generated by $p_2=a^2b$, $q_2=b$.

That S is not a matrix of bisimple inverse semigroups with identity can also be (more easily) deduced from our results. For suppose it is; then by 2. 3, 3. 10, and 3. 2, $S\cong\mathscr{M}(D; I, \Lambda; P)$, where $D=D^1$ is a bisimple inverse semigroup. Since E_S is a semigroup, by 5. 6 we must have $E_S\cong T\times B$, where $T=T^1$ is a semilattice and B is a rectangular band. Since E_S consists of two chains, we must have $|B|=2$, and the set $\{ba, ab\}$ must be either a left or a right zero semigroup. However, $(ba)(ab)=ba^2b\notin\{ba, ab\}$.

Using the theory developed in the previous section, whenever the structure of a class of bisimple inverse semigroups with identity is described by means of some construction involving the group of units, the structure of 0-bisimple semigroups which are 0-matrices of these semigroups is readily available. For example, if S is a 0-bisimple (or, equivalently, regular; 3. 11) semigroup which is a 0-matrix of bisimple ω-semigroups introduced by REILLY [6], then $S\cong\mathscr{M}^0(D; I; \Lambda, P)$, where D is a bisimple ω-semigroup. In fact, S can be represented as the set $(G\times N\times N\times I\times \Lambda)\cup 0$, where G is a group, N is the set of nonnegative integers, with multiplication

$$(g, m, n, i, \lambda)(h, p, q, j, \mu) = (g\alpha^{p-r} q_{\lambda j}\alpha^{n+p-r} h\alpha^{n-r}, m+p-r, n+q-r, i, \mu)$$

if $q_{\lambda j} \neq 0$, otherwise equal to zero, where $Q = (q_{\lambda i})$ is a regular $\Lambda \times I$-matrix over G°, α is a fixed endomorphism of G, α^t is the t-th iterate of α, with α° the identity transformation, and $r = \min\{n, p\}$. S can also be characterized by using 5. 1 (for special cases, see 5. 5, 5. 6, and 5. 7).

The case of a matrix of semigroups (or an r-composition, section 2) as treated in section 3, serves the same purpose as described above for bisimple inverse semigroups with identity, for a much larger class of semigroups (composable semigroups, see examples in section 2).

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