On the operator equation $S^*XT=X$ and related topics

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1. If U_+ is the unilateral shift of multiplicity one, then BROWN and HALMOS showed in [1] that the identity $U_+^*XU_+=X$ characterizes the class of Toeplitz operators. In this paper we determine the class of solutions to $S^*XT=X$ for arbitrary contractions S and T on Hilbert space. We show first in § 2 that we can reduce to the case of isometries and then in § 3 we determine the solutions for such. The form the latter solution takes is the same as for the case of the unilateral shift, namely, the class of solutions consists of the compressions of the intertwining operators between their unitary extensions. In § 4 we investigate when intertwining maps exist between unitary operators. In § 5 we investigate the inequalities $T^*XT\geqq X$ and $T^*XT\leqq X$ for a contraction T and Hermitian operators X. We show first that we can reduce a solution of either to a "pure" positive solution of the latter. These we study with the aid of a construction of SZ.-NAGY and FOIAŞ [9] and a recent result they proved on the intertwining maps for contractions [10]. As corollaries we obtain results analogous to those obtained in §§ 2 and 3. We also obtain a result due to PUTNAM [8] and certain facts about hyponormal operators.

In § 6 we investigate these same equations in the presence of various hypotheses of compactness. As corollaries we obtain a lemma of DYE [2], a generalization of the result that the only compact Toeplitz operator is 0, and a further proof of the result that a compact hyponormal operator is normal. In the last section we briefly explore the form our results take when T is identified as the Cayley transform of an accretive operator.

2. We make use of some of the more elementary aspects of the theory for contractions due to SZ.-NAGY and FOIAŞ [9] and begin by introducing a few of their ideas.

Let T be a contraction on the complex Hilbert space $\mathfrak{H}$. From the inequality $T^{*n}(I-T^*T)T^n\geqq 0$ it follows that the sequence $\{T^{*n}T^n\}$ is monotonically decreasing and hence converges strongly to a positive contraction. If we denote the unique positive square root of this contraction by A_T, then A_T is 0 if and only if the sequence $\{T^n\}$ converges to 0 in the strong operator topology. Moreover, since $T^*A_T^2T=A_T^2$ we see that A_T^2 is a solution to the equation $T^*XT=X$.

Let $\mathfrak{M}_T$ denote the closure of the range of A_T and define V_T by $V_T A_T x = A_T T x$ on the range of A_T. From the identity

$$\|V_T A_T x\|^2 = \|A_T T x\|^2 = (T^* A_T^2 T x, x) = (A_T^2 x, x) = \|A_T x\|^2$$

it follows that V_T is well defined and can be uniquely extended to an isometry on $\mathfrak{M}_T$ which we also denote by V_T. If by abuse of language we allow A_T to denote operators from $\mathfrak{M}_T$ to $\mathfrak{H}$ and from $\mathfrak{H}$ to $\mathfrak{M}_T$ as well as an operatoı from $\mathfrak{H}$ to $\mathfrak{H}$, then the identities $V_T A_T = A_T T$ and $A_T V_T^* = T^* A_T$ can be seen to hold. This convention will be extremely useful and should cause no confusion.

If S is a contraction on $\mathfrak{H}$ and T is a contraction on $\mathfrak{K}$, we denote by $\mathfrak{S}(S, T)$ the collection of operators X in $\mathfrak{L}(\mathfrak{K}, \mathfrak{H})$, the space of bounded operators from $\mathfrak{K}$ to $\mathfrak{H}$, satisfying the equation $S^* X T = X$. It is easy to verify that $\mathfrak{S}(S, T)$ is a subspace of $\mathfrak{L}(\mathfrak{K}, \mathfrak{H})$, which is closed in the weak operator topology. In the special case $S = T$, the subspace $\mathfrak{S}(T, T)\ (= \mathfrak{S}_T)$ is closed under the adjoint operation so that it is spanned by its Hermitian elements. (We shall see that it is also spanned by its positive elements.)

In the following theorem we show how to reduce the solution of the equation $S^* X T = X$ to that of $V_S^* Y V_T = Y$.

Theorem 1. *Let S be a contraction on $\mathfrak{H}$ and T be a contraction on $\mathfrak{K}$. Then $\mathfrak{S}(S, T) = A_S \mathfrak{S}(V_S, V_T) A_T$. Moreover, every X in $\mathfrak{S}(S, T)$ can be represented in the form $X = A_S Y A_T$ with Y in $\mathfrak{S}(V_S, V_T)$ such that $\|Y\| = \|X\|$.*

Proof. Let X be a contraction in $\mathfrak{L}(\mathfrak{K}, \mathfrak{H})$ so that $S^* X T = X$. Then $S^* X X^* S \geqq \geqq S^* X T T^* X^* S = X X^*$ so that by induction we obtain $S^{*n} X X^* S^n \geqq X X^*$ for all n. Thus $S^{*n} S^n \geqq X X^*$ for all n and from the definition of A_S it follows that $A_S^2 \geqq X X^*$. Hence there exists a contraction C_0 from $\mathfrak{M}_S$ to $\mathfrak{H}$ such that $X^* = C_0 A_S$ and, taking adjoints, a contraction $C\ (= C_0^*)$ from $\mathfrak{H}$ to $\mathfrak{M}_S$ so that $X = A_S C$. Substituting this in our equation we obtain $A_S V_S^* C T = S^* A_S C T = A_S C$. Since the range of both V_S^* and C is contained in $\mathfrak{M}_S$ and A_S is one-to-one on $\mathfrak{M}_S$, we obtain $V_S^* C T = C$. Repeating our previous argument we have $T^* C^* C T \geqq T^* C^* V_S V_S^* C T = C^* C$ from which it follows as before that $A_T^2 \geqq C^* C$. Hence there exists a contraction Y from $\mathfrak{M}_T$ to $\mathfrak{H}$ so that $C = Y A_T$. Substituting we have $V_S^* Y V_T A_T = V_S^* Y A_T T = V_S^* C T = = C = Y A_T$, and hence $V_S^* Y V_T = Y$. Thus, the operatoı X can be written $X = A_S Y A_T$, with Y in $\mathfrak{S}(V_S, V_T)$, so we have shown $\mathfrak{S}(S, T)$ is contained in $A_S \mathfrak{S}(V_S, V_T) A_T$.

To prove the converse suppose that Y is in $\mathfrak{S}(V_S, V_T)$. Then we find that $S^* A_S Y A_T T = A_S V_S^* Y V_T A_T = A_S Y A_T$ so that $X = A_S Y A_T$ is in $\mathfrak{S}(S, T)$. Moreover since we have shown that if X is a contraction, then Y can be taken to be a contraction, we have then for a general X in $\mathfrak{S}(S, T)$ that Y can be represented in the form $X = A_S Y A_T$ with Y in $\mathfrak{S}(V_S, V_T)$ and such that $\|Y\| = \|X\|$. This completes the proof that $\mathfrak{S}(S,T)$ is equal to $A_S \mathfrak{S}(V_S, V_T) A_T$.

From this it follows that if either A_T or A_S is 0, then $\mathfrak{S}(S, T)=(0)$. The question of necessary and sufficient conditions for $\mathfrak{S}(S, T)\neq(0)$ must wait for a detailed study of the case of isometries. We can at this point determine the situation in case $S=T$.

Corollary 2. 1. *Let T be a contraction on $\mathfrak{H}$. Then $\mathfrak{S}(T, T)$ is (0) if and only if $A_T=0$.*

Proof. From the theorem we have that $A_T=0$ implies $\mathfrak{S}(T, T)=(0)$. If $A_T\neq 0$, then since $0\neq A_T^2$ is in $\mathfrak{S}(T, T)$ it follows that $\mathfrak{S}(T, T)\neq(0)$.

3. For a Hilbert space $\mathfrak{D}$ we let $\boldsymbol{H}_{\mathfrak{D}}$ denote the space of functions f from the non negative integers $\boldsymbol{Z}^+$ to $\mathfrak{D}$ so that $\sum_{n=0}^{\infty}\|f(n)\|^2<\infty$. The space $\boldsymbol{H}_{\mathfrak{D}}$ is a Hilbert space with respect to pointwise addition and scalar multiplication and the inner product $\langle f, g\rangle=\sum_{n=0}^{\infty}(f(n), g(n))$. The unilateral shift U_+ is defined on $\boldsymbol{H}_{\mathfrak{D}}$ so that $(U_+f)(n)=\begin{cases} 0 & (n=0) \\ f(n-1) & (n>0)\end{cases}$, for f in $\boldsymbol{H}_{\mathfrak{D}}$. The operator U_+ is an isometry and its adjoint, the backward shift, satisfies $(U_+^*f)(n)=f(n+1)$ for f in $\boldsymbol{H}_{\mathfrak{D}}$. The sequence $\{U_+^{*n}\}$ converges strongly to 0. The minimal unitary extension U of U_+ is the bilateral shift defined on $\boldsymbol{L}_{\mathfrak{D}}$, where $\boldsymbol{L}_{\mathfrak{D}}$ is the space of functions f from the integeis $\boldsymbol{Z}$ to $\mathfrak{D}$ so that $\sum_{n=-\infty}^{\infty}\|f(n)\|^2<\infty$ and U is defined $(Uf)(n)=f(n-1)$ for f in $\boldsymbol{L}_{\mathfrak{D}}$. It is easily verified that U is unitary and if we identify $\boldsymbol{H}_{\mathfrak{D}}$ as a subspace of $\boldsymbol{L}_{\mathfrak{D}}$ in the obvious way, then $U_+=U|\boldsymbol{H}_{\mathfrak{D}}$.

A result due to von Neumann [6] states that every isometry V on $\mathfrak{H}$ is of the form $V=U_+\oplus V_0$ on $\mathfrak{H}=\boldsymbol{H}_{\mathfrak{D}}\oplus\mathfrak{H}_0$, where U_+ is the unilateral shift on $\boldsymbol{H}_{\mathfrak{D}}$ and V_0 is a unitary operator on $\mathfrak{H}_0$. Then $W=U\oplus V_0$ on $\mathfrak{K}=\boldsymbol{L}_{\mathfrak{D}}\oplus\mathfrak{H}_0$ is a unitary extension of V so that the smallest reducing subspace for W containing $\mathfrak{H}$ is $\mathfrak{K}$. This extension is unique to an isomorphism (cf. [3] or [9]). Let P denote the projection of $\mathfrak{K}$ onto $\mathfrak{H}$. As in the case of A_T we find it convenient to allow P to denote operators from $\mathfrak{H}$ to $\mathfrak{K}$ and $\mathfrak{K}$ to $\mathfrak{H}$ as well as from $\mathfrak{K}$ to $\mathfrak{K}$.

The following theorem reduces the solution of the equation $V_1^*XV_2=X$ for isometries V_1 and V_2 to the case of unitaries. In case $V_1=V_2$ a proof could be given based on a result of Lebow [5, p. 68]. The following proof is based in part on a proof due to Brown and Halmos [1].

Theorem 2. *For $i=1,2$, let V_i be an isometry on $\mathfrak{H}_i$ with minimal unitary extension W_i on $\mathfrak{K}_i$ and let P_i be the projection of $\mathfrak{K}_i$ onto $\mathfrak{H}_i$. Then $\mathfrak{S}(V_1, V_2)=P_1\mathfrak{S}(W_1, W_2)P_2$. Moreover, any X in $\mathfrak{S}(V_1, V_2)$ can be represented in the form $X=P_1YP_2$ with a Y in $\mathfrak{S}(W_1, W_2)$ such that $\|Y\|=\|X\|$.*

Proof. If B is an operator from $\mathfrak{K}_2$ to $\mathfrak{K}_1$ so that $W_1^* B W_2 = B$, then $V_1^* P_1 B P_2 V_2 = P_1 W_1^* P_1 B P_2 W_2 P_2 = P_1 W_1^* B W_2 P_2 = P_1 B P_2$ where the identities $P_1 W_1^* P_1 = P_1 W_1^*$ and $P_2 W_2 P_2 = W_2 P_2$ follow from the fact that $\mathfrak{H}_1$ and $\mathfrak{H}_2$ are invariant subspaces for W_1 and W_2, respectively. Thus $P_1 \mathfrak{S}(W_1, W_2) P_2$ is contained in $\mathfrak{S}(V_1, V_2)$. Note that $\|P_1 B P_2\| \leqq \|B\|$.

Conversely, suppose C is in $\mathfrak{S}(V_1, V_2)$; we want to define B from $\mathfrak{K}_2$ to $\mathfrak{K}_1$ so that $C = P_1 B P_2$ and $W_2 B W_1 = B$. The operator B will be obtained as the strong limit of the sequence $\{B_n\}$, where $B_n = W_1^{*n} P_1 C P_2 W_2^n$.

An elementary computation shows for $i = 1, 2$, that $P_{i,n} = W_i^{*n} P W_i^n$ is the projection of $\mathfrak{K}_i$ onto $W_i^{*n} \mathfrak{H}_i$ and that the sequence $\{P_{i,n}\}_n$ is monotonically increasing and converges strongly to the identity on $\mathfrak{K}_i$.

Observe now that since $\|B_n\| = \|W_1^{*n} P_1 C P_2 W_2^n\| \leqq \|C\|$, the sequence in uniformly bounded. Moreover, for $n \geqq m \geqq 0$ we have

$$P_{1,m} B_n P_{2,m} = W_1^{*m} P_1 W_1^m W_1^{*n} P_1 C P_2 W_2^n W_2^{*m} P_2 W_2^m =$$
$$= W_1^{*m} P_1 W^{*n-m} C W_2^{n-m} P_2 W_2^m = W_1^{*m} P_1 V_1^{*n-m} C V_2^{n-m} P_2 W_2^m =$$
$$= W_1^{*m} P_1 C P_2 W_2^m = B_m$$

so that $P_{1,m} B_n P_{2,m}$ is independent of n so long as n is greater than m. Thus for x in $P_{2,m}\mathfrak{K}_2$ and y in $P_{1,m}\mathfrak{K}_1$ we have

$$\lim_{n\to\infty} (B_n x, y) = \lim_{n\to\infty} (P_{1,m} B_n P_{2,m} x, y) = (B_m x, y).$$

Thus, $\lim_{n\to\infty} (B_n x, y)$ exists for x in the dense subset $\bigcup_m P_{2,m}\mathfrak{K}_2$ of $\mathfrak{K}_2$ and for y in the dense subset $\bigcup_m P_{1,m}\mathfrak{K}_1$ of $\mathfrak{K}_1$. Since the sequence is uniformly bounded, we have that the sequence $\{B_n\}$ converges weakly to an operator B in $\mathfrak{L}(\mathfrak{K}_2, \mathfrak{K}_1)$. That $P_1 B P_2 = C$ and $W_1^* C W_2 = C$ are obvious. Thus we have completed the proof that $\mathfrak{S}(V_1, V_2)$ is equal to $P_1 \mathfrak{S}(W_1, W_2) P_2$.

In the preceding argument if we notice that we also have $B_n = P_{1,n} B P_{2,n}$, then using the fact that the sequences $\{P_{1,n}\}$ and $\{P_{2,n}\}$ converge strongly to the identity operators on $\mathfrak{K}_1$ and $\mathfrak{K}_2$, respectively, we see that the sequence $\{B_n\}$ converges strongly to B, hence $\|B\| \leqq \|C\|$. From this it follows that any C in $\mathfrak{S}(V_1, V_2)$ can be represented in the form $C = P_1 B P_2$ with a B in $\mathfrak{S}(W_1, W_2)$ such that $\|B\| = \|C\|$.

4. We next study the space $\mathfrak{S}(W_1, W_2)$ for unitary operators W_1 and W_2 defined on the spaces $\mathfrak{K}_1$ and $\mathfrak{K}_2$, respectively. We begin with a lemma which is a mild generalization of a result due to PUTNAM [7]. We state the result for normal operators which necessitates the use of the Putnam—Fuglede Theorem. The same result for unitary operators has an elementary proof.

Lemma 4.1. *Let M and N be normal operators on the spaces $\mathfrak{H}$ and $\mathfrak{K}$, respectively, and let B an operator in $\mathfrak{L}(\mathfrak{H}, \mathfrak{K})$ satisfying $BM=NB$. If $\mathfrak{M}$ denotes the orthogonal complement in $\mathfrak{H}$ of the kernel of B and $\mathfrak{N}$ denotes the closure in $\mathfrak{K}$ of the range of B, then $\mathfrak{M}$ reduces M, $\mathfrak{N}$ reduces N, and $M|\mathfrak{M}$ is unitarily equivalent to $N|\mathfrak{N}$.*

Proof. Let $B=PU$ be the polar decomposition for B with U a partial isometry in $\mathfrak{L}(\mathfrak{H}, \mathfrak{K})$ and P a positive operator on $\mathfrak{K}$ so that the range of U is equal to $\mathfrak{N}$. Since $BM=NB$, the Putnam—Fuglede Theorem [8] implies $BM^*=N^*B$. These two equations imply that $\mathfrak{N}$ reduces N. Taking adjoints we have $M^*B^*=B^*N^*$ and $MB^*=B^*N$ which imply that $\mathfrak{M}$ reduces M.

Substituting we obtain $P^2N=BB^*N=BMB^*=NBB^*=NP^2$ so that P^2 commutes with N. Hence the positive square root of P^2 commutes with N so that $PUM=NPU=PNU$. This latter identity implies $UM=NU$ in view of the fact that the range of both UM and NU are contained in $\mathfrak{N}$ on which P is one-to-one. It now follows that $M|\mathfrak{M}$ and $N|\mathfrak{N}$ are unitarily equivalent with the isometry $U|\mathfrak{M}$ with range $\mathfrak{N}$ effecting this equivalence.

Returning to the situation of W_1 and W_2 unitary on $\mathfrak{K}_1$ and $\mathfrak{K}_2$ what we would like to do is to describe the space $\mathfrak{S}(W_1, W_2)$ more or less explicity. To do this, however, would take us too far afield. We content ourselves with determining when $\mathfrak{S}(W_1, W_2)\neq(0)$. Let $E(\delta)$ and $F(\delta)$ be the spectral measures for W_1 and W_2, respectively (cf. [3]). The unitary operators W_1 and W_2 are said to be *relatively singular* if the measures $\mu(\delta)=(E(\delta)x, x)$ and $\nu(\delta)=(F(\delta)y,y)$ are relatively singular for vectors x in $\mathfrak{K}_1$ and y in $\mathfrak{K}_2$.

Theorem 3. *If for $i=1,2$, W_i is a unitary operator on $\mathfrak{K}_i$, then $\mathfrak{S}(W_1, W_2)=(0)$ if and only if W_1 and W_2 are relatively singular.*

Proof. Suppose B is in $\mathfrak{S}(W_1, W_2)$ and $\mathfrak{M}$ and $\mathfrak{N}$ are defined as in the lemma. Then the operators $W_1|\mathfrak{M}$ and $W_2|\mathfrak{N}$ are unitarily equivalent. If U is an isometry from $\mathfrak{M}$ onto $\mathfrak{N}$ effecting this equivalence and x is any vector in $\mathfrak{M}$, then the measures $(E(\delta)x, x)$ and $(F(\delta)Ux, Ux)$ are identical. If $B\neq 0$, then $\mathfrak{M}\neq(0)$, so we can choose $x\neq 0$. Thus, $\mathfrak{S}(W_1, W_2)\neq(0)$ implies that W_1 and W_2 are not relatively singular.

If W_1 and W_2 are not relatively singular, then there exists vectors x in $\mathfrak{K}_1$ and y in $\mathfrak{K}_2$ so that the measures $\mu(\delta)=(E(\delta)x, x)$ and $\nu(\delta)=(F(\delta)y, y)$ are not relatively singular. Let $\mathfrak{M}_x$ and $\mathfrak{M}_y$ be the cyclic reducing subspaces generated by x and y for the operators W_1 and W_2. It follows from the multiplicity theory for normal operators (cf. [3]) that there exist vectors x_0 in $\mathfrak{M}_x$ and y_0 in $\mathfrak{N}_y$ so that the measures $\mu_0(\delta)=(E(\delta)x_0, x_0)$ and $\nu_0(\delta)=(F(\delta)y_0, y_0)$ are mutually absolutely continuous. Thus the unitary operators $W_1|\mathfrak{M}_{x_0}$ and $W_2|\mathfrak{N}_{x_0}$ are unitarily equivalent. Let V be an isometry from $\mathfrak{M}_{x_0}$ onto $\mathfrak{N}_{y_0}$ so that $(W_1|\mathfrak{M}_{x_0})=V^*(W_2|\mathfrak{N}_{y_0})V$. If we define the operator B in $\mathfrak{L}(\mathfrak{K}_1, \mathfrak{K}_2)$ so that $Bw = Vw$ for w in $\mathfrak{M}_{x_0}$ and $Bw = 0$ for w orthogonal to $\mathfrak{M}_{x_0}$, then B is in $\mathfrak{S}(W_1, W_2)$. Thus the proof is complete.

Implicit in the proofs of lemma 4. 1 and the preceding theorem is a recipe for constructing the elements of $\mathfrak{S}(W_1, W_2)$. We will not elaborate on this.

5. We now consider the operator inequalities $T^*XT \leqq X$ and $T^*XT \geqq X$ for a given contraction T and unknown Hermitian operator X. We show first that we can restrict our attention to the first inequality and consider only positive solutions. Before stating this result we introduce the following terminology. A positive operator Q satisfying $T^*QT \leqq Q$ is said to be a *pure solution* if the sequence $\{T^{*n}QT^n\}$ converges strongly to 0 and we let $\mathfrak{Q}_T$ denote the set of pure positive solutions to $T^*QT \leqq Q$.

Theorem 4. *Let T be a contraction on $\mathfrak{H}$ and H (or K) be a Hermitian operator on $\mathfrak{H}$ so that $T^*HT \geqq H$ ($T^*KT \leqq K$). Then there exist Hermitian operators R and Q so that $H = R - Q$ ($K = R + Q$), $T^*RT = R$, $TQT^* \leqq Q$ and Q is pure. Moreover, this decomposition is unique.*

Proof. Since setting $H = -K$ reduces the second case to the first we consider only the case of a Hermitian H so that $T^*HT \geqq H$. Then the sequence $\{T^{*n}HT^n\}$ is a bounded monotonically increasing sequence of Hermitian operators. Thus it converges strongly to a Hermitian operator R. It is clear that $T^*RT = R$. Setting $Q = R - H$ we see that Q is positive and $T^*QT = T^*RT - T^*HT = R - T^*HT \leqq \leqq R - H = Q$ or $T^*QT \leqq Q$. Moreover, since $T^{*n}QT^n = R - T^{*n}HT^n$ we see that Q is pure. Lastly, suppose $H = R_1 - Q_1$ with $T^*R_1T = R_1$, $T^*Q_1T \leqq Q_1$ and Q_1 is pure. Then $R_1 - R = T^{*n}(R_1 - R)T^n = T^{*n}(Q - Q_1)T^n$ and since the latter sequence converges strongly to 0 we have $R_1 = R$ and $Q_1 = Q$.

This result reduces the solution of the inequalities $T^*XT \geqq X$ and $T^*XT \leqq X$ to the study of the pure positive solutions to the latter inequality. This we shall do in two steps. Firstly, we characterize the pure positive solutions for $T^*QT \leqq Q$ using a construction due to Sz.-Nagy and Foiaş [9, p. 199] who used it for the case in which T is a co-isometry. This will reduce the study of this inequality to that of a commutation identity. Secondly, we make use of a recent result [10] of the same authors to study the obtained commutation identity.

Theorem 5. *Let T be a contraction on $\mathfrak{H}$. A positive operator Q on $\mathfrak{H}$ is a pure solution to $T^*QT \leqq Q$ if and only if there exists a unilateral shift U_+ on a space $\mathbf{H}_{\mathfrak{D}}$ and an operator C from $\mathfrak{H}$ to $\mathbf{H}_{\mathfrak{D}}$ so that $Q = C^*C$ and $CT = U_+^*C$.*

Proof. Suppose Q is a pure solution and set $R^2 = Q - T^*QT$. Then $Q = \sum_{n=0}^{\infty} T^{*n}(Q - T^*QT)T^n = \sum_{n=0}^{\infty} T^{*n}R^2T^n$, where the sum converges in the strong topology. We let $\mathfrak{D}$ be the closure of the range of R and consider U_+^* on $\boldsymbol{H}_{\mathfrak{D}}$. For x in $\mathfrak{H}$ the function f defined on $\boldsymbol{Z}^+$ so that $f(n)=RT^nx$ is in $\boldsymbol{H}_{\mathfrak{D}}$ since

$$\sum_{n=0}^{\infty}\|f(n)\|^2 = \sum_{n=0}^{\infty}(T^{*n}R^2T^nx, x) = \|Q^{1/2}x\|^2.$$

Moreover, the map from $\mathfrak{H}$ to $\boldsymbol{H}_{\mathfrak{D}}$ defined by $Cx=f$ is bounded, $(C^*Cx, x)= =\|Cx\|^2=\|f\|^2=\|Q^{1/2}x\|^2=(Qx, x)$ so that $Q=C^*C$ and $(CTx)(n)=RT^{n+1}x= =[U_*^+(Cx)](n)$ so that $CT=U_+^*C$. Thus the result is proved one way.

If U_*^+ is the backward shift on some $\boldsymbol{H}_{\mathfrak{D}}$ and C is an operator from $\mathfrak{H}$ to $\boldsymbol{H}_{\mathfrak{D}}$ so that $CT=U_+^*C$, then $T^*C^*CT=C^*U_+U_+^*C\leqq C^*C$ so that $Q=C^*C$ satisfies $T^*QT\leqq Q$. Further, $T^{*n}QT^n=C^*U_+^nU_+^{*n}C$ and the latter sequence converges strongly to 0 since $\{U_+^{*n}\}$ does.

We next state a recent result due to Sz.-Nagy and Foiaş [10] which determines the operators satisfying the conclusion of Theorem 5. Recall that for a contraction T on $\mathfrak{H}$, there exists a unique co-isometry V^* on a space $\mathfrak{R}_+$ containing $\mathfrak{H}$ so that $\mathfrak{H}$ is an invariant subspace for V^*, $T=V^*|\mathfrak{H}$, and the smallest reducing subspace for V^* containing $\mathfrak{H}$ is $\mathfrak{R}_+$; V is the minimal isometric dilation of T^* (cf. [9, p. 11]). Call V^* the canonical co-isometry of T. The minimal unitary extension W on $\mathfrak{R}$ of V is the minimal unitary dilation for T^*, that is, if P denotes the projection of $\mathfrak{R}$ onto $\mathfrak{H}$, then $T^{*n}=PW^n|\mathfrak{H}$ for all positive n and the smallest reducing subspace for W containing $\mathfrak{H}$ is $\mathfrak{R}$.

The theorem of Sz.-Nagy and Foiaş [10] can be stated (by taking adjoints) as follows.

Theorem 6. *For $i=1,2$, let T_i be a contraction on $\mathfrak{H}_i$ with canonical co-isometry V_i^* on $\mathfrak{R}_{i+}$. An operator C in $\mathfrak{L}(\mathfrak{H}_1, \mathfrak{H}_2)$ satisfies $CT_1=T_2C$ if and only if there exists an operator D in $\mathfrak{L}(\mathfrak{R}_{1+}, \mathfrak{R}_{2+})$ so that $DV_1^*=V_2^*D$ and $C=D|\mathfrak{H}$ Moreover, D can be chosen such that $\|D\|=\|C\|$.*

Let us remark the following. Suppose T is an isometry in $\mathfrak{H}$ and let W denote its minimal unitary extension in $\mathfrak{R}$. Minimality means that

$$\mathfrak{R} = \bigvee_{0}^{\infty} W^{-n}\mathfrak{H}. \tag{1}$$

From $T^n=W^n|\mathfrak{H}$ we have $T^{*n}=P_{\mathfrak{H}}W^{*n}|\mathfrak{H}$ $(n\geqq 0)$; thus W^* is an isometric (in fact, unitary) dilation of T^*. Moreover, (1) shows that W^* is the *minimal* isometric dilation of T^*. Thus W is the canonical co-isometry of T, as asserted.

Applying Theorem 6 of Sz.-Nagy and Foiaş to the case of isometric T_1 and T_2 we get the following

Corollary 5.1. *For $i=1,2$, let V_i be an isometry on $\mathfrak{H}_i$ with minimal unitary extension W_i on $\mathfrak{R}_i$. An operator C in $\mathfrak{L}(\mathfrak{H}_2, \mathfrak{H}_1)$ satisfies $V_1C=CV_2$ if and only* *if there exists B in $\mathfrak{L}(\mathfrak{K}_2, \mathfrak{K}_1)$ satisfying $W_1 B = BW_2$ and $C = B|\mathfrak{H}_2$. Moreover B can be chosen such that $\|B\| = \|C\|$.*

This result also follows from Theorem 2.

As corollaries to Theorems 5 and 6 we obtain results analogous with those of § 2 and 3.

Corollary 5. 2. *Let T be a contraction on $\mathfrak{H}$ with canonical co-isometry V^* on $\mathfrak{K}_+$ and let $P_{\mathfrak{H}}^+$ be the projection of $\mathfrak{K}_+$ on $\mathfrak{H}$. Then $\mathfrak{Q}_T = P_{\mathfrak{H}}^+ \mathfrak{Q}_{V^*} P_{\mathfrak{H}}^+$. Moreover, every X in $\mathfrak{Q}_T$ can be represented in the form $P_{\mathfrak{H}}^+ Y P_{\mathfrak{H}}^+$ with Y in $\mathfrak{Q}_{V^*}$ such that $\|X\| = \|Y\|$.*

Proof. If Q is in $\mathfrak{Q}_{V^*}$, then

$$T^* P_{\mathfrak{H}}^+ Q P_{\mathfrak{H}}^+ T = P_{\mathfrak{H}}^+ V Q V^* P_{\mathfrak{H}}^+ \leqq P_{\mathfrak{H}}^+ Q P_{\mathfrak{H}}^+$$

so that $P_{\mathfrak{H}}^+ \mathfrak{Q}_{V^*} P_{\mathfrak{H}}^+$ is contained in $\mathfrak{Q}_T$.

If Q is in $\mathfrak{Q}_T$, then from Theorem 5 it follows that there exists a backward shift U_+^* on $H_{\mathfrak{D}}$ and an operator C in $\mathfrak{L}(\mathfrak{H}, H_{\mathfrak{D}})$ so that $CT = U_+^* C$ and $Q = C^* C$. From Theorem 6 of Sz.-Nagy and Foiaş (case $T_2^* = U_+^*$) we obtain an operator D in $\mathfrak{L}(\mathfrak{K}_+, H_{\mathfrak{D}})$ so that $DV^* = U_+^* D$ and $C = D|\mathfrak{H}$. Thus again using Theorem 5 we have $D^* D$ is in $\mathfrak{Q}_{V^*}$. Moreover, $Q = C^* C = P_{\mathfrak{H}}^+ D^* D P_{\mathfrak{H}}^+$ and the proof is complete.

Corollary 5. 3. *Let T be a contraction on $\mathfrak{H}$ with minimal unitary dilation W on $\mathfrak{K}$ and let $P_{\mathfrak{H}}$ denote the projection of $\mathfrak{K}$ onto $\mathfrak{H}$. Then $\mathfrak{Q}_T = P_{\mathfrak{H}} \mathfrak{Q}_W P_{\mathfrak{H}}$. Moreover, every X in $\mathfrak{Q}_T$ can be represented in the form $P_{\mathfrak{H}} Z P_{\mathfrak{H}}$ with Z in $\mathfrak{Q}_W$ such that $\|X\| = \|Z\|$.*

Proof. If V^* is the canonical co-isometry on $\mathfrak{K}_+$ then by the preceding corollary $\mathfrak{Q}_T = P_{\mathfrak{H}}^+ \mathfrak{Q}_{V^*} P_{\mathfrak{H}}^+$. If D satisfyes $DV^* = U_+^* D$, then $VD^* = D^* U_+$ so that from Corollary 5. 1 it follows that there exists E in $\mathfrak{L}(L_{\mathfrak{D}}, \mathfrak{K})$ so that $D^* = E|H_{\mathfrak{D}}$ and $EU = W^* E$. Thus we have $D^* D = P_{\mathfrak{D}_{\mathfrak{H}}} E^* P_{\mathfrak{D}_{\mathfrak{H}}}$ and $W^* E P_{\mathfrak{D}_{\mathfrak{H}}} E^* W = EUP_{\mathfrak{D}_{\mathfrak{H}}} U^* E^* \leqq$ $\leqq EP_{\mathfrak{D}_{\mathfrak{H}}} E^*$ so that $EP_{\mathfrak{D}_{\mathfrak{H}}} E^*$ is in $\mathfrak{Q}_W$ and $\mathfrak{Q}_T$ is seen to be contained in $P_{\mathfrak{H}} \mathfrak{K}_W P_{\mathfrak{H}}$.

Conversely, if Q is in $\mathfrak{Q}_W$ and if R denotes the projection of $\mathfrak{K}$ onto $\mathfrak{K}_+$, then $VRQRV^* = RW^* RQRWR = RW^* QWR \leqq RQR$ so that RQR is in $\mathfrak{Q}_{V^*}$. Using the preceding corollary we have $P_{\mathfrak{H}} Q P_{\mathfrak{H}} = P_{\mathfrak{H}} RQR P_{\mathfrak{H}}$ is in $\mathfrak{Q}_T$ and the proof is complete.

Implicit in the preceding proof is a characterization of the operators in $\mathfrak{Q}_W$ for a unitary operator W. We state it without further proof.

Corollary 5. 4. *Let W be a unitary operator on $\mathfrak{K}$. Then Q is in $\mathfrak{Q}_W$ if and only if there exists a Hilbert space $\mathfrak{D}$ and an operator E in $\mathfrak{L}(\mathfrak{K}, L_{\mathfrak{D}})$ so that $EW = UE$ and $Q = E^* P_{H_{\mathfrak{D}}} E$.*

We illustrate how the preceding results can be applied to obtain a result due to PUTNAM [8, Theorem 2. 3. 2]. Before stating it we need to recall the following. If W is a unitary operator on $\mathfrak{K}$ with spectral measure $E(\delta)$, then W is said to be *absolutely continuous* [*singular*] if the measure $\mu(\delta)=(E(\delta)x, x)$ is absolutely continuous [singular] for each vector x in $\mathfrak{K}$. If W is a unitary operator on $\mathfrak{K}_1$ then $\mathfrak{K} = \mathfrak{K}_a \oplus \mathfrak{K}_s$, where $\mathfrak{K}_a$ and $\mathfrak{K}_s$ are reducing subspaces for W so that $W|\mathfrak{K}_a$ is absolutely continuous while $W|\mathfrak{K}_s$ is singular. The operator $W|\mathfrak{K}_a$ is said to be the absolute continuous part of W. (See [3] for details and proofs.)

Corollary 5. 5. *Let W be a unitary operator on $\mathfrak{H}$ and Q be a pure positive solution to $W^*QW \leqq Q$. Then the range of Q is contained in the absolutely continuous part of W.*

Proof. From the preceding theorem it follows that there exists a backward shift U_+ on some $H_{\mathfrak{D}}$ and an operator C from $\mathfrak{H}$ to $H_{\mathfrak{D}}$ so that $Q=C^*C$ and $CW= U_+^*C$. Thus there exists by Corollary 5. 1 an operator D from $\mathfrak{H}$ to $L_{\mathfrak{D}}$ so that $D=C^*|H_{\mathfrak{D}}$ and $W^*D=DU$. Moreover, since $Q=C^*C$, the closure of the range of Q is equal to the closure of the range of C^* which in turn is equal to the closure of $DH_{\mathfrak{D}}$. Thus our problem is reduced to showing that $DH_{\mathfrak{D}}$ is contained in the absolutely continuous part of W.

Using lemma 4. 1 we have that W restricted to the closure of the range of D is unitarily equivalent to U restricted to the orthogonal complement of the kernel of D. The latter unitary operator is a part of the bilateral shift and so must be absolutely continuous. (We can compute the spectral measure in this case.) Thus $DH_{\mathfrak{D}}$ is contained in the absolutely continuous part of W and the proof is complete.

Corollary 5. 6. *Let W be a singular unitary operator on $\mathfrak{H}$ and H be a Hermitian operator on $\mathfrak{H}$ so that $W^*HW \geqq H$. Then W commutes with H.*

Proof. From Theorem 4 we have that $H = R-Q$ where R commutes with H and Q is a pure positive solution to $W^*QW \leqq Q$. From the preceding corollary we have the range of Q is contained in the absolutely continuous part of W which in this case has been assumed to be (0). Thus $Q=0$ and the proof is complete.

Recall that an operator T on $\mathfrak{H}$ is said to be hyponormal if $T^*T \geqq TT^*$ and completely non normal if for no subspace $\mathfrak{M}$ reducing T is $T|\mathfrak{M}$ normal.

Corollary 5. 7. *If T is an invertible completely non normal hyponormal operator on $\mathfrak{H}$ with polar decomposition $T=PU$, then U is absolutely continuous.*

Proof. Since T is invertible, the operator U is unitary and $U^*P^2U = T^*T \geqq TT^* = P^2$. Thus from Theorem 4 it follows that $P^2=R-Q$, where R and Q are positive, U commutes with R and Q is a pure solution to $U^*QU \leqq Q$. Thus from the preceding corollary it follows that the range of Q is contained in the absolutely continuous part of U. If E is the spectral projection for U onto the singular part of U, then $EP^2 = ER + EQ = ER = RE = P^2E$. Thus $T|E\mathfrak{H} = (EPE)(EUE)|E\mathfrak{H}$, where EPE is positive, EUE is unitary and EPE commutes with EUE. Thus $T|E\mathfrak{H}$ is normal implying by hypothesis that $E=0$ and the proof is complete.

This is related to the result that every compact hyponormal operator is normal (cf. [4]). We offer a proof of this result in § 6.

We conclude this section with a further remark concerning the inequality $T^*XT \geqq X$ for positive operators X. In Theorem 1 we showed that solutions for the equation $T^*XT = X$ could be obtained from solutions to $V_T^*XV_T = X$ where V_T is the isometry associated with the contraction T. This isometry is only part of the minimal unitary dilation for T to which we reduced the study of $T^*XT \leqq X$. It is therefore of interest that the study of $T^*XT \geqq X$ can be reduced to that of $V_T^*XV_T \geqq X$.

For T a contraction let $\mathfrak{P}_T$ denote the class of positive operators P so that $T^*PT \geqq P$.

Theorem 7. *Let T be a contraction on $\mathfrak{H}$ and A_T and V_T as in Theorem 1. Then $\mathfrak{P}_T = (0)$ if and only if $A_T = 0$ and $\mathfrak{P}_T = A_T\mathfrak{P}_{V_T}A_T$. Moreover, every X in $\mathfrak{P}_T$ can be represented in the form A_TYA_T with Y in $\mathfrak{P}_{V_T}$ such that $\|X\| = \|Y\|$.*

The proof is the same as that of Theorem 1.

6. We now obtain some special results in the presence of a compactness hypothesis. Before we can state our result we need a lemma concerning the subspace $\mathfrak{U}_T$ spanned by the eigenvectors of a contraction which belong to an eigenvalue of modulus one. See [9, pp. 8—9] for the proof.

Lemma 6. 1. *If T is a contraction on $\mathfrak{H}$, then $\mathfrak{U}_T$ reduces T and $T|\mathfrak{U}_T$ is a unitary operator with pure point spectrum.*

Our main result in this section is the following.

Theorem 8. *Let T be a contraction on $\mathfrak{H}$. If Q is a compact positive operator in $\mathfrak{P}_T$, then Q is in $\mathfrak{S}_T$. Further, if A is a compact operator in $\mathfrak{S}_T$, then A and A^* commute with T, $\mathfrak{U}_T$ reduces A, and $A|\mathfrak{U}_T^{\perp} = 0$.*

Proof. Suppose Q is positive, compact, and so that $T^*QT \geqq Q$. Let $\lambda_1 > \lambda_2 > \cdots$ be the non zero eigenvalues of Q, and let $\mathfrak{J}_1, \mathfrak{J}_2, \cdots$ be the corresponding eigenspaces. Each of these eigenspaces is finite dimensional and, denoting by P_n the (orthogonal) projection of $\mathfrak{H}$ onto $\mathfrak{J}_n$, we have

$$Qx = \sum_n \lambda_n P_n x \quad \text{for all} \quad x \in \mathfrak{H}.$$

We shall prove that each $\mathfrak{J}_n$ reduces T, and that $T|\mathfrak{J}_n$ is unitary. We do this by induction on n. Suppose this is true for all n less than some $m (\geqq 1)$ (for $m = 1$

this hypothesis being void). For x in $\mathfrak{I}_m$, Tx satisfies then the condition of being orthogonal to each $\mathfrak{I}_n$ with $n<m$ (this condition being void if $m=1$); so we have

$$\lambda_m\|Tx\|^2 \geqq (QTx, Tx) = (T^*QTx, x) \geqq (Qx, x) = \lambda_m\|x\|^2.$$

Since T is a contraction, this implies $\|Tx\|=\|x\|$ and $(QTx, Tx)=\lambda_m\|Tx\|^2$. Thus $T\mathfrak{I}_m\subset\mathfrak{I}_m$, and $T_m=T|\mathfrak{I}_m$ is an isometry. However, since $\mathfrak{I}_m$ is finite dimensional it follows that T_m is unitary. Then so is T_m^*, which is equal to $P_mT^*|\mathfrak{I}_m$. So we have for x in $\mathfrak{I}_m$

$$\|x\| = \|T_m^*x\| = \|P_mT^*x\| \leqq \|T^*x\| \leqq \|x\|,$$

and this implies $P_mT^*x=T^*x$. Hence $T^*\mathfrak{I}_m\subset\mathfrak{I}_m$ so that $\mathfrak{I}_m$ reduces T.

So we have shown that each $\mathfrak{I}_n$ $(n=1, 2, \cdots)$ reduces T to a unitary operator. It follows for an arbitrary x in $\mathfrak{H}$

$$T^*P_nTx=T^*TP_nx=P_nx \qquad (n=1, 2, \cdots),$$

and hence

$$T^*QTx = \sum_n \lambda_n T^*P_nTx = \sum_n \lambda_n P_n x = Qx.$$

Thus Q is in $\mathfrak{S}_T$.

Consider now a compact operator A in $\mathfrak{S}_T$.

If $T^*AT=A$, taking adjoints we obtain $T^*A^*T=A^*$ so that if $A = H+iK$ are the real and imaginary parts of A, then $T^*HT=H$ and $T^*KT=K$. Thus the proof can be reduced to the case of a Hermitian operator.

If H is Hermitian, then there exists a reducing subspace $\mathfrak{N}$ for H so that $H_1=H|\mathfrak{N}$ and $H_2=-H|\mathfrak{N}^\perp$ are positive operators. Substituting we obtain the equation $T^*H_1T-T^*H_2T = H_1-H_2$, where $T^*H_1T\geqq 0$ and $T^*H_2T\geqq 0$.

If R denotes the projection of $\mathfrak{H}$ onto $\mathfrak{N}$, then

$$(TR)^*H_1(TR) \geqq (TR)^*H_1(TR)-(TR)^*H_2(TR) = RH_1R-RH_2R = H_1$$

so that $(TR)^*H_1(TR)\geqq H_1$. Since H_1 is positive and compact it follows from the above that $\mathfrak{U}_{TR}$ reduces H_1 and $H_1|\mathfrak{U}_{TR}^\perp=0$. If x is in $\mathfrak{U}_{TR}$, then for some $e^{i\theta}$ we have $TRx=e^{i\theta}x$ so that $\|x\|=\|TRx\|\leqq\|Rx\|\leqq\|x\|$. Thus $Rx=x$ which implies $Tx=e^{i\theta}x$ and x is in $\mathfrak{U}_T$. Hence $\mathfrak{U}_{TR}\subset\mathfrak{U}_T$ so that $\mathfrak{U}_T$ reduces H_1 and $H_1|\mathfrak{U}_T^\perp=0$.

Consideration of the identity $T^*(-H)T = (-H)$ yields the corresponding results for H_2. Thus $\mathfrak{U}_T$ reduces H and $H|\mathfrak{U}_T^\perp=0$. Moreover, since $TT^*|\mathfrak{U}_T$ is the identity on $\mathfrak{U}_T$, we obtain $HT=TT^*HT=TH$ and $T^*H=T^*HTT^*=HT^*$. This completes the proof.

A lemma of Dye [2, lemma 3. 1] is an immediate corollary to Theorem 8.

The result of Brown and Halmos concerning compact Toeplitz operators [1] admits the following generalization.

Corollary 6. 1. *If T is a contraction on $\mathfrak{H}$ with no eigenvalues of modulus one and A is a compact operator in $\mathfrak{S}_T$, then $A=0$.*

The following corollary is well known (cf. [4]).

Corollary 6. 2. *If T is a compact hyponormal operator, then T is normal.*

Proof. If $T=QV$ is the polar decomposition for T, then Q is positive and compact. Further, $V^*Q^2V \geqq Q^2$. Theorem 8 now applies to conclude $V^*Q^2V=Q^2$ so that $T^*T=TT^*$ and T is normal.

Corollary 6. 3. *If T is a contraction on $\mathfrak{H}$ so that A_T is compact, then A_T is a finite dimensional projection and $T|A_T\mathfrak{H}$ is unitary.*

Proof. From the definition of A_T it follows that $T^*A_T^2T=A_T^2$. Thus by Theorem 8 we see that $T|A_T^2\mathfrak{H}$ is unitary so that for x in $\mathfrak{H}$ we obtain $\|A_T^2x\| = \lim_{n\to 0}\|T^nA_Tx\|^2=\|A_Tx\|$. Since A_T is a positive contraction we obtain $A_T^2=A_T$. Therefore A_T is a compact projection which implies it is finite dimensional.

We next state a couple of miscellaneous corollaries. Recall that for operators V and W defined on $\mathfrak{H}$ and $\mathfrak{K}$, W is said to be a quasi-affine transform of V is there exists a quasi-affinity S in $\mathfrak{L}(\mathfrak{H}, \mathfrak{K})$, that is, an S with dense range and no null space, so that $VS=SW$ (cf. [9]).

Corollary 6. 4. *If the contraction K on $\mathfrak{H}$ is the quasi-affine transform of the isometry V on $\mathfrak{H}$, where $SK=VS$, and SK is compact, then K and V are unitary and unitarily equivalent.*

Proof. Since SKK^*S^* is positive and compact and $V^*SKK^*S^*V= =V^*VSS^*V^*V=SS^*\geqq SKK^*S^*$, we can apply Theorem 8 to conclude that $S(I-KK^*)S^*=0$. Since S and S^* have no null space we conclude that K^* is an isometry. Lastly, since $VS=SK$ has no null space, neither can K which implies K is unitary. Thus SK has dense range which implies V is unitary. An application of Lemma 4. 1 completes the proof.

We now remark that the preceding corollary contains two different results. Firstly, in order for a compact contraction to be the quasi-affine transform of an isometry, the underlying space must be finite dimensional. Secondly, in order for a contraction to be the quasi-affine transform of an isometry with a compact operator implementing this equivalence, both the contraction and the isometry must be unitary.

Corollary 6. 5. *Let S and T be contractions on $\mathfrak{H}$ and A be a Hermitian compact operator on $\mathfrak{H}$ so that $S^*AT=A$. If $\mathfrak{M}=\overline{A\mathfrak{H}}$, then $S|\mathfrak{M}=V_1$ and $T|\mathfrak{M}=V_2$ are unitary, $V_1=V_2$, and V_1 and V_2 commute with $A|\mathfrak{M}$.*

Proof. From $S^*AT=A$ it follows that $S^*A^2S\geqq S^*ATT^*AS=A^2$ so that it follows from Theorem 8 that S commutes with A^2 and $\mathfrak{M}\subset\mathfrak{U}_S$. Similarly, consideration of the identity $T^*AS=A$ leads to the fact that T commutes with A^2 and $\mathfrak{M}\subset\mathfrak{U}_{T^*}=\mathfrak{U}_T$. The result now follows.

7. We conclude with a few remarks. In this paper we have been considering the equation $S^*XT=X$ and the inequalities $T^*XT\geqq X$ and $T^*XT\leqq X$ for contractions S and T. The assumption that S and T be contractions is crucial for results of this nature to hold.

If T is a contraction on $\mathfrak{H}$ not having 1 as an eigenvalue, then the Cayley transform $A=(I+T)(I-T)^{-1}$ of T can be defined. The set of operators obtained in this manner is the class of maximal accretive operators (cf. [9]). Recall that a densely defined operator A on $\mathfrak{H}$ is said to be *accretive* if $\operatorname{Re}(Ax,x)\geqq 0$ for x in the domain of A, and *maximal accretive* if no proper extension of A is accretive.

If S and T are contractions on $\mathfrak{H}$ and $\mathfrak{K}$ with Cayley transforms A and B, then for X in $\mathfrak{L}(\mathfrak{K},\mathfrak{H})$ the equation $S^*XT=X$ holds if and only if $B^*X=-XA$. Thus this equation is amenable to the technique of §§ 2 and 3 for accretive operators A and B. The inequalities $T^*XT\geqq X$ and $T^*XT\leqq X$ for X Hermitian become $A^*X+XA\geqq 0$ and $A^*X+XA\leqq 0$ and can be solved with the results of § 5. The results of the rest of the paper have similar interpretations in terms of accretive operators.

Further, these results have extensions to one parameter semi-groups of contractions and indeed to other commutative semi-groups of contractions, but we will not pursue them.

Lastly, we conclude with an example. Recall that if N is a normal operator on $\mathfrak{K}$ and $\mathfrak{H}$ is an invariant subspace for N, then the operator $T=N|\mathfrak{H}$ is said to be *subnormal*. If the smallest reducing subspace for N containing $\mathfrak{H}$ is $\mathfrak{K}$, then N is said to be the *minimal normal extension of* T. This is unique to an isomorphism (cf. [4]). In case N is unitary, then T is an isometry and an isometry is subnormal by our previous remarks.

Corollary 5. 1 can be interpreted as stating that all "commuting maps" between isometries "lift" to their minimal normal extensions. We want to show that this is not true for subnormal operators in general. We first prove the following lemma.

Lemma 7. 1. *For $i=1,2$, let T_i be a subnormal operator on $\mathfrak{H}_i$ having minimal normal extension N_i on $\mathfrak{K}_i$. Let A be a quasi-affinity in $\mathfrak{L}(\mathfrak{H}_1,\mathfrak{H}_2)$ so that $T_2A=AT_1$. Then a necessary condition that there exist B in $\mathfrak{L}(\mathfrak{K}_1,\mathfrak{K}_2)$ so that $N_2B=BN_1$ and $A=B|\mathfrak{H}_1$ is for N_1 and N_2 to be unitarily equivalent.*

Proof. Suppose such an operator B exists. Then as in the proof of lemma 4. 1, the closure $\mathfrak{N}$ of the range of B reduces N_2. If P denotes the projection of $\mathfrak{K}_2$ onto $\mathfrak{K}_2\ominus\mathfrak{N}$, then for a dense set of x in $\mathfrak{H}_2$ there exists y in $\mathfrak{H}_1$ so that $Ay=x$ and we have $Px=PAy=PBy=0$. Thus $\mathfrak{H}_2$ is contained in $\mathfrak{N}$ which contradicts the minimality of N_2 unless $\mathfrak{N} = \mathfrak{K}_2$. Similarly, the closure of the range of B^* must be $\mathfrak{K}_1$. From lemma 4. 1 it follows that N_1 and N_2 are unitarily equivalent.

Corollary 7. 2. *There exist subnormal operators T_1 on $\mathfrak{H}_1$ and T_2 on $\mathfrak{H}_2$ with minimal normal extensions N_1 on $\mathfrak{K}_1$ and N_2 on $\mathfrak{K}_2$, respectively, and an operator A in $\mathfrak{L}(\mathfrak{H}_1, \mathfrak{H}_2)$ satisfying $AT_1 = T_2A$ for which there is no B in $\mathfrak{L}(\mathfrak{K}_1, \mathfrak{K}_2)$ satisfying $BN_1 = N_2B$ and $A = B|\mathfrak{K}_1$.*

Proof. There is an example in [4] due to SARASON of similar subnormal operators T_1 on $\mathfrak{H}_1$ and T_2 on $\mathfrak{H}_2$ so that their minimal normal extensions N_1 on $\mathfrak{K}_1$ and N_2 on $\mathfrak{K}_2$ are not unitarily equivalent. If A is the invertible operator so that $T_2A = AT_1$, then if follows from the preceding lemma that there exists no B in $\mathfrak{L}(\mathfrak{K}_1, \mathfrak{K}_2)$ satisfying $N_2B = BN_1$ and $A = B|\mathfrak{H}_1$.

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