# Nilpotent groups and automorphisms*)

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I. First an arbitrary endomorphism $A \times V \to A_1 \times V_1$ of semi-direct products $A \times V$, $A_1 \times V_1$ of arbitrary groups $A$, $V$, $A_1$, $V_1$ is described by four functions $f_1 : A \to A_1, f_2 : V \to A_1$, $c_1 : A \to V_1$, and $c_2 : V \to V_1$. Under additional hypotheses, automorphisms of $A \times V$ leaving the subgroup $1 \times V \lhd A \times V$ invariant are studied.

II. If $K$ is any field, set $V = K^n$. Let $A$ be the group of all upper triangular matrices $\alpha = \|a_{ij}\|$ $(0 \leqq i, j \leqq n;\ a_{ij} \in K;\ a_{ij} = 0$ for $i > j;\ a_{ii} \neq 0)$. Form the semi-direct product $A \times V$:

$$(\beta, w)(\alpha, v) = (\beta\alpha, w\alpha + v) \qquad (\alpha, \beta \in A;\ v, w \in K^n);$$

$$w\alpha = (w_1, \ldots, w_n)\|a_{ij}\|, \qquad (\alpha = \|a_{ij}\| \in A;\ w = (w_1, \ldots, w_n) \in K^n).$$

Secondly, the general methods of I are used to compute the automorphism group Aut $A \times V$. Modulo all the inner automorphisms, there is exactly one non-inner automorphism $\sigma: A \times V \to A \times V$ with $\sigma(1 \times V) \neq 1 \times V$; $\sigma$ is found explicitly.

III. The quotients of the descending central series of the commutator subgroup $N = [A \times V, A \times V]$ are $K$-vector spaces. Lastly, all normal subgroups $W \lhd N$ whose image in each quotient of the descending central series is a one dimensional vector space are determined.

The automorphism group Aut $A \times V$ of the holomorph $A \times V$ of a group $V$ has received considerable attention (see [6], [7], [11] and [12]). In all of the above papers, those automorphisms of $A \times V$ which leave the normal subgroup $1 \times V \lhd$ $\lhd A \times V$ invariant, play a significant role. Suppose $V$ is any abelian $o$-group which is divisible by 2 and $A^+$ the group of all order preserving transformations of $V$. It has been shown (see [HARVEY; Theorem 2. 1, p. 24]), that if $A^+$ can be ordered in any manner whatever so that it becomes an $o$-group, then:

(i) $1 \times V$ is $o$-characteristic in the $o$-holomorph $A^+ \times V$;

(ii) every $o$-automorphism of $A^+ \times V$ is inner.

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Let $A$ and $V$ be as in II of the introduction with $K$ an ordered field. Let $A^+ \triangleleft A$ be all $\|a_{ij}\| \in A$ with $a_{ii} > 0$ for $i = 1, \dots, n$. If $V = K^n$ is ordered lexicographically, then $A^+$ is a group of order preserving transformations of $V$. Now take $K$ to be the rationals. Then $A^+$ is precisely all order preserving transformations of $V$. Since clearly $A^+$ can be naturally ordered so that it becomes an $o$-group, any automorphism $\sigma$ of $A^+ \times V$ which does not leave $1 \times V$ invariant satisfies:

(i) $\sigma$ does not preserve the order of $V$;

(ii) $\sigma$ is not inner.

If $N \triangleleft A^+ \times V$ is the commutator subgroup of $A^+ \times V$, then the quotients of the descending central series of $N$ are vector spaces. Since the image of $1 \times V$ in each quotient is a one dimensional vector space, if $\sigma$ is any automorphism of $A^+ \times V$, $\sigma(1 \times V)$ should have the same property. These considerations were the motivation for classifying all normal subgroups $W \triangleleft N$ with one dimensional images in each quotient. If in the above example $K$ is taken as the reals, then the group of all order preserving transformations of $V$ consists of matrices having zeroes below the diagonal, strictly positive entries on the diagonal, and rational-linear (in general discontinuous) linear maps of $K$ into $K$ as the entries above the diagonal. Due to our inability to handle such groups, this note deals with groups of the above general kind, where the entries of the matrices are in an arbitrary field $K$ (sometimes assumed to be not of characteristic 2).

## 1. Automorphisms of semi-direct products

The main objective of this first section is to determine all those automorphisms $F$ of a semi-direct product $A \times V$ of two groups $A$ and $V$ having the property that $F[1 \times V] = 1 \times V$. Most of the propositions are established in greater generality than later needed. In fact, for the most part it is not even necessary to assume that $V$ is abelian — let alone a vector space, or even a finite dimensional one. However, it has to be assumed that $A$ is a group of automorphisms of $V$ and that the inner automorphisms by elements of $V$ belong to $A$.

**1. 1.** Notation. If $A$ and $V$ are any groups, then an *action* of $A$ on $V$ is a map $A \times V \to V$, $(v, \alpha) \to v\alpha$, with the properties

$$(v + w, \alpha) = v\alpha + w\alpha \quad \text{and} \quad (v, \alpha\beta) = (v\alpha)\beta, \quad (\alpha, \beta \in A;\ v, w \in V).$$

Although written additively, $V$ is not assumed to be abelian. With respect to any fixed action of $A$ on $V$, the semi-direct product $A \times V$ will be written as follows:

$$(\alpha, v)(\beta, w) = (\alpha\beta, v\beta + w) \qquad (\alpha, \beta \in A;\ v, w \in V).$$

The identity elements of $V$ and $A$ will be denoted by 0 and 1. Inner automorphisms and commutators in any group will always be written as $\beta^{\alpha}=\alpha^{-1}\beta\alpha$ and $[\alpha, \beta]==\alpha^{-1}\beta^{-1}\alpha\beta$. If $A_1$, $V_1$ are two other such groups, then the functions $f_i$, $c_i$ $(i=1, 2)$ are defined by

$$F[(\alpha, 0)]=(f_1(\alpha), c_1(\alpha)),\ F[(1, v)]=(f_2(v), c_2(v)) \qquad (\alpha\in A, v\in V).$$

If $f\colon B\to A$ is any homomorphism of any group $B$ into $A$, then a map $\tau\colon B\to V$ is a *crossed homomorphism with respect to f provided* $(\alpha\beta)\tau=(\alpha\tau)f(\beta)+\beta\tau$ holds for all $\alpha, \beta\in B$. It is *inner* if there is a $y\in V$ for which $\alpha\tau = -yf(\alpha)+y$ for all $\alpha\in B$.

For any arbitrary group $V$, Aut $V$ will denote the group of all automorphisms of $V$. The centralizer and normalizer of a subgroup $A$ in Aut $V$ will be denoted by $C(A<\text{Aut } V)$ and $N(A<\text{Aut } V)=\{T\in \text{Aut } V | T^{-1}AT=TAT^{-1}=A\}$. Every element $v\in V$ gives rise to an inner automorphism $\bar{v}\in \text{Aut } V$.

Remark. If $A$ acts on an abelian group $V$, the crossed homomorphisms form a group $Z^1(A, V)$ under pointwise addition. The inner automorphisms form a subgroup $B^1(A, V)$. The factor group $Z^1(A, V)/B^1(A, V)$ is the first cohomology group of $A$ with respect to the given action of $A$ on $V$.

In the next proposition $c_1$ and $c_2$ are crossed homomorphisms with respect to $f_1$ and $f_2$. Note that equation (iv) implies that $A$ leaves the kernel of $f_2$ invariant.

Proposition 1.2. *Let* $F\colon A\times V\to A_1\times V_1$ *and* $f_i, c_i, i=1, 2,$ *be as above arbitrary semi-direct products. Then the following hold for all* $\alpha, \beta\in A$ *and* $v, w\in V$:

(i) $F[(\alpha, v)]=(f_1(\alpha)f_2(v), c_1(\alpha)f_2(v)+c_2(v))$;

(ii) $f_1$ *and* $f_2$ *are homomorphisms*;

(iii) $c_1(\alpha\beta)=c_1(\alpha)f_1(\beta)+c_1(\beta)$; $\quad c_2(v+w)=c_2(v)f_2(w)+c_2(w)$;

(iv) $f_2(w\beta)=f_1(\beta)^{-1}f_2(w)f_1(\beta)$;

(v) $c_2(w\beta) = -c_1(\beta)f_2(w\beta)+c_2(w)f_1(\beta)+c_1(\beta)$.

*Conversely, if* $f_1, f_2, c_1$, *and* $c_2$ *are any functions satisfying* (ii)—(v), *then* $F$ *defined by equation* (i) *is an endomorphism.*

Proof. As an illustration, (iv) and (v) will be proved. The proofs of (i)—(iii) are similar and even simpler. Computing $F[(\beta, 0)]F[(1, w\beta)]$ and $F[(1, w)]F[(\beta, 0)]$ by (i) and then equating the first and second components, we obtain

(iv) $f_2(w\beta)=f_2(w)^{f_1(\beta)}$,

(v) $c_1(\beta)f_2(w\beta)+c_2(w\beta) = c_2(w)f_1(\beta)+c_1(\beta)$.

Since it is not clear that (ii)—(v) are all the relations that interrelate the functions $f_i$, $c_i$, the proof of the converse will be indicated. Equation (i) shows that

$$F[(\alpha\beta, v\beta+w)]=(f_1(\alpha\beta)f_2(v\beta+w), c_1(\alpha\beta)f_2(v\beta+w)+c_2(v\beta+w)),$$

$$F[(\alpha, v)]\ F[(\beta, w)] = (f_1(\alpha)f_2(v)f_1(\beta)f_2(w), c_1(\alpha)f_2(v)f_1(\beta)f_2(w)+ + c_2(v)f_1(\beta)f_2(w)+c_1(\beta)f_2(w)+c_2(w)).$$

By (iv) and (ii) the first components of the above two equations are equal. Use of (iv) and (iii) gives that

$$c_1(\alpha)f_2(v)f_1(\beta)f_2(w) = c_1(\alpha\beta)f_2(v\beta+w) - c_1(\beta)f_2(v\beta+w).$$

Thus it only remains to show that

$$c_2(v\beta+w) = -c_1(\beta)f_2(v\beta+w) + c_2(v)f_1(\beta)f_2(w) + c_1(\beta)f_2(w) + c_2(w).$$

But this follows from (iii) and (v), since

$$c_2(v\beta+w) = [-c_1(\beta)f_2(v\beta) + c_2(v)f_1(\beta) + c_1(\beta)]f_2(w) + c_2(w).$$

From now on, three simplifications will be assumed throughout. First $V=V_1$, $A=A_1$; secondly $A$ will be taken in $A \subseteqq \operatorname{Aut} V$; and thirdly, only automorphisms $F$ of $A \times V$ will be considered.

The proof given in [Harvey; p. 7] of the next corollary for the case when $V$ is abelian generalizes to non-abelian $V$.

Corollary 1.3. *If $\tau: A \to V$ is a crossed homomorphism with $\tau(A) \subseteqq$ center $V$, then*

$$F: A \times V \to A \times V, \quad F[(\alpha, v)] = (\alpha, \alpha\tau + v) \qquad ((\alpha, v) \in A \times V)$$

*is an automorphism of $A \times V$ leaving $1 \times V$ elementwise fixed. Conversely, every automorphism of $A \times V$ leaving $1 \times V$ elementwise fixed is necessarily of the above form. Furthermore, suppose $\tau(\alpha) = -y\alpha + y$, for all $\alpha \in A$ and some $y \in$ center $V$. Then $F[(\alpha, v)] = (\alpha, \alpha\tau + v) = (0, y)^{-1}(\alpha, v)(0, y)$ for $(\alpha, v) \in A \times V$.*

Note that the converse of (i) of the next corollary is also true, i.e., $f_2 \equiv 1$ if and only if $F[1 \times V] \subseteqq 1 \times V$.

Corollary 1.4. *In Proposition 1.2 assume $V = V_1$, $A = A_1$, and that $F$ is an automorphism of $A \times V$ onto itself. Suppose $F[1 \times V] \subseteqq 1 \times V$. Then*

(i) $f_2 \equiv 1$,
(ii) $f_1(A) = A$,
(iii) $c_2$ *is an injective homomorphism.*

Corollary 1.5. *Let $T: V \to V$ be defined by $vT = c_2(v)$ for all $v \in V$. Assume that:*
(a) $F: A \times V \to A \times V$ *is an automorphism,*
(b) $F[1 \times V] = 1 \times V$,
(c) $\{\bar{v} | v \in V\} \subseteqq A$.
*Then:*
(i) $f_2 \equiv 1$;
(ii) $f_1$ *is an isomorphism of $A$ onto $A$; $F^{-1}[(\alpha, 0)] = (f_1^{-1}(\alpha), 0)$ for all $\alpha \in A$;*

(iii) $F^{-1}[(1, v)] = (1, vT^{-1})$ *for all* $v \in V$;

(iv) $f_1(\beta) = T^{-1}\beta T\overline{c_1(\beta)}^{-1}$;

(v) $T^{-1}AT = TAT^{-1} = A$; $T \in N(A < \text{Aut } V)$.

Proof. Conclusions (i)—(iv) are immediate consequences of Proposition 1. 2. It follows from (iii), (iv), and (*c*) that $T^{-1}AT \subseteqq A$ and $TAT^{-1} \subseteqq A$ and hence $T^{-1}AT = TAT^{-1} = A$.

The next lemma shows how to construct automorphisms of $A \times V$ which leave $1 \times V$ invariant.

Lemma 1. 6. *Consider any group V (not assumed to be abelian) and any subgroup* $A \subseteqq \text{Aut } V$.

(i) *For any* $S \in N(A < \text{Aut } V)$ *and any* $y \in V$, *the map* $F: A \times V \to A \times V$ *defined by*

$$F[(\beta, w)] = (\beta^s, -y\beta^s + wS + y) \qquad (\beta \in A, w \in V)$$

*is an automorphism.*

(ii) *For S and y as in* (i),

$$(S, y) \in \text{Aut } V \times V \quad \text{and} \quad F[(\beta, w)] = (S, y)^{-1}(\beta, w)(S, y).$$

*Thus the automorphism in* (i) *is inner if and only if* $S \in A$.

(iii) *In addition assume that* $\{\bar{v} | v \in V\} \subseteqq A$ *and let* $T \in \text{Aut } V$. *Then T extends to an automorphism* $F: A \times V \to A \times V$ *if and only if* $T \in N(A < \text{Aut } V)$.

Proof. (i) and (ii). Conclusion (ii) proves (i). (iii) If $T$ is obtained from an automorphism $F$ of $A \times V$ by $F[(1, v)] = (1, vT)$ for $v \in V$, then by Corollary 1. 5 (v) $T \in N(A < \text{Aut } V)$. Conversely, if $T \in N(A < \text{Aut } V)$, then the map $F[(\beta, w)] = (\beta^T, wT)$ for $(\beta, w) \in A \times V$ is an automorphism by (i) of this lemma.

From now on the group $V$ will be abelian, later a vector space, and, finally, a finite dimensional one. The following lemma is well known (see [Harvey; p. 11)]; its proof is omitted.

Lemma 1. 7. *Let V be any abelian group,* $A \subseteqq \text{Aut } V$ *any subgroup, and* $f_1: A \to A$ *a homomorphism of A into A. Assume that:*

(a) *The map* $\bar{2}: V \to V$, $v \to 2v$, *is an isomorphism of V onto V*;

(b) $\bar{2} \in A$;

(c) $f_1(\bar{2}) = \bar{2}$.

*Then every crossed homomorphism* $\tau$ *with respect to the action* $f_1$, *is inner. In fact,* $\alpha\tau = -yf_1(\alpha) + y$, *where* $y = -\bar{2}\tau$.

Definition 1. 8. Consider a vector space $V$ over a field $K$ and a field automorphism $\mu\colon K\to K$. Any $K$-basis $\{v(\lambda)|\lambda\in\Lambda\}$, where $\Lambda$ is an indexing set, defines a map $\tilde{\mu}\colon V\to V$ by:

$$\text{if}\quad v=\Sigma\{k(\lambda)v(\lambda)|k(\lambda)\in K,\ \lambda\in\Lambda\},\quad \text{define}\quad v\tilde{\mu}=\Sigma k(\lambda)\mu v(\lambda).$$

Suppose $A\subseteqq \operatorname{Aut} V$ is any subgroup having the property that

$$A=\{\tilde{\mu}^{-1}\alpha\tilde{\mu}|\alpha\in A\}=\{\tilde{\mu}\alpha\tilde{\mu}^{-1}|\alpha\in A\}.$$

Then an automorphism $\bar{\mu}\colon A\times V\to A\times V$ may be defined by

$$(\beta,w)\bar{\mu}=(\tilde{\mu}^{-1}\beta\tilde{\mu},w\tilde{\mu})\qquad(\beta\in A,\ w\in V).$$

The subgroup of Aut $V$ consisting of all $K$-linear automorphisms of $V$ will be denoted by $\operatorname{Aut}_K V$.

Remarks 1. If in the above definition $\alpha\in A$ is $K$-linear, then so is $\tilde{\mu}^{-1}\alpha\tilde{\mu}$. The matrix with respect to the basis $\{v(\lambda)|\lambda\in\Lambda\}$ of $\tilde{\mu}^{-1}\alpha\tilde{\mu}$ is obtained by applying $\mu$ to each entry of the matrix of $\alpha$.

2. The automorphism $\tilde{\mu}$ depends upon the choice of basis; whether $\tilde{\mu}\in N(A<\operatorname{Aut} V)$ may also depend upon the choice of the basis.

Lemma 1. 9. *Consider a vector space $V$ over a field $K$ and a subgroup $A\subseteqq\operatorname{Aut} V$. Let $E\colon V\to V$ be the identity map. Assume that*

(a) *$F\colon A\times V\to A\times V$ is an automorphism;*
(b) $F[1\times V]=1\times V$;
(c) $C(A<\operatorname{Aut} V)=\{kE|k\in K\setminus\{0\}\}$.

*Then:*

(i) *There is a (bijective) field automorphism $\mu\colon K\to K$ such that*

$$(cv)T=(c\mu^{-1})(vT)\qquad(v\in V,\ c\in K).$$

(ii) *In addition assume that for some choice of basis in $V$, $\tilde{\mu}\in N(A<\operatorname{Aut} V)$. Then the automorphism $F\circ\bar{\mu}\colon A\times V\to A\times V$ satisfies:*

$$F\circ\bar{\mu}[(1,v)]=(1,v\tilde{\mu}T),\quad (cv)(\tilde{\mu}T)=c(v\tilde{\mu}T)\qquad(v\in V,\ c\in K).$$

Proof (i) It follows from equation (v) of Proposition 1. 2, that for any $\beta\in A$, $w\in V$, and $c\in K$, we have

$$w\beta(cE)T=wTf_1(\beta)f_1(cE),\quad w(cE)\beta T=wTf_1(cE)f_1(\beta).$$

Since by Corollary 1. 5 (ii) the map $T$ is surjective, it follows that $f_1(\beta)f_1(cE)=$ $=f_1(cE)f_1(\beta)$ and $f_1(cE)\in C(A<\operatorname{Aut} V)$. Thus there is a map $\nu\colon K\to K$ such that

$f_1(cE)=(c\nu)E$. It is easily seen that $\nu$ is an injective homomorphism. There is a similar map $\mu: K\to K$ associated with the automorphism $F^{-1}$ which satisfies $\nu\circ\mu=\mu\circ\nu=$identity. Thus $\nu$ and $\mu$ are epimorphic and $\nu=\mu^{-1}$. Conclusion (ii) follows immediately.

## 2. Automorphisms of linear groups

In this section the general facts about automorphisms of semi-direct products developed in section 1 are used to find the automorphism groups of a certain class of groups. The next definition gives this class of groups as well as various subgroups which will be of major interest throughout the rest of the discussion.

Definition 2. 1. Let $K$ be an arbitrary field and $G$ the group of all $(n+1)\times\times(n+1)$ upper triangular matrices $P$ with entries from $K$ of the form

$$P=\|a_{ij}\| \ (0\leqq i,j\leqq n); \quad a_{ij}=0 \quad \text{if} \quad i>j;\ a_{ii}\neq 0$$

for all $i$.

Two normal subgroups $N\subset G^1\subset G$ of $G$ are defined by:

$$N=\{P\in G|a_{ii}=1,\ i=0,\ \dots,\ n\}; \quad G^1=\{P\in G|a_{00}=1\}.$$

The normal subgroups $\Gamma$ and $\Gamma^1$ of $G$ are defined by:

$\Gamma=\{P\in N|a_{ij}=0$ for $i<j$, except in the first row and the last column$\}$,

$\Gamma^1=\{P\in N|a_{ij}=0$ for $i<j$, except in the first two rows and the last column$\}$.

Let $\alpha\in K$ be any scalar. Two normal subgroups $B^1$ and $B^1(\alpha)$ of $N$ are defined as follows:

$$B^1=\{P\in N|a_{ij}=0 \quad \text{for } i<j \quad \text{unless} \quad i=0\},$$

$$B^1(\alpha)=\{P\in N|a_{ij}=0 \ \text{for } i<j \ \text{unless } i=0, \text{ or } (i,j)=(1,n) \text{ and } a_{1n}=\alpha a_{01}\}.$$

The groups $B^1$ and $N$ are normal in $G$. Note that $B^1(0)=B^1$. For $\alpha\neq 0$, $B^1(\alpha)$ is normal in $N$ but not in $G^1$. By taking transposes of all elements of $B^1, B^1(\alpha), \Gamma^1$ around the second diagonal, we obtain three other groups $B_1$, $B_1(\alpha)$, $\Gamma_1$. The subgroup of $G$ consisting of all diagonal matrices is denoted by $D$. The element of $D$ whose diagonal entries are $\lambda_0, \dots, \lambda_n$ will be denoted by diag $(\lambda_0, \dots, \lambda_n)$. Set $D^1=G^1\cap D$.

If $N_0=N=[G, G]$ and $N_j=[N_{j-1}, N], j=1, \dots, n-1$; then $N$ is a group of nilpotency class $n$ having the same descending and ascending central series

$$[G, G]=N=N_0\supset N_1\supset \dots \supset N_{n-1}\supset N_n=1.$$

The group $N_j$ consists precisely of all matrices having $j$ strings of zeroes parallel to the main diagonal. The entries of a matrix in the $(j+1)$-st string parallel to the main diagonal will be referred to as the $(j+1)$-st *layer*. The center $N_{n-1}$ of $N$ will be denoted by $Z$ for simplicity. The usual matrix unit with all zeroes except for a one in the $p$-th row and $q$-th column will be denoted by $E_{pq}$, $0\leqq p, q\leqq n$. For

$j=0, \ldots, n$, there is an isomorphism $\alpha_j$ of $K$-vector spaces

$$\alpha_j : K^{n-j} \to N_j/N_{j+1}, \qquad \alpha_j[(x_1,\ldots,x_{n-j})] = I + \sum_{i=1}^{n-j} x_i E_{i-1,\,j+i}.$$

For the reader's convenience we include schematic diagrams showing the forms of the elements in the various groups where the $a_{ij} \in K$ are arbitrary

$$B^1 : \begin{vmatrix} 1 & a_{01} & a_{02} & & a_{0,n-1} & a_{0n} \\ & & & & & \\ 0 & 0 & 0 & & 1 & 0 \\ 0 & 0 & 0 & & 0 & 1 \end{vmatrix} \qquad B^1(\alpha) : \begin{vmatrix} 1 & a_{01} & a_{02} & & a_{0,n-1} & a_{0n} \\ 0 & 1 & 0 & & 0 & \alpha a_{01} \\ 0 & 0 & 0 & & 1 & 0 \\ 0 & 0 & 0 & & 0 & 1 \end{vmatrix}$$

$$\Gamma : \begin{vmatrix} 1 & a_{01} & a_{02} & & a_{0,n} \\ 0 & 1 & 0 & & a_{1,n} \\ & & & & \\ 0 & 0 & 0 & 1 & a_{n-1,n} \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix} \qquad \Gamma^1 : \begin{vmatrix} 1 & a_{01} & a_{02} & a_{0,n-1} & a_{0n} \\ 0 & 1 & a_{12} & a_{1,n-1} & a_{1n} \\ 0 & 0 & 1 & 0 & a_{2n} \\ 0 & 0 & 0 & 1 & a_{n-1,n} \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix}$$

$$\alpha\,[(x_1,\ldots,x_{n-j})] = \begin{vmatrix} 1 & \overbrace{0 \;\; 0 \quad\quad 0}^{j} & x_1 & 0 & 0 & 0 \\ 0 & 1 \;\; 0 \quad\quad 0 & 0 & x_2 & 0 & 0 \\ & & & & & \\ 0 & 0 \;\; 0 & & & 0 & x_{n-j} \\ 0 & 0 \;\; 0 & & & 0 & 1 \end{vmatrix} N_{j+1} \in N_j/N_{j+1} (j=0,\ldots,n-1).$$

*Figure 1.*

Let $n$ be any integer; set $V=K^n$. Let $A$ be the group of all $n \times n$ matrices $\alpha$ with zeroes below the diagonal, arbitrary elements above the diagonal and non-zero elements on the diagonal. Elements of $V$ are viewed as row vectors and in $A \times V$, $A$ acts on these by right multiplication. The group $A \times V$ can be identified as a subgroup of the general linear group $Gl(n+1, K)$ as follows:

$$\alpha = \begin{vmatrix} a_{11} & a_{12} & a_{1n} \\ 0 & a_{22} & a_{2n} \\ & & \\ 0 & 0 & a_{nn} \end{vmatrix} \qquad v = (a_{01}, a_{02}, \ldots, a_{0n}) \qquad (\alpha \in A,\ v \in V);$$

$$(\alpha, v) = \begin{vmatrix} 1 & v \\ 0 & \alpha \end{vmatrix} = \begin{vmatrix} a_{00} & a_{01} & a_{0n} \\ 0 & a_{11} & a_{1n} \\ & & \\ 0 & 0 & a_{nn} \end{vmatrix} \in Gl(n+1, K);$$

$$(\alpha, v)(\alpha', v') = (\alpha\alpha', v\alpha' + v'), \qquad \begin{vmatrix} 1 & v \\ 0 & \alpha \end{vmatrix} \begin{vmatrix} 1 & v' \\ 0 & \alpha' \end{vmatrix} = \begin{vmatrix} 1 & v\alpha' + v' \\ 0 & \alpha\alpha' \end{vmatrix} \qquad (\alpha' \in A,\ v' \in V).$$

Sometimes $G^1 = A \times V$ will be viewed as a matrix subgroup of $Gl(n+1, K)$ and denoted by $G^1$, whereas at other times, when in our considerations its semi-direct product structure plays an important role, it will be written as a semi-direct product $A \times V$. In case $K$ is an ordered field, the normal subgroup of $G$ consisting of all matrices with strictly positive entries on the diagonal will be denoted by $G^+$; set $G^{+1} = G^1 \cap G^+$, $D^1 = D \cap G^1$, and $D^{+1} = D^1 \cap G^+$. Similarly $A^+$ will consist of all $\alpha$ having strictly positive diagonal entries. Thus just as $A \times V$ can be identified with $G^1$, so $A^+ \times V$ can be identified with $G^{+1}$.

Next some automorphisms of $G^1$ and $G^{+1}$ are defined. If $\mu: K \to K$ is any field automorphism, then $\tilde{\mu}: V \to V$ will always be defined with respect to the natural basis by

$$v\tilde{\mu} = (v_1\mu, \ldots, v_n\mu) \qquad (v_1, \ldots, v_n) \in K^n.$$

Let $\tilde{V}$ denote the subgroup of Aut $A \times V$ consisting of all automorphisms $F: A \times V \to A \times V$ such that the restriction $F|1 \times V \in \mathrm{Aut}_K V$, where $\mathrm{Aut}_K V$ was the group of all $K$-linear isomorphisms of $V$.

For the remainder of this definition suppose now that $K$ is an ordered field. The subgroup of all $\tilde{\mu}$ obtained from order preserving automorphisms will be denoted by $U$. The element $F_i = I - 2E_{ii} \in G$ $(i = 0, 1, \ldots, n)$ defines an automorphism $\bar{F}_i: G^{+1} \to G^{+1}$ by

$$\bar{F}_i(g) = F_i^{-1} g F_i \qquad (g \in G^{+1}; \quad i = 0, 1, \ldots, n).$$

Note that $F_i \in G^1$ for $i = 1, \ldots, n$, that $F_0 \notin G^1$. However, $F_0 = -F_1 \ldots F_n$ and hence $\bar{F}_0 = \bar{F}_1 \ldots \bar{F}_n$. The $\bar{F}_1, \ldots, \bar{F}_n$ generate a subgroup $\mathscr{F}$ of Aut $A^+ \times V$. The group of inner automorphisms of Aut $A^+ \times V$ will be denoted by $J$.

The objective is to find all automorphisms of $G^{+1}$.

Proposition 2.2. *Consider the group $G^{+1} = A^+ \times V$ of Definition* 2.1 *and any automorphism $F: A^+ \times V \to A^+ \times V$ such that $F[1 \times V] = 1 \times V$. Let $c_2: V \to V$ be defined by $F[(1, v)] = (1, c_2(v))$ for $v \in V$. Then:*

(i) *There is an order preserving field automorphism $\mu: K \to K$ such that*

$$F[(1, cv)] = (1, (c\mu^{-1})c_2(v)) \qquad (c \in K, v \in V).$$

(ii) *If $T: V \to V$ is defined by $F \circ \tilde{\mu}[(1, v)] = (1, vT)$ $(v \in V)$, then* $T \in N(A^+ < \mathrm{Aut}\, V) = A$.

(iii) *There is a $y \in V$ such that for any $(\alpha, v) \in A^+ \times V$,*

$$F \circ \tilde{\mu}[(\alpha, v)] = (\alpha^T, -y\alpha^T + vT + y) = \begin{vmatrix} 1 & y \\ 0 & T \end{vmatrix}^{-1} \begin{vmatrix} 1 & v \\ 0 & \alpha \end{vmatrix} \begin{vmatrix} 1 & y \\ 0 & T \end{vmatrix}.$$

(iv) $\tilde{V} = \mathscr{F} J$.

Proof. (i) First, it is easy to see that $C(A^+ < \mathrm{Aut}\, V)$ are the scalar operators. The automorphisms $\mu$ and $\mu^{-1}$ given by Lemma 1. 9 (i) clearly preserve the order of $K$. (ii) It is well known and easy to prove that $N(A^+ < \mathrm{Aut}_K V) = A$. In order to show that $T \in \mathrm{Aut}_K V$, take $c \in K$ and $(v_1, \ldots, v_n) \in V$. Then

$$\big(1, (cv)T\big) = F[(1, (cv)\tilde{\mu}] = F[1, (c\mu)(v\tilde{\mu})] = \big(1, cc_2(v\tilde{\mu})\big) = \big(1, c(vT)\big).$$

Thus $T \in \mathrm{Aut}_K V$ and it follows from Corollary 1. 5 (v) that $T \in N(A^+ < \mathrm{Aut}_K V)$.

(iii) By Proposition 1. 2 there are functions $f_1, f_2 \equiv 1, c_1, c_2 \equiv T$ corresponding to the automorphism $F \circ \bar{\mu}$ such that

$$F \circ \bar{\mu}[(\alpha, v)] = \big(f_1(\alpha), c_1(\alpha) + vT\big) \qquad (\alpha, v) \in A \times V.$$

By Corollary 1. 5 (iv), $f_1(\alpha) = T^{-1}\alpha T$ for $\alpha \in A$. By Lemma 1. 7, $c_1$ is of the form $c_1(\alpha) = -y\alpha^T + y$ for some $y \in V$. Then the above equation becomes

$$F \circ \bar{\mu}[(\alpha, v)] = (\alpha^T, -y\alpha^T + v^T + y).$$

(iv) The automorphism $F$ can be realized as an inner automorphism by the elements $(1, y) \in A^+ \times V$ and $(T, 0) \in A \times V$ as follows:

$$F \circ \bar{\mu}[(\alpha, v)] = (T, y)^{-1}(\alpha, v)(T, y) = (1, y)^{-1}(T, 0)^{-1}(\alpha, v)(T, 0)(1, y)$$
$$\big((\alpha, v) \in A \times V\big).$$

However, $(T, 0)$ equals a product of some of $F_1, \ldots, F_n$ times an element of $A^+ \times V$. Thus $\tilde{V} = \mathscr{F}J$.

Remark. The last Proposition 2. 2 remains valid verbatim if $A^+$ is replaced by $A$ throughout.

The second step in determining the automorphism group of $G^{+1}$ is to show that any automorphism $F\colon G^1 \to G^1$ maps either $F(B^1) = B^1$ or $F(B^1) = B_1$. Since the previous Proposition 2. 2 completely determines all automorphisms $F\colon G^1 \to G^1$ satisfying $F(B^1) = B^1$, the final step will be the construction of an automorphism $\sigma\colon G^1 \to G^1$ satisfying $\sigma(B^1) = B_1$. The next definition and sequence of lemmas is needed in order to accomplish the second step.

Definition 2. 3. For $0 \leqq p < \varkappa \leqq n$, let $A_{p\varkappa}$ be the set of all $\|a_{\alpha\beta}\| \in N$ with $a_{\alpha\beta} = 0$ for $\alpha \geqq p+1$ and for $\beta \leqq \varkappa - 1$.

Remarks. 1. The group $A_{p\varkappa}$ is abelian and $A_{p\varkappa} \lhd G^1$.

2. It can be shown that $A_{p,\, p+1}$ $(0 \leqq p \leqq n-1)$ is a maximal normal abelian subgroup of $N$. It is our conjecture that there are no others.

3. Note that $A_{01} = B^1$, $A_{n-1,n} = B_1$; also $A_{0j} \subseteqq B^1$ and $A_{in} \subseteqq B_1$ for all $j$ and $i$.

4. If $j = \varkappa - p - 1$, then $N_{j-1} \subset A_{p\varkappa} \subseteqq N_j$; $A_{p\varkappa} = N_j$ if and only if $A_{0n} = N_{n-1} = Z$.

Suppose $\|a_{\alpha\beta}\| \in N_j$ with $a_{p\varkappa} \neq 0$ where $\varkappa - p - 1 = j$, $0 \leqq j \leqq n-1$. Suppose $i$ and $j$ satisfy $i \leqq p$ and $\varkappa \leqq j$. The next sequence of lemmas will show that by inner automorphisms from $G^1$, $\|a_{\alpha\beta}\|$ can be transformed into $I + cE_{ij}$, where $c \in K$. In the diagram below, $\|a_{\alpha\beta}\|$ has non-trivial entries in the triangular region in the upper right hand corner. The group $A_{p\varkappa}$ consists of all elements having non-trivial entries in the rectangle region inside the triangular region. The $j$-th layer is represented by the line connecting $(0, \varkappa - p)$ and $(n - \varkappa + p, n)$ entries.

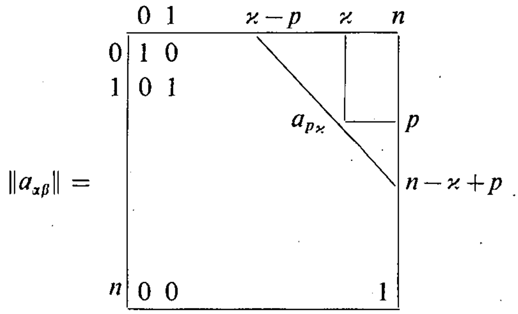

*Figure 2.*

The next lemma describes the types of elements that can be constructed by application of inner automorphisms.

Lemma 2.4. *Let* $\|\beta_{ij}\|$, $\beta_{ij} \in K$, $0 \leqq i, j \leqq n$, *be any matrix.*

(i) *If* $\|\beta'_{ij}\|$ *is defined by*

$$\|\beta'_{ij}\| = (I - cE_{pq})\|\beta_{ij}\|(I + cE_{pq}) \qquad (0 \leqq p,\ q \leqq n),$$

*then* $\|\beta'_{ij}\|$ *is obtained from* $\|\beta_{ij}\|$ *by subtracting c times the q-th row from the p-th row and adding c-times the p-th column to the q-th column.*.

(ii) *If* $S = \operatorname{diag}(\lambda_0, \ldots, \lambda_n)$, *where* $\lambda_0, \ldots, \lambda_n \in K \setminus \{0\}$, *and if* $S^{-1}\|\beta_{ij}\|S = \|\beta'_{ij}\|$, *then* $\beta'_{ij} = \lambda_i^{-1}\beta_{ij}\lambda_j$ $(0 \leqq i, j \leqq n)$.

(iii) *If* $S = \operatorname{diag}(1, \lambda, \lambda^2, \ldots, \lambda^n)$, *and if* $S^{-1}\|\beta_{ij}\|S = \|\beta'_{ij}\|$, *then* $\beta'_{\varkappa, j+\varkappa} = \beta_{\varkappa, j+\varkappa}\lambda^j$ $(\varkappa = 0, \ldots, n-j)$; *i.e.* $S^{-1}\|\beta_{ij}\|S$ *is obtained from* $\|\beta_{ij}\|$ *by multiplying the j-th layer of* $\|\beta_{ij}\|$ *by* $\lambda^j$.

Remark. The inner automorphism by the diagonal element $S$ in (iii) of the above Lemma 2.4 induces scalar multiplication by $\lambda^{j+1}$ on $N_j/N_{j+1} = K^{n-j}$. If the ground field $K$ does not contain all the $j$-th roots of its elements for $2 \leqq j \leqq n$, then t may be impossible to obtain all scalar multiplications on $N_j/N_{j+1}$ from inner automorphisms. The following fact will not be later used. If $d = \operatorname{diag}(1, \lambda, \lambda^2, \ldots,$ $\ldots, \lambda^{j-1}, 1, \lambda, \lambda^2, \ldots, \lambda^{n-j})$ and if $(\beta_{0, j+1}, \ldots, \beta_{n-j-1, n})$ is the $(j+1)$-st layer of $\|\beta_{ij}\|$, then $(\lambda\beta_{0,j+1},\dots,\lambda\beta_{j-1,2j},\lambda^{j+1}\beta_{j,2j+1},\dots,\lambda^{j+1}\beta_{n-j-1,n})$ is the $(j+1)$-st layer of $d^{-1}\|\beta_{ij}\|d$.

Lemma 2. 5. *Let* $P=\|a_{ij}\|$ $(0\leqq i,j\leqq n)$ *be any matrix of the form* $P=\begin{vmatrix} B & C\\ 0 & D\end{vmatrix}$ *where* $B$ *is* $(j+1)\times(j+1)$, $C$ *is* $(j+1)\times(n-j)$, $D$ *is* $(n-j)\times(n-j)$, *and where* $B$ *and* $D$ *have inverses* $B^{-1}, D^{-1}$. *For* $\lambda\in K$ *set* $d(j,\lambda)=E_{00}+\dots+E_{jj}+ +\lambda(E_{j+1,j+1}+\dots+E_{nn})$. *Then:*

$$\text{(i)}\quad P^{-1}=\begin{vmatrix} B^{-1} & -B^{-1}CD^{-1}\\ 0 & D^{-1}\end{vmatrix},$$

$$\text{(ii)}\quad d(j,\lambda)^{-1}Pd(j,\lambda)=\begin{vmatrix} B & \lambda C\\ 0 & D\end{vmatrix},$$

$$\text{(iii)}\quad [P,d(j,\lambda)]=P^{-1}d(j,\lambda)^{-1}Pd(j,\lambda)=\begin{vmatrix} I & (\lambda-1)B^{-1}C\\ 0 & I\end{vmatrix}.$$

(iv) *If* $B$ *has ones on the diagonal, zeroes below and if the last* $t$ *rows of* $B$ *are those of the identity matrix, then the last* $t$ *rows of* $B^{-1}C$ *are those of* $C$.

Proof. Conclusions (i), (ii), and (iii) are immediate, while (iv) is a consequence of (i) with $D$ an $t\times t$ matrix.

Lemma 2. 6. *For any subgroup* $W\lhd G^1$, *if for some* $j$, $N_j\cap W$ *contains an element* $\|a_{\alpha\beta}\|$ *with* $a_{p\varkappa}\neq 0$ *for* $p$ *and* $\varkappa$ *satisfying* $\varkappa-p-1=j$, *then there is an element* $\|a''_{ij}\|\in \in A_{p\varkappa}\cap W$ *with* $a''_{p\varkappa}\neq 0$.

Proof. Applying the previous Lemma 2. 5 with $j=\varkappa-1$, we get a matrix $P=\|a'_{\alpha\beta}\|$ as follows

$$P_1=[P,d(\varkappa-1,\lambda)]=\begin{vmatrix} I & (\lambda-1)B^{-1}C\\ 0 & I\end{vmatrix},\qquad a'_{p\varkappa}=(\lambda-1)\,a_{p\varkappa}.$$

Due to the fact that from the $p$-th row on (and including the $p$-th row) the entries of $B$ are those of the identity matrix, also the matrices $B^{-1}C$ and $C$ agree in the $p$-th and all subsequent rows. Lemma 2. 5 will be applied a second time to $P_1$ with $j=p$; in the decomposition

$$P_1=\begin{vmatrix} B_1 & C_1\\ 0 & D_1\end{vmatrix},\qquad C_1 \text{ is } (p+1)\times(n-p);$$

$B_1=I$ and columns $p+1$ to $\varkappa-1$ inclusive of $C_1$ are zero. For $\lambda\in K$ let $P_2=\|a''_{ij}\|\in A_{p\varkappa}$ be defined by

$$P_2=[P_1,d(p,\lambda_1)]=\begin{vmatrix} I & (\lambda_1-1)C_1\\ 0 & I\end{vmatrix},\qquad a''_{p\varkappa}=(\lambda_1-1)(\lambda-1)\,a_{p\varkappa}.$$

The $p$-th row of $P_2$ from the $\varkappa$-th column on inclusive is that of $P$ multiplied by $(\lambda_1-1)(\lambda-1)$. (In fact, the same is true for the rows $p, \ldots, \varkappa-1$.)

The proof of the next lemma follows from Lemma 2. 4. It should be observed that the next two lemmas require the use of inner automorphisms from $N$ but not from $G^1$.

Lemma 2. 7. *If $c \in K$ and $P=\|a_{\alpha\beta}\| \in A_{p\varkappa}$ are arbitrary then:*

(i) *The inner automorphism by $I+cE_{ip}$ $(0 \leqq i \leqq p-1)$ subtracts c times the p-th row from the i-th (with the exception that the $(i, p)$ entry remains unchanged).*

(ii) *The inner automorphism by $I+cE_{\varkappa j}$, $\varkappa+1 \leqq j \leqq n$, adds c-times the $\varkappa$-th column to the j-th (except for the $(\varkappa, j)$ entry which remains unchanged).*

(iii) *Consequently, if $a_{p\varkappa} \neq 0$, and if $a_1, a_2 \in K$ are any non-zero scalars, then $\|a_{\alpha\beta}\|$ can be transformed by inner automorphisms from N into elements $Q$, $Q_1$, and $Q_2$ of the form*

$$Q = I+a_{p\varkappa}E_{p\varkappa}+T, \quad T = \Sigma\{b_{ij}E_{ij} | i \leqq p-1;\ \varkappa+1 \leqq j\},$$

$$Q_1 = I+a_{p\varkappa}E_{p\varkappa}+a_1E_{p,\varkappa+1}+T, \quad Q_2 = I+a_{p\varkappa}E_{p\varkappa}+a_2E_{p-1,\varkappa}+T.$$

Lemma 2. 8. *If $\|a_{\alpha\beta}\| \in A_{p\varkappa}$ is an element with $a_{p\varkappa} \neq 0$, then the normal subgroup of N generated by $\|a_{\alpha\beta}\|$ is precisely $A_{p\varkappa}$.*

Proof. It suffices to show that the subgroup of $N$ generated by $\|a_{\alpha\beta}\|$ contains all elements of the form $I+cE_{ij}$, where $0 \neq c \in K$ and $i \leqq \varkappa, p \leqq j$. By application of inner automorphisms from $N$, $\|a_{\alpha\beta}\|$ can be transformed into elements $Q$, $Q_1$, and $Q_2$ as in the last Lemma 2. 7. But then

$$Q^{-1}Q_1 = I+a_1E_{p,\varkappa+1}, \ Q^{-1}Q_2 = I+a_2E_{p-1,\varkappa} \qquad (a_1, a_2 \in K;\ a_1 \neq 0,\ a_2 \neq 0).$$

It is now clear that by a finite number of applications of the above process, the element $I+cE_{ij}$ can be obtained.

The previous lemmas imply the next proposition. It is false if the hypothesis that $W \lhd G^1$ is weakened to $W \lhd N$. (See Figure 2.)

Proposition 2. 9. *For a subgroup $W \lhd G^1$, if for some $j=0, \ldots, n-1$, the group $N_j \cap W$ contains an element $\|a_{\alpha\beta}\|$ with $a_{p\varkappa} \neq 0$ for p and $\varkappa$ satisfying $\varkappa-p-1=j$, then $A_{p\varkappa} \subseteqq W$. In particular, if $\|a_{\alpha\beta}\| \in W$ and if for some i and j, $a_{ij} \neq 0$, then $I+cE_{ij} \in W$ for all $c \in K$.*

Remark. The previous Proposition 2. 9 has the following interesting consequence. Suppose $W \subseteqq N$, $W \lhd G^1$, and $\|a_{\alpha\beta}\| \in W$. If $\|b_{\alpha\beta}\|$ is obtained from $\|a_{\alpha\beta}\|$ by replacing all $a_{\alpha\beta} \neq 0$, $\alpha < \beta$, with arbitrary scalars $b_{\alpha\beta}$, then also $\|b_{\alpha\beta}\| \in W$.

Corollary 2. 10. *If $F\colon G^1 \to G^1$ is any automorphism, then either $F(B^1)=B^1$ or $F(B^1)=B_1$.*

Proof. If $F(B^1)\nsubseteq \Gamma$, let $j$ be the smallest integer such that $N_j \cap F(B^1)$ contains an element $\|a_{\alpha\beta}\|$ with $a_{p\varkappa} \neq 0$, $\varkappa - p - 1 = j$, and with $p \neq 0$, $\varkappa \neq n$. Then the second center of $N$ is $N_{n-2} \subseteq A_{p\varkappa} \subseteq F(B^1)$. Thus the nilpotency class of $G^1/F(B^1) \leqq \leqq n-2$, whereas class of $G^1/B^1$ is $n-1$. This is a contradiction, since $G^1/B^1 \cong \cong G^1/F(B^1)$. Thus $F(B^1) \subseteq \Gamma$. Since $F$ induces an automorphism on $N_0/N_1$, there is an $\|a_{\alpha\beta}\| \in F(B^1)$ with either $a_{01} \neq 0$ or $a_{n-1,n} \neq 0$. Thus either $B^1 \subseteq F(B^1)$ or $B_1 \subseteq F(B^1)$. If $B^1 \subseteq F(B^1)$, but $B^1 \neq F(B^1)$, then by the last Proposition 2.9, $F(B^1)$ would have to contain a group of the form $A_{pn}$ with $1 \leqq p$. However, then $F(B^1)$ would not be abelian. A similar argument applies if $B_1 \subseteq F(B^1)$ and $B_1 \neq F(B^1)$.

The last step in determining the group of automorphisms of $G^1$ is to construct an automorphism $\sigma\colon G^1 \to G^1$ such that $\sigma(B^1) = B_1$.

**2.11.** Homomorphisms of semi-direct products. The group $B_1$ is embedded in a semi-direct product $K \times B_1 \lhd G^1$ consisting of all $(\lambda, b)$ of the form:

$$(\lambda, b) = \begin{vmatrix} I & b \\ 0 & \lambda \end{vmatrix}, \quad b = \begin{vmatrix} b_0 \\ \vdots \\ b_{n-1} \end{vmatrix} \in K^n$$

$$\bigl(0 \neq \lambda \in K; \quad (\lambda, b)(\lambda', b') = (\lambda\lambda', b\lambda' + b'); \quad (\lambda', b') \in K \times B_1\bigr).$$

Similarly $B^1$ is embedded in another semi-direct product $K \times B^1 \lhd G^1$ consisting of all $[\lambda, a]$

$$[\lambda, a] = \operatorname{diag}(1, \lambda, \ldots, \lambda) \begin{vmatrix} 1 & a \\ 0 & I \end{vmatrix}$$

$$\bigl(a = (a_1, \ldots, a_n) \in K^n; \quad 0 \neq \lambda \in K; \quad [\lambda, a][\lambda', a'] = [\lambda\lambda, a' + \lambda' a], \qquad [\lambda', a'] \in K \times B^1\bigr).$$

The map $K \times B_1 \to K \times B^1$, $(\lambda, b) \to [\lambda, b]$ is an isomorphism.

The group $G$ is a direct product $G = K \oplus G^1$. Define a map $p\colon G \to G^1$ by:

$$g = \|a_{ij}\| \in G, \; p(g) = \|a_{ij} a_{00}^{-1}\| = a_{00}^{-1} I \|a_{ij}\|.$$

Note that both $G^1$ and $G$ are semi-direct products $G = D \times N$ and $G^1 = D^1 \times N$.

Definition 2.12. An anti-automorphism $\tau'\colon G \to G$ is defined by transposing around the second diagonal, i.e. by

$$g = \|a_{ij}\| \in G, \; \tau'(g) = \|b_{ij}\|; \; b_{ij} = a_{n-i,\, n-j}$$

$(0 \leqq i, j \leqq n)$. An automorphism $\tau\colon G \to G$ is defined by $\tau(g) = \tau'(g^{-1})$ for $g \in G$. Thirdly, by use of the map $p$ of 2.11, a map $\sigma\colon G \to G^1$ is defined by $\sigma(g) = p[\sigma(g)]$, $g \in G$.

The matrix with zeroes everywhere but ones on the second diagonal will be denoted by $P$. A superscript $t$ denotes the transpose of a matrix; $-t$ denotes the inverse of the transpose.

Remarks. 1. It is asserted that $\tau'(g)=Pg^tP$, $\tau(g)=Pg^{-t}P$, $\tau'(g^{-1})=(\tau'(g))^{-1}$ $(g\in G)$. For any matrix $g$, the matrix $Pg$ is obtained simply by rotating $g$ hundred eighty degrees about its horizontal axis of symmetry. Similarly, $gP$ is obtained by rotating $g$ hundred eighty degrees about its vertical axis of symmetry. Thus $PgP$ is obtained by transposing $g$ about both of its diagonals, the order being immaterial. Thus $\tau'(g)=Pg^tP$. The other two equations follow from the fact that $P^2=I$.

2. Since $G=D\times N$ and $G^1=D^1\times N$ are semi-direct products, for $g=dn$; $n\in N$, $d=\text{diag}(\lambda_0,\ldots,\lambda_n)$, $\lambda_i\in K$; $\sigma(g)=p[\tau(d)]\tau(n)$, where $p[\tau(d)]=\text{diag}(1, \lambda_n\lambda_{n-1}^{-1}\ldots, \lambda_n\lambda_0^{-1})$. Thus all elements of the form $\text{diag}(1,\lambda,\lambda^2,\ldots,\lambda^n)$ $(\lambda\in K)$ are left invariant by $\sigma$.

The main properties of the map $\sigma$ are given by the next proposition.

Proposition 2. 13. *Let the notation be as in* 2. 11 *and* 2. 12. *Then:*

(i) *For any* $g=\|a_{ij}\|\in G$, $\sigma(g)=(a_{nn}I)Pg^{-t}P$.

(ii) *The restriction* $\sigma|G^1\colon G^1\to G^1$ *is an automorphism.*

(iii) $\sigma(K\times B^1)=K\times B_1$.

(iv) *There does not exist an* $M\in Gl(n+1;K)$ *such that for all* $h\in G^1$,

$$\sigma(h)=M^{-1}hM \quad or \quad \sigma(h)=M^{-1}h^{-t}M.$$

Proof. Conclusions (i)—(iii) are easily verified using the formula for $\sigma$ given in Remark 2 above. For $g=\text{diag}(1,\lambda_1,\ldots,\lambda_n)n\in G^1$, with $n\in N$, determinant $g=(\lambda_1\ldots\lambda_{n-1})^{-1}\lambda_n^n$. Thus (iv) follows.

Finally, we are in a position to combine Propositions 2. 2, 2. 9, and 2. 13 to find all automorphisms of $G^{+1}$.

Theorem I. *Let $K$ be any ordered field and* $G^{+1}=A^+\times V$; $\bar F_i$, $i=0,\ldots,n$; $\mathscr{F}$, $\bar\mu$, $U$, *and $J$ as in Definition* 2. 1 *and as in* 2. 12. *Then:*

(i) $\sigma^2=1$; $\mathscr{F}$ *is abelian*; $\bar F_0=\bar F_1\ldots\bar F_n$; $\bar F_i^2=1$; $\sigma^{-1}\bar F_i\sigma=\bar F_{n-i}$ $(i=0,\ldots,n)$.

(ii) *The following subgroups of* Aut $A^+\times V$ *are semi-direct products:*

$$\{\sigma\}\times\mathscr{F},\ \{\sigma\}\times J,\ \mathscr{F}\times J,\ \{\sigma\}\times\mathscr{F}J;$$

$$\text{Aut } A^+\times V=U\oplus\{\sigma\}\times[\mathscr{F}\times J].$$

*In particular,*

$$\frac{\text{Aut } A^+\times V}{J}\cong U\oplus\{\sigma\}\times\mathscr{F}.$$

Proof. Conclusion (i) is clear. (i) Since $F_iF_j=F_jF_i$, $0\leqq i,j\leqq n$, and since $F_0=-F_1\ldots F_n$, it follows that $\mathscr{F}$ is abelian and that $\bar F_0=\bar F_1\ldots\bar F_n$. The geometric characterization of $PF_i$ and $F_iP$ shows that $PF_i=F_{n-i}P$, $i=0,\ldots,n$. For $g=\|a_{ij}\|\in G^{+1}$,

$$\bar F_i[\sigma(g)]=F_i(a_{nn}I)(Pg^{-t}P)F_i=(a_{nn}I)P(F_{n-i}gF_{n-i})^{-t}P=\sigma[\bar F_{n-i}(g)].$$

Thus $\sigma^{-1}\bar F_i\sigma=\bar F_{n-i}$, $i=0,\ldots,n-i$. Thus (i) follows; it immediately implies (ii).

The situation becomes somewhat simpler if the group $A^+$ of the last theorem is replaced by $A$ as is done in the next corollary. Its proof is an immediate consequence of the remark following Proposition 2. 2.

Corollary 2. 14. *For any field $K$, let $G^1 = A \times V$, $\bar{\mu}$, and $\sigma$ be as in* 2. 1 *and* 2. 12. *Let $\bar{U} \subseteqq$ Aut $A \times V$ be the subgroup generated by all $\bar{\mu}$; $\bar{J}$ denotes the group of inner automorphisms of $A \times V$. Then:*

(i) $\sigma^2 = 1$,

(ii) $\{\sigma\} \times \bar{J}$ *is a semi-direct product*; Aut $A \times V = \bar{u} \oplus \{\sigma\} \times \bar{J}$.

## 3. Subgroups whose images in the quotients of a central series are one dimensional

In the notation of the previous section

$$N = N_0 \supset N_1 \supset \ldots \supset N_{n-1} \supset N_n = 1$$

is both the descending and ascending central series of $N$; $N_j/N_{j-1} \cong K^{n-j}$, $j = 0, \ldots, n-1$. The objective of this section is to characterize all normal subgroups $W \subseteqq N$ such that each $(W \cap N_j)N_{j+1}/N_{j+1}$ is a one dimensional $K$-vector space.

Definition 3. 1. A subgroup $W \subseteqq N$ is called *K-linear*, if it has the property that for any $k \in K$ and $P \in W$, if $Pk$ is the matrix obtained by multiplying every non-diagonal entry of $P$ by $k$, then $Pk \in W$. For any subgroup $W \subseteqq N$, (which is not assumed to be $K$-linear) the *rank* of $(W \cap N_j)N_{j+1}/N_{j+1}$, $j = 0, \ldots, n-1$, is defined as the number of linearly independent elements over $K$ that it contains.

Example. Let $n = 4$ and let $a, b, c, d, e \in K$ be arbitrary constants. The most general abelian subgroup $M$ of $N$ consists of all $(x, y, z)$ where for any $x, y, z \in K$,

$$(x, y, z) = \begin{vmatrix} 1 & ax & ay+dx & z \\ 0 & 1 & cx & by+ex \\ 0 & 0 & 1 & bx \\ 0 & 0 & 0 & 1 \end{vmatrix}.$$

Since two elements $(x, y, z)$ and $(x', y', z') \in M$ multiply according to the rule

$$(x, y, z)(x', y', z') = (x+x', y+y'+cxx', z''),$$

(where $z'' = z + z' + ab(xy' + x'y) + (ae + db)xx'$), it follows that for $0 \neq k \in K$, the map $M \to M$, $P \to Pk$ is not a homomorphism. However, this map is in general for any $n$ a homomorphism for all the normal abelian subgroups of $N$ considered here.

Lemma 3. 2. (i) *If $W \subseteqq N$ is a subgroup such that $(W \cap N_j)N_{j+1}/N_{j+1}$ is a K-vector space for all $j = 0, \ldots, n-1$, then $W$ is K-linear.*

(ii) *Let $n$ be the nilpotency class of $G^1$ and suppose that $K$ contains the $j$-th roots of all of its elements for $2 \leqq j \leqq n$. Then if $W$ is a normal subgroup of $G^1$ and $W \subset N$, then $W$ is $K$-linear.*

Proof. Conclusion (i) is easily proved; (ii) follows from (i) and Lemma 2. 4 (iii). The subgroup $W$ in the next lemma need not be $K$-linear.

Lemma 3. 3. *Suppose $W \subseteqq N$ is any subgroup such that:*

(a) *$W$ is invariant under inner automorphisms from $\Gamma^1\Gamma_1$;*

(b) *$(W \cap N_j)N_{j+1}/N_{j+1}$ is of rank at most one for all $j=0, \ldots, n-1$.*

*Then $W \subseteqq \Gamma$.*

Proof. For $n=1$, $N=\Gamma$. Assuming the lemma to be true for $1, \ldots, n-1$, it will be proved for groups $W \subset N$ of $(n+1)\times(n+1)$ matrices. Replacing $N$ by $N/B_1$ and $W$ by $WB_1/B_1$ and using induction, we obtain that $WB_1/B_1 \subseteqq \Gamma/B_1$. Thus $W \subseteqq \Gamma_1$. Similarly, $W \subseteqq \Gamma^1$. Thus $W \subseteqq \Gamma_1 \cap \Gamma^1$. Suppose there is a $g \in W$ of the form

$$g = \begin{vmatrix} 1 & a & a_{n-1} & z \\ & & s & b_1 \\ & & & b \\ & & & 1 \end{vmatrix} \qquad a = (a_1, \ldots, a_{n-2}). \quad b = \begin{vmatrix} b_2 \\ \vdots \\ b_{n-1} \end{vmatrix} \qquad (a_j, b_j \in K; \quad 0 \neq s \in K).$$

For arbitrary $c, k \in K$ we have

$$(I-kE_{n-1,n})(I-cE_{01})g(I+cE_{01})(I+kE_{n-1,n}) = \begin{vmatrix} 1 & a & a_{n-1}-cs & z_0 \\ & & & b_1+ks \\ & & & b \\ & & & 1 \end{vmatrix}$$

(where $z_0 = z - cb_1 + ka_{n-1}$). Since $c$ and $k$ are arbitrary, there are elements $h, f \in W$ of the form

$$h = \begin{vmatrix} 1 & a & 0 & z_1 \\ & & s & b_1 \\ & & & b \\ & & & 1 \end{vmatrix} \qquad f = \begin{vmatrix} 1 & a & a_{n-1} & z_2 \\ & & & 0 \\ & & & b \\ & & & 1 \end{vmatrix}$$

$$(b_1, a_{n-1}, z_1, z_2 \in K; \quad b_1 \neq 0, a_{n-1} \neq 0).$$

Then

$$g^{-1} = \begin{vmatrix} 1 & -a & (a_1 s - a_{n-1}) & z_3 \\ & & -s & (s\, b_{n-1} - b_1) \\ & & & -b \\ & & & 1 \end{vmatrix}$$

(where $z_3 = -z-(a_1b_1+\ldots+a_{n-1}b_{n-1})-a_1sb_{n-1}$), and

$$g^{-1}h=\begin{vmatrix}1 & 0 & 0 & z_4\\ & & 0 & -b_1\\ & & & 0\\ & & & 1\end{vmatrix}\qquad g^{-1}f=\begin{vmatrix}1 & 0 & -a_{n-1} & z_5\\ & & & 0\\ & & & 0\\ & & & 1\end{vmatrix}$$

for some $z_4, z_5\in K$. Thus $s=0$ and $W\subseteqq\Gamma$.

Notation 3.4. Consider a subgroup $W$ with $Z\subseteqq W\subseteqq\Gamma$. Its elements will be written as $[a; z; b]$, where

$$[a;z;b]=\begin{vmatrix}1 & a & z\\ 0 & I & b\\ 0 & 0 & 1\end{vmatrix}\qquad \left(a=(a_1,\ldots,a_{n-1});\quad b=\begin{vmatrix}b_1\\ \vdots\\ b_{n-1}\end{vmatrix};\quad a_j, b_i\in K\right).$$

Then the elements of $W/Z$ and $B^1/Z$ are canonically of the form $[a; 0; b]$ and $[a; 0; 0]$. There is a homomorphism $\pi^1\colon W/Z\to B^1/Z$ defined by $\pi^1([a; 0; b])= =[a; 0; 0]$. Similar remarks apply to $\pi_1$ and $B_1/Z$. Note that

$$[a;z;b]^{-1}=[-a;\, a\cdot b-z;\, -b],\qquad a\cdot b=a_1b_1+\ldots+a_{n-1}b_{n-1},$$

$$[a;0;b]^{-1}=[-a;0;-b]\quad\text{in}\quad B^1/Z.$$

Note that hypothesis (b) of the next lemma implies that $Z\subseteqq W$.

Lemma 3.5. *Suppose the subgroup $W\subseteqq N$ satisfies:*

(a) *$W$ is invariant under inner automorphisms from $\Gamma^1\Gamma_1$;*

(b) *$(W\cap N_j)N_{j+1}/N_{j+1}$ is a one dimensional $K$-vector space, $j=0,\ldots,n-1$.*

(c) *There exists $[a; z: b]\in W$ with $a_1\neq 0$.*

*Then:*

(i) *$\pi^1$ is a bijective isomorphism.*

(ii) *There exist linear functionals $f_i\colon K^{n-1}\to K$, $i=1,\ldots,n-1$, such that every element of $W$ is of the form*

$$[t;z;f(t)]=\begin{vmatrix}1 & t & z\\ 0 & I & f(t)\\ 0 & 0 & 1\end{vmatrix}$$

$$\left(t=(t_1,\ldots,t_{n-1})\in K^{n-1};\quad f(t)=\begin{vmatrix}f_1(t)\\ f_{n-1}(t)\end{vmatrix}\in K^{n-1};\quad z\in K\right).$$

Proof. (i) In order to show that $\pi^1$ is surjective, it suffices to show that for any $q$ in $2\leqq q\leqq n-1$, there is an element $w\in W$ of the form

$$w=[(0,\ldots,0,c_q,0,\ldots,0);z;b]\qquad b\in K^{n-1}, z\in K$$

for some $0\neq c_q\in K$. For any $c\in K$, the element $(I+ca_1^{-1}E_{1q})^{-1}\,[a;z;b](I+c_qa_1E_{1q})$ has the same entries as $[a;z;b]$, except in positions $(0,q)$ and $(1,n)$, where $(0,q)$ entry $=a_q+c$ $(q=2,\dots,n-1)$ and $(1,n)$ entry $=b_1-ca_1^{-1}b_q$. Thus there are $u, u^{-1}, r, ru^{-1}\in W$ of the form

$$u=[(a_1,0,\dots,0);0;(b_1',b_2,\dots,b_{n-1})],$$

$$u^{-1}=[(-a_1,0,\dots,0);a_1b_1';(-b_1,-b_2,\dots,-b_{n-1})],$$

$$r=[(a_1,0,\dots\ 0,c_q,0,\dots,0);0;(b_1'',b_2,\dots,b_{n-1})],$$

$$ru^{-1}=[(0,\dots\ 0,c_q,0,\dots,0);z;(\bar{b}_1,0,\dots,0)],$$

where $0\neq c_q\in K$ and where, in fact, $b_1'$, $b_1''$, $\bar{b}_1$, and $z$ are

$$b_1'=b_1+a_1^{-1}\,(a_1b_1+\dots+a_nb_n),\qquad b_1''=b_1'-c_qa_1^{-1}b_q,$$

$$\bar{b}_1=-b_1''+b_1=-c_qa_1^{-1}b_q,\ z=-c_qb_q.$$

Since kernel $\pi^1=\{[0;0;b]|b\in K^{n-1}\}$, the hypothesis (b) with $\pi^1(W)=B^1/Z$ imply that $\pi^1$ is a bijective isomorphism.

(ii) Since an arbitrary element $[t;0;b]\in W/Z$ with $t, b\in K^{n-1}$ is uniquely determined by its first component $t$, the functions $f_1,\dots,f_{n-1}\colon K^{n-1}\to K$ are uniquely defined by setting $(f_1(t),\dots,f_{n-1}(t))=b$. Let $f\colon K^{n-1}\to K^{n-1}$ be the map $f(t)=(f_1(t),\dots,f_{n-1}(t))$. Since for any $t, t'\in K^{n-1}$, $[t;0;f(t)][t';0;f(t')]=[t+t';0;f(t)+f(t')]=[t+t';0;f(t+t')]$, we have $f(t+t')=f(t)+f(t')$. Since by assumption (b) and Lemma 3. 2 (i) the group $W$ is $K$-linear, it follows that for any $c\in K$ and any $[t;0;f(t)]\in W/Z$, $[ct;0;cf(t)]\in W/Z$. But $[ct;0;cf(t)]=[ct;0;f(ct)]$; thus $f(ct)=cf(t)$ and the $f_i$ are $K$-linear functionals $f_i\colon K^{n-1}\to K$.

Remark. The assumption (b) of the last Lemma 3. 5 in conjunction with Lemma 3. 3 implies that $W\subseteq\Gamma$. Assumption (b) of the last Lemma 3. 5 guarantees that there is an element $[a;z;b]\in W$ with either $a_1\neq 0$ or $b_1\neq 0$. Thus hypothesis (c) is no real restriction but merely a notational convenience.

Lemma 3. 6. *Assume that the subgroup* $W\subseteq N$ *satisfies* (b) *and* (c) *of the previous Lemma* 3. 5 *and that in addition* $W\lhd N$. *Then:*

(i) *There are* $\alpha, \beta\in K$ *such that every element of* $W$ *is of the form*

$[t;z;(\alpha t_1+\beta t_2,-\beta t_1,0,\dots,0)]$ $(t_1,\dots,t_{n-1})\in K^{n-1}$ $(z\in K)$;

(ii) *If characteristic* $K=2$, $W$ *is abelian.*

(iii) *If characteristic* $K\neq 2$, *and if in addition* $W$ *is abelian, then* $\beta=0$ *and* $W=B^1(\alpha)$.

Proof. (i) Let $[t;z;f(t)]\in W$ be an arbitrary element with $t\in K^{n-1}$ and $z\in K$ For $c\in K$ and any indices $i$ and $r$ satisfying $0\leqq i-r<i\leqq n-1$, let $t\in K^{n-1}$, $z'\in K$ be defined by

(1) $$[t';z';f(t')]=(I-cE_{i-r,\,i})[t;z;f(t)](I+cE_{i-r,i}).$$

The inner automorphism has only changed the $(0, i)$, $(i-r, n)$, and $(0, n)$ entries in the manner indicated in the following diagram:

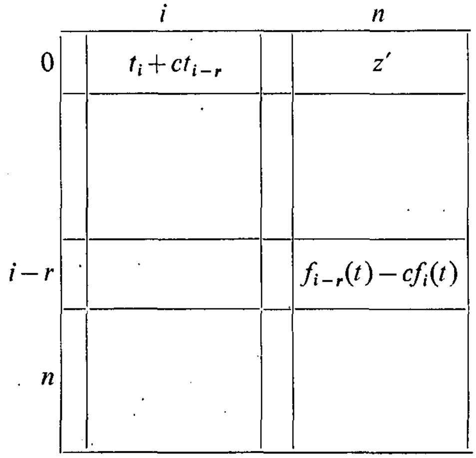


*Figure 3.*

Thus $t'_j = t_j$ for $j \neq i$ and $t_i = t_i + ct_{i-r}$. Then equation (1) shows that

$$f_{i-r}(t') = f_{i-r}(t) - cf_i(t)$$

which in turn implies that

$$f_i(t) = -f_{i-r}(0, \dots, 0, t_{i-r}, 0, \dots, 0) \qquad (0 \leqq i-r < i \leqq n),$$

where $t_{i-r}$ is in the $i$-th position. Suppose $i \geqq 3$; for $r = 1, 2$ the above becomes

$$(2) \qquad r = 1: f_i(t) = -f_{i-1}(0, \dots, 0, t_{i-1}, 0, \dots, 0),$$

$$r = 2: f_i(t) = -f_{i-2}(0, \dots, 0, t_{i-2}, 0, \dots, 0).$$

where $t_{i-1}$ and $t_{i-2}$ are in the $i$-th position. Since for arbitrary $t_{i-1}, t_{i-2} \in K$, $f_{i-1}(0, \dots, 0, t_{i-1}, 0, \dots, 0) = f_{i-2}(0, \dots, 0, t_{i-2}, 0, \dots, 0)$, it follows that both of these are identically zero for all choices of $t_{i-1}, t_{i-2} \in K$; consequently $f_i \equiv 0$ for $i \geqq 3$. The equation $f_k(t') = f_k(t)$, $k \neq i-r$ implies that

$$f_k(0, \dots, 0, t_{i-r}, 0, \dots, 0) = 0 \qquad (k \neq i-r,\ 1 \leqq k \leqq n),$$

where the $t_{i-r}$ is in the $i$-th position. Take a fixed $k$, $1 \leqq k \leqq n$ and $r = 1$; then the above equation holds for all $i$ except $i = k+1$ and $i = 1$. Since $f_k$ is linear and $t_{i-r} \in K$ is arbitrary, this implies that

$$f_k(t) = f_k(t_1, \dots, t_{n-1}) = f_k(t_1, 0, \dots, 0, t_{k+1}, 0, \dots, 0),$$

where the $t_{k+1}$ is in the $(k+1)$-st position. In particular, for $k=1$ there are $\alpha, \beta \in K$ such that $f_1(t) = \alpha t_1 + \beta t_2$. But now equation (2) with $i=2$ becomes $f_2(t) = -f_1(0, t_1, 0, ..., 0) = -\beta t_1$. Thus (i) has been proved.

(ii) and (iii) If $[t; z; f(t)]$ and $[t'; z'; f(t')]$ are arbitrary elements of $W$, then a necessary and sufficient condition that they commute is that $2\beta(t_1 t_2' - t_1' t_2) = 0$.

Corollary 3. 7. *If $K_0, K_1, ..., K_n$ are any multiplicative subgroups of $K \setminus \{0\}$, let $G(K_0, K_1, ..., K_n)$ denote the subgroup of $G$ consisting of all matrices $\|a_{ij}\|$ with $a_{ii} \in K_i$ $(i=0, 1, ..., n)$. Suppose the subgroup $W \lhd N$ satisfies the hypotheses of the last Lemma* 3. 6. *Then $\alpha=\beta=0$ and $W=B^1$ if either one of the following two conditions hold:*

(i) $K_0=K_1=...=K_{n-1}=\{1\}$, $K_n \neq \{1\}$;

(ii) $K_0=K_2=...=K_n=\{1\}$; $K_1 \neq \{1\}, \{1, -1\}$.

*In particular, if $W \lhd G^1$, then $W=B^1$.*

Proof. Let $d(\varkappa)=\operatorname{diag}(1, ..., 1, \lambda, 1, ..., 1)$ where $0 \neq \lambda \in K$ is located in the $\varkappa$-th row and column. Let $w=[t; z; (\alpha t_1+\beta t_2, -\beta t_1, 0, ..., 0)] \in W$, where $t \in K^{n-1}$, $z \in K$. Then

$$d(n)^{-1} w d(n) = [t; \lambda z; (\lambda\alpha t_1 + \lambda\beta t_2, -\beta t_1, 0, ..., 0)],$$

$$d(1)^{-1} w d(1) = [(\lambda t_1, t_2, ..., t_n); z(\lambda^{-1}\alpha t_1 + \lambda^{-1}\beta t_2, -\beta t_1, 0, ..., 0)].$$

Thus $\alpha t_1 + \beta t_2 = \lambda(\alpha t_1 + \beta t_2)$, $\lambda \neq 1$ for all $t_1; t_2 \in K$ implies that $\alpha=\beta=0$. Similarly, $\lambda(\alpha t_1+\beta t_2) = \lambda^{-1}(\alpha t_1+\beta t_2)$ and $\lambda^2-1 \neq 0$ also implies that $\alpha=\beta=0$.

The next lemma is proved by tedious but straightforward computations; its proof is omitted.

Lemma 3. 8. *For any constant $\alpha \in K$, $B^1(\alpha)$ and $B_1(\alpha)$ are maximal normal abelian subgroups of $N$; $B^1$ and $B_1$ are maximal normal abelian subgroups of $G^1$.*

The results of this section are summarized in the next theorem.

Theorem II. *Let the notation be as in* 2. 1 *and* 3. 4. *Suppose $W \subseteqq N$ is a subgroup satisfying:*

(a) $W \lhd N$, (b) $(W \cap N_j)N_{j+1}/N_{j+1}$ *is a one dimensional $K$-vector space for each $j=0, ..., n-1$. Then $W \subseteqq \Gamma$ and consequently there exists an element $[a; z; b] \in W$ with either $a_1 \neq 0$ or $b_1 \neq 0$. Assume:*

(c) $[a; z; b] \in W$, $a_1 \neq 0$.

*Then:*

(i) *There exist $\alpha, \beta \in K$ such that $W$ consists of all elements of the form*

$$[t; z; (\alpha t_1 + \beta t_2, -\beta t_1, 0, ..., 0)] \qquad (t \in K^{n-1}, z \in K).$$

(ii) *If characteristic $K=2$, $W$ is abelian.*

(iii) *If characteristic* $K \neq 2$, *and if* $W$ *is abelian, then necessarily* $\beta = 0$ *and* $W = B^1(\alpha)$.

(iv) *If, in addition to* (a), (b), *and* (c), $W$ *also satisfies* (d) $W \triangleleft G^1$, *then* $W = B^1$. *In particular,* $W$ *is abelian. In the other case when* $b_1 \neq 0$, *the obvious analogues of* (i)—(iv) *hold.*

(v) *For any* $\alpha \in K$ ($\alpha = 0$ *is not excluded*), $B^1(\alpha)$ *and* $B_1(\alpha)$ *are maximal normal abelian groups of* $N$; $B^1$ *and* $B_1$ *are maximal normal abelian subgroups of* $G^1$.

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