On the power-bounded operators of Sz.-Nagy and Foiaş

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1. In [6] Sz.-Nagy and Foiaş considered, for each $\varrho > 0$, the class C_ϱ of operators T on a given complex Hilbert space $\mathfrak{H}$ having the following property: for some Hilbert space $\mathfrak{K}$ containing $\mathfrak{H}$ as a subspace and some *unitary* operator U on $\mathfrak{K}$, $T^n = \varrho P_{\mathfrak{H}} U^n$ $(n = 1, 2, 3, \ldots)$, where $P_{\mathfrak{H}}$ denotes the orthogonal projection of $\mathfrak{K}$ onto $\mathfrak{H}$. It had been shown previously that $C_1 = \{T: \|T\| \leqq 1\}$ (see Sz.-Nagy [5]) and that $C_2 = \{T: w(T) \leqq 1\}$ (see Berger [1]), where $w(T)$ denotes the "numerical radius" of T, namely $\sup\{|(Th, h)|: h \in \mathfrak{H} \text{ and } \|h\| \leqq 1\}$. It seemed natural to us to introduce the functions w_ϱ defined on the space $\mathscr{L}(\mathfrak{H})$ of operators on $\mathfrak{H}$ in such a way that (a) w_ϱ is homogeneous $\big(w_\varrho(zT) = |z| w_\varrho(T)\big)$, and (b) $w_\varrho(T) \leqq 1 \Leftrightarrow T \in C_\varrho$. In this way we obtain a family of "operator radii" which includes the familiar norms $\|\cdot\| \, (= w_1(\cdot))$ and $w(\cdot) (= w_2(\cdot))$ and which has a number of interesting properties. Recently we received from J. P. Williams a preprint of [8] where he, too, introduces the functions w_ϱ, stressing properties different from those which concern us here.

One can, of course, show that $w_\varrho(T^n) \leqq \big(w_\varrho(T)\big)^n$ for all $\varrho > 0$ and all $n \geqq 1$ (recall the "power inequality" $w(T^n) \leqq \big(w(T)\big)^n$ of Berger); here however we shall deal with somewhat different kinds of multiplicative behavior in the operator radii $w_\varrho(\cdot)$ (see § 4 and § 6 below). A basic result of this nature is the inequality $w_{\varrho\sigma}(TS) \leqq$ $\leqq w_\varrho(T) w_\sigma(S)$, holding whenever T and S double commute.

We shall also show that another well-known "operator radius", namely the spectral radius $\nu(\cdot)$ may be adjoined in a natural way to our family $\{w_\varrho(\cdot)\}_{\varrho > 0}$; in fact, if we let $w_\infty(T) = \lim\limits_{\varrho \to 0} w_\varrho(T)$, we find that $w_\infty(T) = \nu(T)$. This result, and others concerning the relationship between $\nu(T)$ and $w_\varrho(T)$ are discussed in § 5.

These techniques may be applied to yield information about the classes C_ϱ themselves. We shall see, for example, that although Sz.-Nagy and Foiaş have shown that $\bigcup\limits_{\varrho > 0} C_\varrho$ does not contain every "power-bounded" operator (see [6], § 4), nevertheless $\bigcup\limits_{\varrho > 0} C_\varrho$ is dense in the class of all power-bounded operators.

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2. We shall use the following two characterizations of the classes C_ϱ. Both of these theorems are immediate consequences of the Theorem of [6] (or of its proof)

For $T\in\mathcal{L}(\mathfrak{H})$ and $\varrho>0$ define $T_\varrho(n)$ as follows:

$$T_\varrho(n)=\frac{1}{\varrho}T^n \text{ if } n=1,2,\dots;\quad T_\varrho(0)=I;\quad T_\varrho(n)=\frac{1}{\varrho}(T^*)^{-n} \text{ if } n=-1,-2,\dots.$$

Theorem 2. 1. *Given $\varrho>0$ and $T\in\mathcal{L}(\mathfrak{H})$ we have $T\in C_\varrho$ if, and only if, $\sum_{n=-\infty}^{\infty} r^{|n|}e^{in\theta}T_\varrho(n)\geqq 0$ for every θ and r such that $0\leqq r<1$. It is understood that the series converges absolutely, i. e., $\sum_{n=-\infty}^{\infty} r^{|n|}\|T_\varrho(n)\|<\infty$, whenever $0\leqq r<1$.*

Theorem 2. 2. *Given $\varrho>0$ and $T\in\mathcal{L}(\mathfrak{H})$, we have $T\in C_\varrho$ if, and only if, $\nu(T)\leqq 1$, and for each $h\in\mathfrak{H}$ and each complex z such that $|z|<1$,*

$$(*)\qquad \operatorname{Re}((I-zT)h,h)\geqq\left(1-\frac{\varrho}{2}\right)\|(I-zT)h\|^2.$$

If $\varrho\leqq 2$, the condition on the spectral radius is redundant [1]).

As Sz.-Nagy and Foiaş point out in [6], it is a simple matter to use Theorem 2. 2 to derive the earlier results of Sz.-Nagy and Berger that $C_1=\{T:\|T\|\leqq 1\}$ and $C_2=\{T: w(T)\leqq 1\}$.

3. For each $p>0$, we define the function w_ϱ on $\mathcal{L}(\mathfrak{H})$ as follows:

$$w_\varrho(T)=\inf\left\{u: u>0, \frac{1}{u}T\in C_\varrho\right\}.$$

Theorem 3.1. *$w_\varrho(\cdot)$ has the following properties:*

(1) $w_\varrho(T)<\infty$;

(2) $w_\varrho(T)>0$ *unless* $T=0$; *in fact*, $w_\varrho(T)\geqq\frac{1}{\varrho}\|T\|$;

(3) $w_\varrho(zT)=|z|w_\varrho(T)$;

(4) $w_\varrho(T)\leqq 1\Leftrightarrow T\in C_\varrho$.

Proof. To prove (1) we need only show that, for some $v>0$, $vT\in C_\varrho$. However, if $0\leqq r<1$ and $z=re^{i\theta}$,

$$\sum_{n=-\infty}^{\infty} r^{|n|}e^{in\theta}(vT)_\varrho(n)=I-\frac{2}{\varrho}\operatorname{Re}\sum_{n=1}^{\infty}(zvT)^n\geqq\left(1-\frac{2}{\varrho}\sum_{n=1}^{\infty}(v\|T\|)^n\right)I\geqq O$$

provided $v\|T\|$ is sufficiently small. For such v, then, by Theorem 2. 1, $vT\in C_\varrho$.

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1) By a recent result, it is actually redundant for *any* ϱ; *cf.* Ch. Davis, The shell of a Hilbert-space operator, *Acta Sci. Math.*, **29** (1968), 69—86 (Prop. 8. 3).

(2) follows once we observe that, if $0<u<\frac{1}{\varrho}\|T\|$, we have $\left\|\frac{1}{u}T\right\|>\varrho$ so that we cannot have $\frac{1}{u}T=\varrho P_{\mathfrak{H}}U$ for any unitary operator U.

For the proofs of (3) and (4) we shall need the following result: $T\in C_\varrho$ and $|z|\leqq 1\Rightarrow zT\in C_\varrho$. To see this note if that $T\in C_\varrho$ we have a unitary operator U on $\mathfrak{K}\supset\mathfrak{H}$ such that $T^n=\varrho P_{\mathfrak{H}}U^n$ $(n=1,2,3,\ldots)$; thus $(zT)^n=\varrho P_{\mathfrak{H}}(zU)^n$. But, if $|z|\leqq 1$, then $\|zU\|\leqq 1$ so that $zU\in C_1$ for the new space $\mathfrak{K}$; letting V be a unitary operator on $\mathfrak{K}_1\supset\mathfrak{K}$ such that $(zU)^n=P_{\mathfrak{K}}V^n$, we see that (with the obvious interpretation) $(zT)^n=\varrho P_{\mathfrak{H}}V^n$, so that, indeed, $zT\in C_\varrho$.

Recalling Theorem 2.1, it is clear that $O\in C_\varrho$ for every $\varrho>0$, and it follows easily that $w_\varrho(O)=0$. Thus (3) certainly holds when $|z|=0$. Turning to the case where $|z|>0$, write $z=re^{i\theta}$ and observe that, by the result of the last paragraph, we can assert that, for every $S\in\mathscr{L}(\mathfrak{H})$, $e^{i\theta}S\in C_\varrho\Leftrightarrow S\in C_\varrho$. We may thus perform the following calculation:

$$|z|\,w_\varrho(T)=r\left(\inf\left\{u:u>0,\frac{1}{u}T\in C_\varrho\right\}\right)=$$

$$=\inf\left\{ru:u>0,\frac{1}{ru}rT\in C_\varrho\right\}=\inf\left\{ru:u>0,\frac{1}{ru}re^{i\theta}T\in C_\varrho\right\}=$$

$$=\inf\left\{u:u>0,\frac{1}{u}zT\in C_\varrho\right\}=w_\varrho(zT).$$

The implication $(\Leftarrow)$ in (4) is immediate from the definition of w_ϱ. To prove $(\Rightarrow)$ assume that $w_\varrho(T)\neq 0$ and observe that we always have $u_n>0$ such that $\frac{1}{u_n}T\in C_\varrho$ and $u_n\downarrow w_\varrho(T)$; it follows easily, using Theorem 2.2, that $\left(\lim\frac{1}{u_n}\right)T\in C_\varrho$, i.e., that $\frac{T}{w_\varrho(T)}\in C_\varrho$. If $w_\varrho(T)(=|w_\varrho(T)|)\leqq 1$, we conclude that $T=w_\varrho(T)\left(\frac{T}{w_\varrho(T)}\right)\in C_\varrho$. Finally, if $w_\varrho(T)=0$, then $T=O$ by (2), and, as noted earlier, we always have $O\in C_\varrho$. Q.e.d.

For $\varrho=1$ and $\varrho=2$, of course, $w_\varrho(\cdot)$ is actually a norm; more generally we have the following result.

Theorem 3.2. *The function w_ϱ is a norm on $\mathscr{L}(\mathfrak{H})$ whenever $0<\varrho\leqq 2$.*

Proof. Equivalently, we must show that C_ϱ is a convex body in $\mathscr{L}(\mathfrak{H})$ whenever $\varrho\leqq 2$. Suppose, then, that $T,S\in C_\varrho$; by Theorem 2.2 we have, for every $h\in\mathfrak{H}$ and complex z such that $|z|<1$,

$$\mathrm{Re}\,((I-zT)h,h)\geqq\left(1-\frac{\varrho}{2}\right)\|(I-zT)h\|^2$$

and an analogous inequality for S. It follows that, if $\lambda, \mu \geqq 0$ and $\lambda+\mu=1$, we have

$$\operatorname{Re}\left((I-z(\lambda T+\mu S))h, h\right) \geqq \left(1-\frac{\varrho}{2}\right)[\lambda\|(I-zT)h\|^2+\mu\|(I-zS)h\|^2].$$

For any $x, y \in \mathfrak{H}$ we have $\lambda\|x\|^2+\mu\|y\|^2 \geqq \|\lambda x+\mu y\|^2$, as the following calculation shows: $\lambda\|x\|^2+\mu\|y\|^2-\|\lambda x+\mu y\|^2 = (\lambda-\lambda^2)\|x\|^2+(\mu-\mu^2)\|y\|^2-2\lambda\mu \operatorname{Re}(x, y) \geqq$ $\geqq \lambda(1-\lambda)\|x\|^2+(1-\mu)\mu\|y\|^2-2\lambda\mu\|x\|\cdot\|y\| = \lambda\mu(\|x\|-\|y\|)^2 \geqq 0$. Since $\varrho \leqq 2$, we have $\left(1-\frac{\varrho}{2}\right) \geqq 0$; thus

$$\operatorname{Re}\left((I-z(\lambda T+\mu S))h, h\right) \geqq \left(1-\frac{\varrho}{2}\right)\|\lambda(I-zT)h+\mu(I-zS)h\|^2 =$$
$$= \left(1-\frac{\varrho}{2}\right)\|(I-z(\lambda T+\mu S))h\|^2.$$

Using Theorem 2. 2 again, we conclude that $\lambda T+\mu S \in C_\varrho$. Q.e.d.

As a by-product of the results of § 6, we shall see that $w_\varrho(\cdot)$ fails to be a norm whenever $\varrho > 2$.

4. In this section we discusss some of the basic inequalities governing the operator radii $w_\varrho(\cdot)$.

The following theorem comes as no surprise; it is simply a generalization of BERGER's proof of the "power inequality" $w(T^n) \leqq (w(T))^n$ (a conjecture of HALMOS).

Theorem 4.1. *For each $\varrho>0$ and $T \in \mathscr{L}(\mathfrak{H})$ we have $w_\varrho(T^k) \leqq (w_\varrho(T))^k$ $(k=1, 2, 3, \ldots)$.*

Proof. By Theorem 3.1, $w_\varrho(\cdot)$ is homogeneous so that we need only show that $w_\varrho(T) \leqq 1 \Rightarrow w_\varrho(T^k) \leqq 1$, or equivalently that $T \in C_\varrho \Rightarrow T^k \in C_\varrho$. But if U is a unitary operator on $\mathfrak{K} \supset \mathfrak{H}$ such that $T^n = \varrho P_{\mathfrak{H}} U^n$, then $(T^k)^n = \varrho P_{\mathfrak{H}}(U^k)^n$ and U^k is unitary. Q.e.d.

In the next theorem we derive a different sort of inequality concerning the behavior of the w_ϱ with respect to operator multiplication.

Theorem 4. 2. *If $\varrho, \sigma > 0$ and $T, S \in \mathscr{L}(\mathfrak{H})$, we have $w_{\varrho\sigma}(TS) \leqq w_\varrho(T) \cdot w_\sigma(S)$ provided T and S double commute (i.e., $TS=ST$ and $TS^*=S^*T$).*

Proof. Again it is clear, using Theorem 3.1 ((3) and (4)), that we need only show that $TS \in C_{\varrho\sigma}$ whenever $T \in C_\varrho$ and $S \in C_\sigma$ and T, S double commute.

By Theorem 2.1 we have $\sum_{n=-\infty}^{\infty} r^{|n|} e^{in\theta} T_\varrho(n) \geqq O$ and $\sum_{n=-\infty}^{\infty} r^{|n|} e^{in\theta} S_\sigma(n) \geqq O$ in the sense described in that theorem. Now it is not hard to prove (see [4], Theorem 3.3) that if, in the appropriate sense, $\sum_{-\infty}^{\infty} r^{|n|} e^{in\theta} A_n \geqq O$ and $\sum_{-\infty}^{\infty} r^{|n|} e^{in\theta} B_n \geqq O$, then we also have $\sum_{-\infty}^{\infty} r^{|n|} e^{in\theta} A_n B_n \geqq O$, provided $A_n B_m = B_m A_n$ for all choices of n and m. Since T and S double commute we may apply this result to conclude that $\sum_{-\infty}^{\infty} r^{|n|} e^{in\theta} T_\varrho(n) S_\sigma(n) \geqq O$.

Finally, we note that, for every n, $T_\varrho(n) S_\sigma(n) = (TS)_{\varrho\sigma}(n)$ so that, using Theorem 2.1 once more, we indeed have $TS \in C_{\varrho\sigma}$. Q.e.d.

In connection with the essential fact of the last theorem — namely that $T \in C_\varrho$, $S \in C_\sigma$ and T, S double commute imply $TS \in C_{\varrho\sigma}$ — we wish to mention another proof of this result, sent to us recently by Professor Sz.-Nagy, see [5*]. In that proof the "unitary $\varrho\sigma$-dilation" for TS is given explicitly in the form UV where U and V are commuting unitary ϱ-and σ-dilations of T and S respectively, constructed simultaneously on a space $\mathfrak{K} \supset \mathfrak{H}$.

If $\varrho = 2$, $\sigma = 1$ in the theorem just proved we obtain the inequality $w(TS) \leqq w(T) \cdot \|S\|$ (if T, S double commute). This result occurs in [4], where a number of proofs of the inequality are discussed.

At this point it is important to determine the value of $w_\varrho(I)$ for each $\varrho > 0$.

Theorem 4. 3. *For* $\varrho \geqq 1$, $w_\varrho(I) = 1$; *for* $0 < \varrho < 1$, $w_\varrho(I) = \dfrac{2}{\varrho} - 1$.

Proof. We must determine for which values $u > 0$ we have $\frac{1}{u} I \in C_\varrho$. Using Theorem 2. 2 we see that it is necessary and sufficient that

$$(*) \qquad \operatorname{Re}\left(1 - \frac{z}{u}\right) \geqq \left(1 - \frac{\varrho}{2}\right)\left|1 - \frac{z}{u}\right|^2$$

whenever $|z| < 1$ and that $v\left(\frac{1}{u} I\right) \leqq 1$. The last condition implies that, in any case, $u \geqq 1$.

Rewriting (*) in the form $\left(1 - \frac{\varrho}{2}\right) \leqq \operatorname{Re}\left(1 - \frac{z}{u}\right)^{-1}$, we see that we must consider the values of $\operatorname{Re} w^{-1}$ where w lies inside the circle c_1 of radius $\frac{1}{u}$ centered at 1. Since $\frac{1}{u} \leqq 1$ it is clear that, inverting c_1 in the unit circle, we obtain a circle (or half-plane) c_2 having $\left(1 + \frac{1}{u}\right)^{-1}$ as its most westerly point. Thus, the additional condition imposed on u by (*) is $\left(1 - \frac{\varrho}{2}\right) \leqq \left(1 + \frac{1}{u}\right)^{-1}$; this holds automatically if $\left(1 - \frac{\varrho}{2}\right) \leqq 0$ and otherwise reduces to $u \geqq \frac{2}{\varrho} - 1$.

Thus $\frac{1}{u} I \in C_\varrho \Leftrightarrow u \geqq \max\left(1, \frac{2}{\varrho} - 1\right)$ so that, indeed, $w_\varrho(I) = \max\left(1, \frac{2}{\varrho} - 1\right)$. Q.e.d.

It should be pointed out that the theorem above is included in a result of DURSZT (see [3, Theorem 1]) which, upon introducing the functions $w_\varrho(\cdot)$, amounts to the evaluation of $w_\varrho(T)$ for any normal T. In § 5, on the other hand, we shall see that Theorem 4. 3 combined with some general inequalities yields the theorem of DURSZT in a somewhat extended form.

We can now prove some preliminary results concerning the behavior of $w_\varrho(T)$ for fixed T as ϱ varies.

Theorem 4. 4. *Suppose $T \in \mathcal{L}(\mathfrak{H})$ and $0 < \varrho < \varrho'$. Then $w_{\varrho'}(T) \leqq w_\varrho(T)$ and $w_\varrho(T) \leqq \left(\frac{2\varrho'}{\varrho} - 1\right) w_{\varrho'}(T)$. Thus $w_\varrho(T)$ is continuous and non-increasing as ϱ increases.*

Proof. Simply combine Theorems 4. 2 and 4. 3 as follows:

$$w_{\varrho'}(T) = w_{\left(\frac{\varrho'}{\varrho}\right)\varrho}(IT) \leqq w_{\frac{\varrho'}{\varrho}}(I) \cdot w_\varrho(T) = 1 \cdot w_\varrho(T);$$

$$w_\varrho(T) = w_{\left(\frac{\varrho}{\varrho'}\right)\varrho'}(IT) \leqq w_{\frac{\varrho}{\varrho'}}(I) \cdot w_{\varrho'}(T) = \left(\frac{2\varrho'}{\varrho} - 1\right) \cdot w_{\varrho'}(T). \quad \text{Q. e. d.}$$

In view of Theorem 3.1, the fact that $w_\varrho(T)$ is non-increasing as ϱ increases implies that $C_{\varrho'} \supset C_\varrho$ whenever $\varrho' > \varrho$. In [6] (§ 3) SZ.-NAGY and FOIAŞ discuss the problem of determining when these inclusions are strict. In essence, they consider the operator A defined by the matrix $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ (relative to an orthonormal basis) on a 2-dimensional subspace of $\mathfrak{H}$ (and vanishing on the orthogonal complement) and show that $\varrho A \in C_{\varrho+1} \setminus C_{\varrho-\varepsilon}$ whenever $\varepsilon > 0$ and $\varrho \geqq 1$, and that $\frac{\varrho}{2-\varrho} A \in C_\varrho \setminus C_{\left(\frac{\varrho}{2-\varrho} - \varepsilon\right)}$ whenever $\varepsilon > 0$ and $\varrho < 1$. Actually, as DURSZT was the first to point out (see [3, Theorem 2]), we can show that $\varrho A \in C_\varrho \setminus C_{\varrho-\varepsilon}$ for every $\varrho > 0$ and $\varepsilon > 0$, so that the classes C_ϱ form a *strictly* increasing scale (as ϱ increases). By Theorem 3.1, it is sufficient to show that $w_\varrho(\varrho A) = 1$ and $w_{\varrho-\varepsilon}(\varrho A) > 1$; but it is clear that $w_\varrho(A) = \frac{1}{\varrho}$, for every $\varrho > 0$, by means of the following observation, which we shall have occasion to use several times again.

Theorem 4. 5. *Suppose $T \in \mathcal{L}(\mathfrak{H})$, $\|T\| = 1$, and $T^2 = O$. Then, for every $\varrho > 0$, $w_\varrho(T) = \frac{1}{\varrho}$.*

Proof. As $w_1(T) = \|T\| = 1$ we have $T \in C_1$, i.e., for some unitary operator U on $\mathfrak{K} \supset \mathfrak{H}$ we have $T^n = P_{\mathfrak{H}} U^n$ $(n = 1, 2, 3, \ldots)$. Since $T^2 = O$, $(\varrho T)^n = \varrho T^n$ $(n = 1, 2,$

3, ...), so that we have $(\varrho T)^n = \varrho P_{\mathfrak{H}} U^n$ $(n=1, 2, 3, \ldots)$, i.e., $\varrho T \in C_\varrho$. Thus $w_\varrho(\varrho T) \leqq 1$ and $w_\varrho(T) \leqq \frac{1}{\varrho}$. But, by Theorem 3.1 (2), $w_\varrho(T) \geqq \frac{1}{\varrho}\|T\| = \frac{1}{\varrho}$. Q.e.d.

As we have noted above, we have as an immediate consequence the following fact.

Corollary 4.6 (Durszt). *Provided $\mathfrak{H}$ is at least 2-dimensional, we have $C_{\varrho'} \supset C_\varrho$ strictly whenever $\varrho' > \varrho$ (>0).*

5. In this section we discuss the relationship between the spectral radius $v(T)$ and the operator radii $w_\varrho(T)$.

Since $w_\varrho(T)$ decreases with increasing ϱ and is always non-negative, we may define, for each $T \in \mathscr{L}(\mathfrak{H})$, $w_\infty(T) = \lim_{\varrho \to \infty} w_\varrho(T)$.

Theorem 5.1. *For every $T \in \mathscr{L}(\mathfrak{H})$, $w_\infty(T) = v(T)$.*

Proof. We have $\frac{T}{w_\varrho(T)} \in C_\varrho$ so that, by Theorem 2. 2, $v\left(\frac{T}{w_\varrho(T)}\right) \leqq 1$; thus $v(T) \leqq w_\varrho(T)$ for every ϱ.

On the other hand, suppose that $v(T) < 1$. For some $s > 1$ we also have $v(sT) < 1$ and since, by the spectral radius formula, $\|(sT)^n\|^{\frac{1}{n}} \to v(sT)$, we see that for some $B < \infty$ we have $s^n \|T^n\| \leqq B$ $(n=1, 2, 3, \ldots)$. Thus, if $|z| < 1$, $\left\|\sum_{n=1}^{\infty} (zT)^n\right\| \leqq \sum_{n=1}^{\infty} \|T^n\| \leqq$ $\leqq \sum_{n=1}^{\infty} \frac{B}{s^n} = M(<\infty)$. It follows that if $0 \leqq r < 1$ we have, setting $z = re^{i\theta}$,

$$\sum_{n=-\infty}^{\infty} r^{|n|} e^{in\theta} T_\varrho(n) = I + \frac{2}{\varrho} \operatorname{Re}\left(\sum_{n=1}^{\infty} (zT)^n\right) \geqq \left(1 - \frac{2}{\varrho}\left\|\sum_{n=1}^{\infty} (zT)^n\right\|\right) I \geqq \left(1 - \frac{2}{\varrho} M\right) I \geqq o$$

as soon as $\varrho \geqq 2M$.

Using Theorem 2.1, it is clear that, whenever $v(T) < 1$, there is some ϱ such that $T \in C_\varrho$, i.e., $w_\varrho(T) \leqq 1$. Now if $v(T) \neq 0$, and $\varepsilon > 0$ we have $v\left(\frac{T}{(1+\varepsilon)v(T)}\right) =$ $= \frac{1}{1+\varepsilon} < 1$ so that, for some ϱ, $w_\varrho\left(\frac{T}{(1+\varepsilon)v(T)}\right) \leqq 1$, i.e., $(1+\varepsilon)v(T) \geqq w_\varrho(T) (\geqq v(T))$. Clearly, then, $w_\infty(T) = v(T)$ in this case. If $v(T) = 0$, then for any n $v(nT) = 0 < 1$ so that for some ϱ $w_\varrho(nT) \leqq 1$, i.e., $w_\varrho(T) \leqq \frac{1}{n}$. Thus $w_\infty(T) = 0$ $(= v(T))$. Q.e.d.

An operator T in any one of the operator classes C_ϱ is "power bounded", i.e., the sequence $\{\|T^n\|\}_1^\infty$ is bounded; in fact, $\|T^n\| = \|\varrho P_{\mathfrak{H}} U^n\| \leqq \varrho$. Sz.-Nagy and Foiaş show, however, by constructing an example (see [6], §4), that there are power-bounded operators not lying in any of the classes C_ϱ. Nevertheless, we have the following result.

Theorem 5. 2. *The family of power-bounded operators* $\bigcup_{\varrho>0} C_\varrho$ *is dense (with respect to the ordinary operator norm) in the class of all power-bounded operators.*

Proof. If T is power-bounded the $v(T)=\lim \|T^n\|^{\frac{1}{n}} \leqq 1$. Thus, for any r such that $0 \leqq r < 1$, we have $v(rT) < 1$ and hence, by Theorem 5.1, there is some ϱ such that $w_\varrho(rT) \leqq 1$, i.e., $rT \in C_\varrho$, hence the assertion follows.

If $T \in \mathscr{L}(\mathfrak{H})$ and $w(T) = \|T\|$, then we actually have $v(T) = w(T) = \|T\|$, i.e. $w_1(T) = w_2(T) \Rightarrow v(T) = w_1(T)$. We may even replace 1 and 2 in the above statement by *any* distinct values of ϱ. Indeed, we have the following:

Theorem 5. 3. *If* $T \in \mathscr{L}(\mathfrak{H})$ *is such that* $w_{\varrho_0}(T) > v(T)$, *then* $w_\varrho(T)$ *is strictly decreasing at* ϱ_0, *i.e.,* $\varrho > \varrho_0 \Rightarrow w_\varrho(T) < w_{\varrho_0}(T)$.

Proof. We may assume that $w_{\varrho_0}(T) = 1$ and $v(T) < 1$, and prove that, if $\varrho > \varrho_0$, $w_\varrho(T) < 1$. By Theorems 3.1 and 2. 2 we have $T \in C_{\varrho_0}$ and hence, for each $h \in \mathfrak{H}$ and complex z such that $|z| < 1$,

$$(*) \qquad \operatorname{Re}((I - zT)h, h) \geqq \left(1 - \frac{\varrho_0}{2}\right) \|(I - zT)h\|^2.$$

Now $\alpha = \inf\left(\|(I - zT)h\|^2 \colon |z| < 1,\ h \in \mathfrak{H},\ \|h\| = 1\right) > 0$, since we would otherwise have $h_n \in \mathfrak{H}$ and complex z_n such that $\|h_n\| = 1$, $|z_n| < 1$, and $\|(I - z_n T)h_n\| \to 0$; by passing to a subsequence we could assume that $z_n \to z_0$, and it is easy to see that $\|(I - z_0 T)h_n\| \to 0$ in this case: thus we would have $1/z_0$ in the spectrum of T, contradicting the assumption that $v(T) < 1$.

If we choose $b > 1$ such that, whenever $|z| < 1$ and $\|h\| = 1$, we have

$$|\operatorname{Re}((I - zbT)h, h) - \operatorname{Re}((I - zT)h, h)| < \frac{\varrho - \varrho_0}{2} \cdot \frac{\alpha}{2}$$

and

$$\left|\left(1 - \frac{\varrho}{2}\right) \|(I - zbT)h\|^2 - \left(1 - \frac{\varrho}{2}\right) \|(I - zT)h\|^2\right| < \frac{\varrho - \varrho_0}{2} \cdot \frac{\alpha}{2},$$

it is easy to see that $(*)$ implies

$$\operatorname{Re}((I - zbT)h, h) \geqq \left(1 - \frac{\varrho}{2}\right) \|(I - zbT)h\|^2$$

for all such z and h. But this inequality is independent of the value of $\|h\|$, so that, by Theorem 2. 2, we have $bT \in C_\varrho$ provided we have chosen $b(>1)$ small enough so that, in addition, $v(bT) \leqq 1$. In this case $w_\varrho(bT) \leqq 1$, i.e., $w_\varrho(T) \leqq \frac{1}{b} < 1$. Q.e.d.

The following theorem finds its natural place in this section.

Theorem 5. 4. *For any* $T \in \mathscr{L}(\mathfrak{H})$ *and* $\varrho > 0$ *we have* $w_\varrho(T) \geqq w_\varrho(I)\, v(T)$.

This result follows upon recalling Theorem 2. 2 and the fact that T has an approximate eigenvalue λ such that $|\lambda|=v(T)$.

By Theorem 4. 2 we have $w_\varrho(T)\leqq w_\varrho(I)w_1(T)$ and this combined with the last theorem and our evaluation of $w_\varrho(I)$ (i.e., Theorem 4. 3) yields the following extension of a theorem of DURSZT (see [3, Theorem 1]). The extension is implicit in DURSZT's work, and has also been pointed out by BERGER and STAMPFLI (see [2, Theorem 6]).

Theorem 5. 5. *For any $T\in\mathcal{L}(\mathfrak{H})$ such that $v(T)=\|T\|$ (such T have been called "normaloid" operators, and include, of course, the normal operators) we have*

$$w_\varrho(T)=\|T\|\,w_\varrho(I)=\begin{cases}\|T\|\left(\dfrac{2}{\varrho}-1\right), & \text{if } 0<\varrho<1,\\ \|T\|, & \text{if } \varrho\geqq 1.\end{cases}$$

6. Upon considering the "power inequality" of Theorem 4.1 one naturally asks to what extent the operator radii $w_\varrho(\cdot)$ are multiplicative, i.e., under what conditions do we have an inequality of the following type: $w_\varrho(TS)\leqq w_\varrho(T)\cdot w_\varrho(S)$. Although it does not seem possible, except in very special cases, to derive the power inequality from a more general inequality involving a pair of operators we shall describe here some results along these lines.

Let us first observe that in the case where T and S may be quite unrelated, and in the case where they are assumed to double commute, the problem may be settled in a fairly satisfactory way.

Theorem 6.1. *For any $T, S\in\mathcal{L}(\mathfrak{H})$ and $\varrho\geqq 1$ we have $w_\varrho(TS)\leqq\varrho^2 w_\varrho(T)\cdot w_\varrho(S)$; this result is best possible, provided $\mathfrak{H}$ is at least 2-dimensional.*

Proof. Using Theorems 4. 4 and 3.1 (2) we have at once $w_\varrho(TS)\leqq w_1(TS)\leqq$ $\leqq w_1(T)w_1(S)\leqq(\varrho w_\varrho(T))(\varrho w_\varrho(S))$.

On the other hand, if $\dim(\mathfrak{H})\geqq 2$ we may define operators A and B on some 2-dimensional subspace by the matrices (relative to an orthonormal basis) $\begin{bmatrix}0&1\\0&0\end{bmatrix}$ and $\begin{bmatrix}0&0\\1&0\end{bmatrix}$ respectively, and require that A and B vanish on the orthogonal complement. By Theorem 4. 5, $w_\varrho(A)=w_\varrho(B)=\dfrac{1}{\varrho}$. Now AB corresponds to the matrix $\begin{bmatrix}1&0\\0&0\end{bmatrix}$ so that $w_1(AB)=v(AB)=1$, and hence $w_\varrho(AB)=1$ whenever $\varrho\geqq 1$. This example shows that the inequality of the theorem cannot be improved. Q.e.d.

Theorem 6. 2. *If $T, S\in\mathcal{L}(\mathfrak{H})$ and T and S double commute, then $w_\varrho(TS)\leqq$ $\leqq\varrho w_\varrho(T)w_\varrho(S)$ for all $\varrho>0$. This result is best possible, at least if* $\dim(\mathfrak{H})\geqq 4$.

Proof. Using Theorems 4. 2 and 3.1 (2) we have $w_\varrho(TS) \leqq w_1(T)w_\varrho(S) \leqq \leqq (\varrho w_\varrho(T))w_\varrho(S)$.

On the other hand, if $\dim(\mathfrak{H}) \geqq 4$ we may define operators C and D on some 4-dimensional subspace by the matrices (relative to an orthonormal basis) $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ respectively, and require that C and D vanish on the orthogonal complement. It is easy to verify that C and D double commute and that CD corresponds to the matrix $\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Applying Theorem 4. 5, we see that $w_\varrho(C) = w_\varrho(D) = w_\varrho(CD) = \frac{1}{\varrho}$, for every $\varrho > 0$. It follows that our inequality cannot be improved. Q.e.d.

When we simply assume that T and S commute, the situation is much less clear. Since $w_1(\cdot) = \|\cdot\|$ and $w_\infty(\cdot) = v(\cdot)$, we have $w_1(TS) \leqq w_1(T) \cdot w_1(S)$ and, provided T and S commute, $w_\infty(TS) \leqq w_\infty(T) \cdot w_\infty(S)$. The case of $w_2(\cdot)\,(= w(\cdot))$ is settled by the following theorem, which also shows that the constant in Theorem 6.1 can be improved if we assume T and S commute, at least when $\sqrt{2} < \varrho \leqq 2$.

Theorem 6. 3. *If $T, S \in \mathscr{L}(\mathfrak{H})$, T and S commute, and $w_\varrho(\cdot)$ is a norm (and hence, by Theorem 3. 2, whenever $\varrho \leqq 2$), then $w_\varrho(TS) \leqq 2 w_\varrho(T) w_\varrho(S)$. This result is best possible for $\varrho = 2$, at least if* $\dim(\mathfrak{H}) \geqq 4$.

Proof. We may assume that $w_\varrho(T) = w_\varrho(S) = 1$ and prove that $w_\varrho(TS) \leqq 2$. In the following calculation we use both the assumption that $w_\varrho(\cdot)$ is a norm and the "power inequality" of Theorem 4.1:

$$w_\varrho(TS) = w_\varrho\left(\tfrac{1}{4}[(T+S)^2 - (T-S)^2]\right) \leqq$$

$$\leqq \tfrac{1}{4}\left[w_\varrho((T+S)^2) + w_\varrho((T-S)^2)\right] \leqq \tfrac{1}{4}\left[(w_\varrho(T+S))^2 + (w_\varrho(T-S))^2\right] \leqq$$

$$\leqq \tfrac{1}{4}\left[(w_\varrho(T) + w_\varrho(S))^2 + (w_\varrho(T) + w_\varrho(S))^2\right] = 2.$$

To see that the inequality $w_2(TS) \leqq 2 w_2(T) \cdot w_2(S)$ cannot be improved (if $\dim(\mathfrak{H}) \geqq 4$), recall that, by Theorem 6. 2, the inequality is best possible even under the assumption that T and S *double* commute. Q.e.d.

Corollary 6. 4. *For $\varrho > 2$, $w_\varrho(\cdot)$ fails to be a norm on $\mathscr{L}(\mathfrak{H})$.*

Proof. Compare Theorems 6. 3 and 6. 2. Q.e.d.

The following theorem shows that Theorem 6. 3 can be much improved if one of the operators is normal.

Theorem 6. 5. *Suppose T and S are commuting operators in $\mathcal{L}(\mathfrak{H})$ and that T is normal. Then, for all $\varrho>0$, $w_\varrho(TS)\leqq w_\varrho(T)w_\varrho(S)$.*

Proof. Since S commutes with the normal operator T, FUGLEDE's theorem (see ROSENBLUM [7] for a slick proof) tells us that S and T double commute. Hence, by Theorem 4. 2, $w_\varrho(TS)\leqq w_1(T)w_\varrho(S)$. But, as T is normal, $v(T)=\|T\|$ $(=w_1(T))$, so that for all $\varrho>0$ $w_1(T)\leqq w_\varrho(T)$. Thus $w_\varrho(TS)\leqq w_\varrho(T)w_\varrho(S)$. Q.e.d.

While it does not seem clear whether or not the inequalities of Theorems 6. 2 and 4. 2 can be extended to the case where the operators merely commute, it is usually possible to say something more in this case than in the case where the operators are quite arbitrary. Our final theorem is a rather curious example of a result of this nature. Note that for arbitrary $T, S\in\mathcal{L}(\mathfrak{H})$ we have, for $\varrho\geqq 1$, $w_\varrho(TS)\leqq\|TS\|\leqq\leqq\|T\|\cdot\|S\|\leqq\varrho w_\varrho(T)\cdot\|S\|$ (we have used Theorems 4. 4 and 3.1(2)); furthermore we can actually have equality under these conditions (consider the operators A and B introduced in the proof of Theorem 6. 1). Of course, if T and S *double* commute, Theorem 4. 2 tells that $w_\varrho(TS)\leqq w_\varrho(T)\cdot\|S\|$. Whether or not we can say the same if T and S merely commute, we *do* have the following improvement over the case where T and S may be completely unrelated.

Theorem 6. 6. *Suppose $\varrho>1$, and T and S are commuting operators in $\mathcal{L}(\mathfrak{H})$. Then provided $T\neq O$ and $S\neq O$, $w_\varrho(TS)<\varrho w_\varrho(T)\cdot\|S\|$.*

Proof. Since, as we have noted above, we have $w_\varrho(TS)\leqq\|TS\|\leqq\|T\|\cdot\|S\|\leqq\leqq\varrho w_\varrho(T)\cdot\|S\|$, the theorem could fail only if we had $w_\varrho(TS)=\|TS\|$. In this case, by Theorem 5. 3, $w_\varrho(TS)=v(TS)$; but this is impossible because, since T and S commute, we would have $w_\varrho(TS)=v(TS)\leqq v(T)\cdot v(S)\leqq w_\varrho(T)\cdot\|S\|$, as well as $w_\varrho(TS)=\varrho w_\varrho(T)\cdot\|S\|$. Q.e.d.

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