Dini derivatives of semicontinuous functions. I

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1. In a paper [3] NEUGEBAUER has proved that for a continuous real function f, the upper and the lower derivatives of f on the right are respectively equal to the upper and the lower derivatives of f on the left, except a set of the first category. That this result is not true for arbitrary real function is also remarked in [3]. BRUCKNER and GOFFMAN [1] have shown that the result of NEUGEBAUER [3] comes as a corollary to a more general theorem. In the present paper we establish certain results analogous to the results of NEUGEBAUER [3] considering semicontinuous functions. Also some other results analogous to the results of [1] are established. Though the former results are simple consequences of the latter, we have established them independently because of their basic importance.

Throughout the paper f will denote a function from R to R where R is the set of real numbers and $D^+f(x_0)$ etc. will denote, as usual, the Dini derivatives of f at x_0.

2. For convenience let us state the following obvious

Lemma. *If $f(x)$ is upper semicontinuous and $g(x)$ is continuous at ξ, and if $g(\xi)>0$, then $\frac{f(x)}{g(x)}$ is upper semicontinuous at ξ.*

Theorem 1. *If $f: R\to R$ is upper semicontinuous, then the set*

$$\{x: D_-f(x)>D_+f(x)\}\cup\{x: D^-f(x)>D^+f(x)\}$$

is of the first category.

Proof. Let $E=\{x: D_-f(x)>D_+f(x)\}$. For $x\in E$, we choose a rational number r and a positive integer n such that

$$\frac{f(x-h)-f(x)}{-h}>r>D_+f(x) \quad \text{for all } h,\ 0<h<\frac{1}{n}.$$

Let E_{rn} denote the set of all points of E satisfying the above relation. Then

$$E = \bigcup_r \bigcup_n E_{rn},$$

where the union is taken over all rationals r and over all positive integers n. We shall show that E_{rn} is nondense for all r and all n.

If possible, let E_{rn} be dense in some interval (a, b). We choose two points ξ and ξ_2 such that $\xi \in (a, b) \cap E_{rn}$, $\xi_2 \in (a, b)$, $0 < \xi_2 - \xi < \frac{1}{n}$ and

$$\frac{f(\xi_2) - f(\xi)}{\xi_2 - \xi} < r.$$

By the lemma $\frac{f(x) - f(\xi)}{x - \xi}$ is upper semicontinuous at ξ_2 and so there is a $\delta > 0$ such that

$$\frac{f(x) - f(\xi)}{x - \xi} < r \quad \text{whenever } \xi_2 - \delta < x < \xi_2 + \delta.$$

We may choose δ such that $(\xi_2 - \delta, \xi_2 + \delta) \subset (a, b)$ and $\xi < \xi_2 - \delta < \xi_2 + \delta < \xi + \frac{1}{n}$. Since E_{rn} is dense in (a, b), there is a point $\xi_1 \in (\xi_2 - \delta, \xi_2 + \delta) \cap E_{rn}$. So,

$$\frac{f(\xi_1) - f(\xi)}{\xi_1 - \xi} < r, \tag{1}$$

Again, since $\xi_1 \in E_{rn}$ and $0 < \xi_1 - \xi < \frac{1}{n}$, we have

$$\frac{f(\xi) - f(\xi_1)}{\xi - \xi_1} > r. \tag{2}$$

Since (1) and (2) are contradictory, we conclude that the set E_{rn} is nondense and consequently the set E is of the first category. Similarly we can show that the set

$$F = \{x : D^- f(x) > D^+ f(x)\}$$

is of the first category.

In a similar manner we get the following theorem.

Theorem 2. *If $f \colon R \to R$ is lower semicontinuous then the set*

$$\{x : D_- f(x) < D_+ f(x)\} \cup \{x : D^- f(x) < D^+ f(x)\}$$

is of the first category.

We remark that the conclusion of Theorems 1 and 2 remains valid if we assume the semicontinuity of $f(x)$ in one side only.

Theorem 3. *If $f\colon R\to R$ is upper semicontinuous and r is a real number then each of the sets*

$$E=\{x\colon D^+f(x)<r\} \quad \text{and} \quad F=\{x\colon D_-f(x)>r\}$$

is of the type F_σ.

Proof. We shall prove for the set E; the proof for F is analogous.

For any positive integer n, let

$$E_n=\left\{x\colon \frac{f(y)-f(x)}{y-x}\leqq r-\frac{1}{n},\ \text{for}\ x<y<x+\frac{1}{n}\right\}.$$

Then it is easy to verify that

$$E=\bigcup_n E_n.$$

To show that E is of the type F_σ we have to show that E_n is closed for each n.

Let ξ be a limit point of E_n and let $\{x_i\}$ be a sequence in E_n which converges to ξ. We take a point y_0 such that $\xi<y_0<\xi+\frac{1}{n}$. Since the sequence $\{x_i\}$ converges to ξ there is a positive integer i_0 such that

$$x_i<y_0<x_i+\frac{1}{n}\ \text{whenever}\ i\geqq i_0.$$

Since $x_i\in E_n$, we have

$$\frac{f(y_0)-f(x_i)}{y_0-x_i}\leqq r-\frac{1}{n}\ \text{when}\ i\geqq i_0.$$

Since by the lemma $\dfrac{f(y_0)-f(x)}{y_0-x}$ is lower semicontinuous at ξ, we conclude

$$\frac{f(y_0)-f(\xi)}{y_0-\xi}\leqq r-\frac{1}{n}.$$

Since y_0 is an arbitrary point in $\xi<y<\xi+\frac{1}{n}$, we conclude $\xi\in E_n$. Hence E_n is closed and this completes the proof.

Similarly we get

Theorem 4. *If $f\colon R\to R$ is lower semicontinuous and r is a real number then each of the sets*

$$\{x\colon D_+f(x)>r\} \quad \text{and} \quad \{x\colon D^-f(x)<r\}$$

is of the type F_σ.

3. If φ is a real function of the two variables x and y defined and continuous for $x<y$ i.e. above the line $L\colon x-y=0$, then it is known [1] that for any given

direction θ there is a set of points $\{(\xi, \xi)\}$ residual on L[1]) such that

$$\operatorname*{Lim\,sup}_{(x,y)\to(\xi,\xi)} \varphi(x,y) = \operatorname*{Lim\,sup}_{\theta:(x,y)\to(\xi,\xi)} \varphi(x,y)$$

where $\theta:(x,y)\to(\xi,\xi)$ denotes that (x, y) tends to (ξ, ξ) along the direction θ.

If, however, φ is a lower semicontinuous function of x for every fixed value of y then also the result is true (see remark below the corollary to Theorem 1 [1]) except in the case when θ is the horizontal direction i.e. when y remains fixed. In that case we may not get such a residual set on L on which the above result is true. For, consider the function $\varphi(x, y)$, which equals 1 when y is rational, and 0 when y is irrational.

Then φ is a lower semicontinuous function of x for every fixed value of y; but for each point (ξ, ξ) on L

$$\operatorname*{Lim\,sup}_{(x,y)\to(\xi,\xi)} \varphi(x,y) = 1,$$

whereas

$$\operatorname*{Lim\,sup}_{x<\xi,\ (x,\xi)\to(\xi,\xi)} \varphi(x,y) = 0$$

whenever ξ is irrational. Now if f is an upper semicontinuous real function of a single real variable x then for any two reals x, y where $x<y$, we write

$$\varphi(x,y) = \frac{f(y)-f(x)}{y-x}.$$

Then by the lemma φ is a lower semicontinuous function of x for each fixed value of y. Considering the vertical direction we get, except a set of points (ξ, ξ) of the first category on L,

$$\operatorname*{Lim\,sup}_{x<\xi,\ x\to\xi} \varphi(x,\xi) \leqq \operatorname*{Lim\,sup}_{y>\xi,\ y\to\xi} \varphi(\xi,y)$$

i.e.

$$D^-f(\xi) \leqq D^+f(\xi).$$

Again considering the direction normal to L we get except a set of first category on L

$$\operatorname*{Lim\,sup}_{x+y=2\xi,\ (x,y)\to(\xi,\xi)} \varphi(x,y) = \operatorname*{Lim\,sup}_{y>\xi,\ y\to\xi} \varphi(\xi,y)$$

i.e. $\overline{f^{(1)}}(\xi) = D^+f(\xi)$ where $\overline{f^{(1)}}(\xi)$ is the upper symmetric derivative of f at ξ [2]. Similarly except a set of the first category on L, we have

$$D^-f(\xi) \leqq \overline{f^{(1)}}(\xi).$$

Lastly, the following results are true except a set of the first category:

$$\overline{f^{(1)}}(\xi) = \overline{f^*}(\xi), \qquad \overline{f^*}(\xi) = D^+f(\xi)$$

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[1]) I. e. whose complement is of the first category.

where $\overline{f^*}(\xi)$ denotes the upper unstraddled derivatives of f at ξ, i.e.

$$\overline{f^*}(\xi) = \operatorname*{Lim\,sup}_{x \neq y,\ (x,y) \to (\xi,\xi)} \frac{f(x)-f(y)}{x-y}.$$

Thus we get:

Theorem. *If f is upper semicontinuous then except a set of the first category the following relations are true:*

$$D^- f(x) \leqq \overline{f^{(1)}}(x) = \overline{f^*}(x) = D^+ f(x).$$

References

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