Some algorithms for the representation of natural numbers

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1. Let $1=a_1<a_2<\cdots$ be a sequence of natural numbers. Let further $\mathscr{A}$ denote the set $\{a_n\}$.

Every natural number can be represented in the form

$$n=a_{i_1}+\cdots+a_{i_\nu} \tag{1.1}$$

where $a_{i_j}\in\mathscr{A}$, $a_{i_1}\geqq a_{i_2}\geqq\cdots\geqq a_{i_\nu}$, and a_{i_1} denotes the greatest element of $\mathscr{A}$ which does not exceed n and, in general a_{i_k} denotes the greatest element of $\mathscr{A}$ which does not exceed $n-(a_{i_1}+\cdots+a_{i_{k-1}})$ $(k=2,\cdots,\nu)$.

Let $\alpha(n)$ denote the length of this representation, i.e. $\alpha(n)=\nu$, $\alpha(0)=0$.

In this paper we study the distribution of the values $\alpha(n)$ for some special set $\mathscr{A}$. In the sections 2 and 3 we shall study the cases when the differences of the consecutive elements of $\mathscr{A}$ have a limiting distribution. In the section 4 we investigate the case when $\mathscr{A}$ consists of the square numbers.

2. L t.

$$d_i=a_{i+1}-a_i\ (i=1,2,\ldots);\qquad A(x)=\sum_{a_i\leqq x}1, \tag{2.1)—(2.2}$$

$$\varrho_l(x)=\sum_{\substack{a_i\leqq x\\ d_i=l}}1\qquad(l=1,2,\ldots). \tag{2.3}$$

Set

$$T_k(x)=\sum_{n=0}^{[x]}\alpha^k(n)\qquad(k=0,1,2,\ldots); \tag{2.4}$$

$$S(N,u)=\sum_{n=0}^{N}e^{iu\alpha(n)};\qquad \varphi_N(u)=\frac{1}{N+1}S(N,u). \tag{2.5)—(2.6}$$

We shall prove

Theorem 1. *If* $n^{-1}A(n)\geqq\alpha(>0)$ *for* $n=1,2,\cdots,N$, *then* $n^{-1}T_1(n)\leqq1/\alpha$ *for* $n=1,2,\cdots,N$.

Let us now suppose that the limits

(2. 7)—(2. 8) $\lim_{x\to\infty} x^{-1}A(x) = c(>0), \qquad \lim_{x\to\infty} x^{-1}\varrho_l(x) = \varrho_l \quad (l=1,2\,\ldots)$

exist and the relation

(2. 9) $$\sum_{l=1}^{\infty} l\varrho_l = 1$$

holds.

It is known that (2. 9) is equivalent to

(2. 10) $$\overline{\lim_{x\to\infty}}\, x^{-1} \sum_{l\geqq y} \varrho_l(x) \to 0 \qquad (y\to\infty).$$

Theorem 2. *Under the assumptions* (2. 7), (2. 8), (2. 9) *the following assertions hold:*

a) *The sequence of the characteristic functions* $\varphi_N(u)$ *tends to a limit function* $\varphi(u)$ *as* $N\to\infty$, *uniformly in* u, *and the relation*

(2. 11) $$\varphi(u) = e^{iu} \sum_{l=1}^{\infty} \varphi_{l-1}(u) l\varrho_l$$

holds.

Furthermore the limits

(2. 12) $$\lim_{N\to\infty} \frac{1}{N+1} \sum_{\substack{n\leqq N\\ \alpha(n)=l}} 1 = \tau_l \qquad (l=1,2,\ldots)$$

exist and

$$\sum_{l=1}^{\infty} \tau_l = 1.$$

b) *We have*

(2. 13) $$\lim_{x\to\infty} x^{-1} T_k(x) = 1 + \sum_{\nu=1}^{k} \binom{k}{\nu} \sum_{l=1}^{\infty} T_\nu(l-1)\varrho_l,$$

for $k=1,2,\cdots$, *the sum on the right hand side of* (2. 13) *being convergent.*

3. For the proof of Theorem 1 we use induction on n. Since $1\in\mathscr{A}$, so $T_1(1)/1\leqq 1/\alpha$ evidently holds. Suppose now, that $m^{-1}T_1(m)\leqq 1/\alpha$ for $m=1,\cdots,n-1$, where $1<n\leqq N$. Hence we deduce that $n^{-1}T_1(n)\leqq 1/\alpha$. Indeed we have

$$T_1(n) = \sum_{m\leqq n} \alpha(m) = \sum_{j=2}^{\nu} \sum_{a_{j-1}\leqq m<a_j} \alpha(m) + \sum_{a_\nu\leqq m\leqq n} \alpha(m),$$

where $a_\nu\leqq n<a_{\nu+1}$. Since

$$\sum_{a_{j-1}\leqq m<a_j} \alpha(m) = d_{j-1} + \sum_{\nu=0}^{d_{j-1}-1} \alpha(\nu) = d_{j-1} + T_1(d_{j-1}-1)$$

so we get

$$T_1(n) = n + \sum_{j=2}^{\nu} T_1(d_{j-1}-1) + T_1(n-a_\nu) = \tag{3.1}$$

$$= n + \sum_{d=1}^{\infty} T_1(d-1)\varrho_d(a_{\nu-1}) + T_1(n-a_\nu).$$

If $n^{-1}T_1(n) \leqq \max_{m \leqq n-1} m^{-1}T_1(m)$, then $n^{-1}T_1(n) \leqq 1/\alpha$ evidently holds. Let us now suppose that

$$n^{-1}T_1(n) = \max_{1 \leqq m \leqq n} m^{-1}T_1(m).$$

Then from (3.1) it follows that

$$\frac{T_1(n)}{n} \leqq 1 + \frac{T_1(n)}{n} \cdot \frac{1}{n}\left\{\sum_d \varrho_d(a_{\nu-1})(d-1) + (n-a_\nu)\right\} =$$

$$= 1 + \frac{T_1(n)}{n} \cdot \frac{1}{n}\left\{\sum_{j=1}^{\nu-1}(a_{j+1}-a_j-1) + (n-a_\nu)\right\} =$$

$$= 1 + \frac{T_1(n)}{n} \cdot \frac{1}{n}\{n - A(n)\} = 1 + \frac{T_1(n)}{n}\left(1 - \frac{A(n)}{n}\right),$$

and consequently $\frac{T_1(n)}{n} \cdot \frac{A(n)}{n} \leqq 1$, i. e. $\frac{T_1(n)}{n} \leqq \frac{1}{\alpha}$ holds.

We begin the proof of Theorem 2. Let $a_\nu \leqq N < a_{\nu+1}$.

a) We have

$$S(N,u) = e^{iu\alpha(0)} + \sum_{j=2}^{\nu} \sum_{a_{j-1} \leqq n < a_j} e^{iu\alpha(n)} + \sum_{a_\nu \leqq n < N} e^{iu\alpha(n)} =$$

$$= 1 + e^{iu}\sum_{j=2}^{\nu} S(d_{j-1}, u) + e^{iu}S(N-a_\nu, u) =$$

$$= 1 + e^{iu}\sum_{d=2}^{\infty} \frac{S(d-1,u)}{d}\, d\varrho_d(a_{\nu-1}) + e^{iu}S(N-a_\nu, u).$$

Since the limit $\lim_{x\to\infty} x^{-1}A(x) = c$ exists, so $d_i = o(a_i)$ $(i \to \infty)$, and consequently $|S(N-a_\nu, u)|/N \to 0$. Hence it follows that

$$\varphi_N(u) = \frac{1}{N+1}\,\frac{a_{\nu-1}+1}{N+1}\, e^{iu}\sum_{d=2}^{\infty} \varphi_{d-1}(u)\, d\,\frac{\varrho_d(a_{\nu-1})}{a_{\nu-1}+1} + o(1).$$

Let now $\varphi(u)$ be defined by the relation (2. 11). Then

$$|\varphi_N(u)-\varphi(u)| = (1+o(1))\left|\sum_{d=2}^{\infty}\varphi_{d-1}(u)\left(\frac{d\varrho_d(a_{\nu-1})}{a_{\nu-1}+1}-d\varrho_d\right)\right|+o(1) \leqq$$

$$\leqq 2\sum_{d=2}^{\infty}\left|\frac{d\varrho_d(a_{\nu-1})}{a_{\nu-1}+1}-d\varrho_d\right|+o(1).$$

From (2. 8), (2. 9) it follows that the last sum tends to zero as $N\to\infty$, independently from u.

From (2. 11) it follows that $\varphi(u)$ is a characteristic function. Since $\varphi_N(u)$ and consequently $\varphi(u)$ are periodic functions mod 2π, so $\varphi(u)$ has a Fourier expansion

$$\varphi(u) = \sum_{n=-\infty}^{\infty}\delta_n e^{inu}.$$

Using the uniform convergence of $\varphi_N(u)$ to $\varphi(u)$ we have

$$\delta_l = \frac{1}{2\pi}\int_0^{2\pi}\varphi(u)e^{-ilu}du = \lim_{N\to\infty}\frac{1}{2\pi}\int_0^{2\pi}\varphi_N(u)e^{-ilu}du =$$

$$= \begin{cases}\lim\limits_{N\to\infty}\dfrac{1}{N+1}\sum\limits_{\substack{n\leqq N\\ \alpha(n)=l}}1 = \tau_l & \text{for } l=1,2,\cdots,\\ 0 \quad \text{for } l=0,-1,-2,\cdots.\end{cases}$$

Furthermore

$$\sum\tau_l = \sum\delta_l = \varphi(0) = 1.$$

b) We have

$$T_k(x) = \sum_{n\leqq x}\alpha^k(n) = \sum_{a_i\leqq x}\sum_{j=0}^{d_i-1}{}'(\alpha(j)+1)^k,$$

where the dash means that for $a_i\leqq x<a_{i+1}$ we sum over those j for which $j\leqq x-a_i$. Hence it follows that

$$T_k(x) = \sum_{a_i\leqq x}\sum_{j=0}^{d_i-1}\binom{k}{\nu}\alpha^\nu(j) = \sum_{\nu=0}^{k}\binom{k}{\nu}\sum_{a_i\leqq x}{}'T_\nu(d_i-1) = \sum_{\nu=0}^{k}\binom{k}{\nu}\sum_{l=1}^{\infty}{}'T_\nu(l-1)\varrho_l(x).$$

The fulfilment of the relation (2. 13) would follow from the boundedness of the sums $T_\nu(l)/l$ $(l=1,2,\cdots;\ \nu=1,\cdots,k)$ by (2. 10) immediately. The boundedness of $T_1(x)/x$ follows from Theorem 1. The proof of the general case is similar and so it can be omitted.

4. Let $\mathcal{A}$ be the set of square numbers. Introduce the notation $\log_2 x=\log\log x$ where the base of the logarithm is 2.

It is easy to prove that

$$\alpha(n)\leqq\log_2 n+5. \tag{4. 1}$$

Indeed, if

$$A(x) = \max_{n \leqq x} \alpha(n),$$

then from the inequality $n-[\sqrt{n}]^2 \leqq 2\sqrt{n}$ it follows that

$$A(x) \leqq 1 + A(2\sqrt{x}).$$

Iterating this inequality k times we have

$$A(x) \leqq k + A(2^{1+\frac{1}{2}+\dots+1/2^{k-1}} x^{\frac{1}{2}k}) \leqq k + A(4x^{\frac{1}{2}k}).$$

Let k be the smallest integer for which $x^{\frac{1}{2}k} \leqq 2$, i.e. $k=[\log_2 x]+1$. Since $A(8)=4$ we have

$$A(x) \leqq \log_2 x + 5.$$

Set

$$T_k(x) = \sum_{n \leqq x} \alpha^k(n) \quad \text{and} \quad \Delta_k(x) = \sum_{n \leqq x} |\alpha(n) - \log_2 x|^k.$$

Theorem 3. *We have*

$$T_k(x) = x(\log_2 x)^k + O(x(\log_2 x)^{k-1}), \tag{4. 2}$$

$$\Delta_k(x) = O(x), \tag{4. 3}$$

where the constants in the O terms depend on k only.

Proof. It is evident that (4. 2) follows from (4. 3). For the proof of (4. 3) we use induction on k. The relation holds for $k=0$. Let now suppose that (4. 3) holds for $k=0, 1, \cdots, K-1$. Then we deduce the inequality (4. 3) for $k=K$.

We have

$$\Delta_K(N) \leqq \sum_{\nu^2 \leqq N} \sum_{\nu^2 \leqq n < (\nu+1)^2} |\alpha(n) - \log_2 N|^K = \sum_{\nu^2 \leqq N} \sum_{j=0}^{2\nu} |\alpha(j) + 1 - \log_2 N|^K = \sum_{\nu \leqq \sqrt{N}} B_\nu.$$

Using the inequality

$$|a+b|^K \leqq |a|^K + \sum_{l=1}^{K} \binom{K}{l} |a|^{K-l} |b|^l$$

and consequently that

$$|\alpha(j) + 1 - \log_2 N|^K \leqq |\alpha(j) - \log_2 2\nu|^K + \sum_{l=1}^{K} \binom{K}{l} |\alpha(j) - \log_2 2\nu|^{K-l} \left| \log \frac{\log 2\nu}{\log \sqrt{N}} \right|^l$$

we obtain

$$B_\nu \leqq \Delta_K(2\nu) + \sum_{l=1}^{K} \binom{K}{l} \Delta_{K-l}(2\nu) \left| \log \frac{\log \sqrt{N}}{\log 2\nu} \right|^l.$$

Using our assumption that $\Delta_k(2\nu)\ll 2\nu$ for $k\leqq K-1$ we get

$$\Delta_K(N)\leqq \sum_{\nu\leqq\sqrt{N}}\Delta_K(2\nu)+O(\textstyle\sum_1),$$

where

$$\sum{}_1\ll\sum_{\nu\leqq\sqrt{N}}\nu\left\{\left|\log\frac{\log\sqrt{N}}{\log 2\nu}\right|+\left|\log\frac{\log\sqrt{N}}{\log 2\nu}\right|^K\right\}.$$

Dividing the interval of summation $[1,\sqrt{N}]$ into subintervals of type $\left[\frac{\sqrt{N}}{2^{j+1}},\frac{\sqrt{N}}{2^j}\right]$ we easily obtain the inequality

$$\sum{}_1\ll\sum_{2^j\leqq\sqrt{N}}\left(\frac{\sqrt{N}}{2^j}\right)^2\left\{\frac{j}{\log\sqrt{N}}+\left(\frac{j}{\log\sqrt{N}}\right)^K\right\}\ll\frac{N}{\log N}.$$

Hence

$$\Delta_K(N)\leqq\sum_{\nu\leqq\sqrt{N}}\Delta_K(2\nu)+O\left(\frac{N}{\log N}\right) \tag{4.4}$$

follows.

Introduce now the notation

$$\Delta_K(N)=\varepsilon(N)N.$$

We prove that $\varepsilon(N)$ is bounded; hence the inequality (4. 3) follows for $k=K$, and this will finish the proof of our theorem.

Let

$$\beta_j=\max_{2^{j-1}\leqq m\leqq 2^j}\varepsilon(m)\qquad(j=1,2,\ldots).$$

From (4. 4)

$$\varepsilon(N)\leqq\frac{1}{N}\sum_{\nu\leqq\sqrt{N}}\varepsilon(2\nu)2\nu+c/\log N$$

follows with a suitable constant c. Hence

$$\beta_{2l}\leqq\max_{j\leqq l+2}\beta_j+\frac{c}{l};\qquad\beta_{2l+1}\leqq\max_{j\leqq l+2}\beta_j+\frac{c}{l}.$$

Define the non-decreasing sequence of positive numbers $\gamma_4,\gamma_5,\ldots$ as follows: Let

$$\gamma_4=\gamma_5=\max(\beta_1,\beta_2)+\frac{c}{2};\qquad\gamma_{2l}=\gamma_{2l+1}=\max_{j\leqq l+2}\gamma_j+\frac{c}{l}\quad(l=3,\ldots). \tag{4.5}$$

Clearly, $\beta_j\leqq\gamma_j$ for $j\geqq 4$. So it is enough to prove that γ_n is bounded. Let

$$B(x)=\sum_{j\leqq x}\gamma_j.$$

From (4. 5) it follows that

$$B(2x)\leqq 2B(x)+2\gamma_{[x]}+c\log x.$$

Furthermore from (4. 1) we can easily see that $\varepsilon(N) \ll (\log_2 N)^K$. Hence $\beta_j \ll (\log j)^K$, and so

$$\gamma_{[x]} \ll \beta_{[x]} + \log x \ll (\log x)^K$$

follows.

Set $\varphi(x) = \frac{B(x)}{x}$. Then $\varphi(2x) \leqq \varphi(x) + c_1 \frac{(\log x)^K}{x}$. So the sequence $\varphi(2^m)$ $(m=1, 2, \cdots)$ is bounded. Hence $B(x) < cx$ follows for every x. Since $\{\gamma_n\}$ is non-decreasing we have

$$\gamma_l \leqq \frac{\gamma_{l+1} + \cdots + \gamma_{2l}}{l} \leqq \frac{B_{2l}}{l} < 2c,$$

i.e. γ_l is bounded.

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