On the sum $\sum dd(f(n))$

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1.

Let $f(n)$ denote an irreducible polynomial with integer coefficients. We assume that $f(n) > 0$ for $n \geqq 1$. Suppose further that $f(n) \not\equiv cn$. Let $d(n)$ denote the number of divisors of n, and $dd(n)$ the number of divisors of $d(n)$. The letters $p, q, p_1, p_2, \cdots,$ $q_1, q_2, \cdots$ stand for prime numbers. For the sake of brevity we write $x_1 = \log x$, $x_2 = \log x_1, \cdots$. We shall prove the following results.

Theorem 1. *If the degree of $f(n)$ is $\leqq 3$, then*

$$\sum_{n \leqq x} dd(f(n)) = cxx_2 + O(x\sqrt{x_2}), \tag{1.1}$$

where c is a positive constant.

Theorem 2. *If the degree of $f(n)$ is $\leqq 2$, then*

$$\sum_{p \leqq x} dd(f(p)) = c' \operatorname{li} x \cdot x_2 + O(\operatorname{li} x \cdot \sqrt{x_2 x_3}), \tag{1.2}$$

where c' is a positive constant.

Remarks. It seems probable that the relations (1.1)—(1.2) hold without any restriction on the degree of $f(n)$. For the proof of (1.1) in the case $r=3$ we use a result of C. Hooley concerning the power-free values of polynomials [1]. (This question previously was investigated by P. Erdős in [2].) For the proof of (1.2) we use some well-known theorems on the distribution of prime numbers in arithmetical progressions.

2. Notation

The function $U(n)$ is the number of distinct prime factors of n. (a, b) is the highest common factor of a and b. $\varrho(n)$ denotes the number of (incongruent) roots of the congruence $f(\nu)\equiv 0 \pmod{n}$, and $\lambda(n)$ the number of those roots for which $(\nu, n)=1$. The letter m denotes square-free numbers.

We shall say that K is a "square-full" number if it contains every prime-divisors at least on the second power. Let $\mathfrak{U}$ denote the set of the square-full numbers. It is evident, that every integer n can be represented in the form $n=Km$, where $K\in\mathfrak{U}$, $(m, K)=1$. This representation is unique. We say that K is the square-full part and m is the square-free part of n. Let $\mathscr{B}_K$ denote the set of n's, square-full part of which is K.

Let $\mu(n)$ denote the Möbius-function.

For $K\in\mathfrak{U}$ we introduce the notation:

$$(2.1)\qquad k=d(K),\quad k=2^\alpha k_1 (k_1 \text{ is odd}),\quad k_2=d(k),\quad k_3=d(k_1);$$

$$(2.2)\qquad a(K)=k_2-U(K)k_3.$$

Thus for $f(n)\in B_K$ we have

$$(2.3)\qquad ddf(n)=k_3U(f(n))+a(K).$$

Let $B_K(x)$ (resp. $\bar{B}_K(x)$) the number of n's (resp. p's) in the interval $[1, x]$ for which $f(n)$ (resp. $f(p)$) belongs to $\mathscr{B}_K$. Let $C_l(x, \eta)$ (resp. $\bar{C}_l(x, \eta)$) the number of n's (resp. p's) in the interval $[1, x]$ for which $f(n)\equiv 0 \pmod{l}$ but $f(n)\not\equiv 0 \pmod{q^2}$ (resp. $f(p)\equiv 0 \pmod{l}$ but $f(p)\not\equiv 0 \pmod{q^2}$), when $1\leqq q\leqq\xi$ and $q\nmid l$. Let $C_l(x)=C_l(x,\infty)$, $\bar{C}_l(x)=\bar{C}_l(x,\infty)$.

The following relations obviously hold:

$$(2.4)\qquad B_K(x)=\sum_{\nu|K}\mu(\nu)C_{K\nu}(x),$$

$$(2.5)\qquad \bar{B}_K(x)=\sum_{\nu|K}\mu(\nu)\bar{C}_{K\nu}(x).$$

$\varepsilon_1, \varepsilon_2, \varepsilon_3$ denote sufficiently small positive constants. We use the symbol $\ll$ in VINOGRADOV's sense.

3. Lemmas

Lemma 1. [3] *The following relations hold:* a) $\varrho(ab)=\varrho(a)\varrho(b)$, *if* $(a, b)=1$; b) $\varrho(p^\alpha)\ll\alpha$; c) $\varrho(p^\alpha)=\varrho(p)$, *if* $p\nmid D$ *(D denotes the discriminant of* $f(n)$*). Further* $\varrho(p^\alpha)=\lambda(p^\alpha)$, *when p is sufficiently large.*

We shall use the following result of P. TURÁN.

Lemma 2. [4]
$$\sum_{n\leqq x}(U(f(n))-x_2)^2\ll xx_2.$$

Combining the method of TURÁN with the Rodossky—Tatuzawa theorems, we can prove the following

Lemma 3.

$$\sum_{p\leq x}(U(f(p))-x_2)^2 \ll \frac{x}{x_1}x_2\log x_2. \tag{3. 1}$$

Using additionally the result of BOMBIERI in the theory of large sieve [6], we could prove that the left hand side of (3. 1) has the order $xx_1^{-1}x_2$.

Lemma 4. ([8])

$$\sum_{n\leq x}[d(f(n))]^\alpha \ll x\cdot x_1^{c(\alpha)} \quad \textit{if} \quad \alpha\geqq 1.$$

$c(\alpha)$ is a suitable constant which depends only on α and f.

Corollary.

$$\sum_{\substack{n\leq x\\ U(f(n))>\beta x_2}} d(f(n)) \ll \frac{x}{x_1^2}, \textit{ if } \beta \textit{ is large enough.}$$

Let $N(x, y)$ denote the number of those n's in $1\leqq n\leqq x$, for which $p^2|f(n)$ with some $p>y$.

C. HOOLEY proved

Lemma 5. ([1])

$$N(x, x_1)\ll x\cdot x_1^{-A/x_3} \qquad (A>0, \textit{ suitable constant}).$$

Lemma 6. *Let $b_n\ll n_4^\varepsilon$ be a sequence of positive numbers. Then*

$$\sum_{K>y}\frac{b_K}{K} \ll y^{-\frac{1}{2}+\varepsilon} \quad \textit{for} \quad y\to\infty.$$

The proof is simple and so can be omitted.

Applying the sieve method, we can prove the following

Lemma 7.

$$C_h(x, x) = x\frac{\varrho(h)}{h}\prod_{p\nmid h}\left(1-\frac{\varrho(p^2)}{p^2}\right)+O(xx_1^{-1})$$

uniformly for $1\leqq h\leqq x_1^2$.

Lemma 8. *Let $f(n)$ be an irreducible polynomial of degree 2. Then for fixed h the number of the solutions of $f(n)=hs^2$ ($1\leqq n\leqq x$, n, s integers) is at most $O(x_1)$ uniformly in h.*

For the proof see [7], Lemma 2.

Lemma 9.

$$\bar{C}_h(x, x^{1/2}) = \text{li}\, x \cdot \frac{\lambda(h)}{h} \prod_{p \nmid h} \left(1 - \frac{\lambda(p^2)}{p^2}\right) + O(x x_1^{-2}),$$

uniformly in $1 \leqq h \leqq x_1$.

The proof goes with the standard application of the sieve method using in addition the prime number theorems in the form:

$$\pi(x, k, l) = \frac{\text{li}\, x}{\varphi(k)} (1 + O(x_1^{-2})), \tag{3.2}$$

uniformly for $1 \leqq k \leqq x_1^3$, $(k, l) = 1$ (see [5], and the Brun—Titchmarsh inequality stating that

$$\pi(x, k, l) < C_\delta \frac{\text{li}\, x}{\varphi(k)}, \quad \text{for} \quad k < x^{1-\delta} \qquad (\delta > 0) \text{ ([5])}. \tag{3.3}$$

4. The proof of Theorem 1

$$\sum\nolimits_K = \sum_{\substack{n \leqq x \\ f(n) \in B_K}} ddf(n); \quad \sum\nolimits_{K,A} = \sum_{\substack{n \leqq x \\ f(n) \in B_K}} U(f(n)).$$

Using (2. 3) we have

$$\sum\nolimits_K = k_3 \sum\nolimits_{K,A} + a(K) B_K(x).$$

Let $\xi = x_1^\delta$, and let δ be a sufficiently small positive constant. First we prove that

$$\sum_{K > \xi} \sum\nolimits_K \ll x. \tag{4.1}$$

Applying the Corollary to Lemma 4, it is enough to prove that

$$\sum_{K > \xi} (x_2 k_3 + k_2) B_K(x) \ll x.$$

Since $B_K(x) \ll \frac{x \varrho(K)}{K} + \varrho(K)$, by Lemma 6 we obtain

$$\sum_{\xi \leqq K \leqq x} (k_3 x_2 + k_2) B_K(x) \ll x x_2 \sum_{\xi \leqq K \leqq x} \frac{k_3 \varrho(K)}{K} + x \sum_{K \geqq \xi} \frac{k_2 \varrho(K)}{K} \ll x x_2 \xi^{-1/3} \ll x.$$

Let now $K > x$. $K = p_1^{\alpha_1} \cdots p_1^{\alpha_r}$, $p_1 < p_2 < \cdots < p_j \leqq x^{1/4} < p_{j+1} < \cdots < p_r$. Let $K = K_1 K_2$, $K_1 = p_1^{\alpha_1} \cdots p_j^{\alpha_j}$.

Let

$$\sum_{K > x} (x_2 k_3 + k_2) B_K(x) = \sum\nolimits_a + \sum\nolimits_b + \sum\nolimits_c$$

where in the sums $\sum_a$, $\sum_b$, $\sum_c$ we sum over those K for which: a) $K_1 \leqq \xi$; b) $\xi < K_1 \leqq x$; c) $K_1 > x$ holds, respectively.

Since for $K_1 \leqq \xi$ the inequality

$$(k_3 \leqq) k_2 \leqq dd(K) \ll dd(K_1) \ll [d(K_1)]^{\varepsilon} \ll \exp\left(2\varepsilon \frac{\log \xi}{\log\log \xi}\right) \ll \exp\left(\varepsilon_1 \frac{x_2}{x_3}\right)$$

holds, by Lemma 5 we have

$$\sum\nolimits_a \ll x_2 \exp\left(\varepsilon_1 \frac{x_2}{x_3}\right) N(x, x^{1/4}) \ll xx_2 \exp\left(-\frac{A}{2}\frac{x_2}{x_3}\right) \ll x.$$

For $\sum_b$ we have

$$\sum\nolimits_b \ll \sum_{\xi \leqq K_1 \leqq x} (k_3 x_2 + k_2) C_{K_1}(x) \ll xx_2 \sum_{\xi < K_1 < x} \frac{d(K_1)\varrho(K_1)}{K} \ll xx_2 \xi^{-1/3} \ll x.$$

For the estimation of $\sum_c$ let K_3 denote the maximal square-full divisor of K_1 in the interval $x^{1/4} \leqq K_3 \leqq x$. ($K_3$ exists since the greatest prime factor of K_1 is $\leqq x^{1/4}$.) Consequently, we have

$$\sum\nolimits_c \ll x^{1+\varepsilon} \sum_{x^{1/4} < K_3 \leqq x} \frac{\varrho(K)}{K} \ll x.$$

So (4. 1) holds.

Since

$$\sum_{K \leqq \xi} a(K) B_K(x) \ll x \sum_{K \leqq \xi} \frac{K^{\varepsilon}\varrho(K)}{K} \ll x,$$

for the proof of (1. 1) it is enough to prove that

$$\sum_{K \leqq \xi} k_3 \sum\nolimits_{K,A} = cxx_2 + O(x\sqrt{x_2}).$$

By the Cauchy—Schwarz inequality we have

$$T = \sum_{K \leqq \xi} k_3 \{\sum\nolimits_{K,A} - x_2 B_K(x)\} \ll \sum_{K \leqq \xi} \sum_{f(n) \in \mathcal{B}_K} k_3 |U(f(n)) - x_2| \ll$$

$$\ll \Big(\sum_{K \leqq \xi} k_3^{1/2} B_K(x)\Big)^{1/2} \Big(\sum_{n \leqq x} |U(f(n)) - x_2|^2\Big)^{1/2} = \sum\nolimits_1^{1/2} \cdot \sum\nolimits_2^{1/2}.$$

Since

$$\sum\nolimits_1 \ll x \sum_{K \leqq \xi} k_3^{1/2} \frac{\varrho(K)}{K} \ll x$$

and by Lemma 2 $\sum_2 \ll xx_2$, we have $T \ll xx_2^{1/2}$.

Now we prove that

$$\sum_{K \leqq \xi} k_3 B_K(x) = cx + O\left(x \exp\left(-\frac{A}{2}\frac{x_2}{x_3}\right)\right), \tag{4.2}$$

hence Theorem 1 follows.

Applying (2. 4) we have

$$\sum_{K \leqq \xi} k_3 B_K(x) = \sum_{K \leqq \xi} k_3 \sum_{\nu | K} \mu(\nu) C_{K\nu}(x) = \sum_{K \leqq \xi} k_3 \sum_{\nu | K} \mu(\nu) C_{K\nu}(x, x) +$$

$$+ O\Big(\sum_{K \leqq \xi} k_3 \sum_{\nu | K} |\mu(\delta)| |C_{K\nu}(x, x) - C_{K\nu}(x)|\Big) = \sum\nolimits_3 + O(\sum\nolimits_4).$$

Since in the sum $\sum_4$ the relations $d(K) \ll \exp\left(3\delta \frac{x_2}{x_3}\right)$, $k_3 \leqq d^{\varepsilon}(K)$ hold, by Lemma 5

$$\sum\nolimits_4 \ll \exp(4\delta x_2/x_3) N(x, x) \ll x x_1^{-A/2x_3},$$

if δ is small enough.

Further by Lemma 7

$$\sum\nolimits_3 = cx + O(x x_1^{-1} \xi^2) = cx + O(x x_1^{-A/2x_3}),$$

where

$$c = \sum_K \frac{k_3}{K} \left\{ \sum_{\nu | K} \mu(\nu) \frac{\varrho(K\nu)}{\nu} \right\} \prod_{p \nmid K} \left(1 - \frac{\varrho(p^2)}{p^2}\right).$$

5. The proof of Theorem 2

Let

$$S_K = \sum_{\substack{p \leqq x \\ f(p) \in B_K}} ddf(p); \qquad S_{K,A} = \sum_{\substack{p \leqq x \\ f(p) \in B_K}} U(f(p)).$$

By (2. 3)

$$S_K = k_3 \, S_{K,A} + a(K) \bar{B}_K(x).$$

Using the Corollary to Lemma 4, we have

$$\sum_{K > \xi} S_K \ll \sum_{K > \xi} (k_3 x_2 + k_2) \bar{B}_K(x) + O(x/x_1^2) = \sum + O\left(\frac{x}{x_1^2}\right).$$

Let

$$\sum = \sum\nolimits_1 + \sum\nolimits_2 + \sum\nolimits_3 + \sum\nolimits_4,$$

where in $\sum_1$: $\xi \leqq K \leqq x^{3/4}$, in $\sum_2$: $x^{3/4} < K \leqq x$, in $\sum_3$: $x \leqq K \leqq x^{7/4}$, and in $\sum_4$: $K \geqq x^{7/4}$.

For $K \leqq x^{3/4}$ we have by (3. 3) that

$$\bar{B}_K(x) \ll \frac{\lambda(K)}{\varphi(K)} \operatorname{li} x.$$

Consequently

$$\sum\nolimits_1 \ll \operatorname{li} x \sum_{K \geqq \xi} \frac{k_3 x_2 + k_2}{\varphi(K)} \ll x_2 \operatorname{li} x \cdot \xi^{-1/3} \ll \operatorname{li} x.$$

For $x^{3/4} < K \leqq x$ we use the trivial estimation

$$\bar{B}_K(x) \leqq B_K(x) \ll x \frac{\varrho(K)}{K},$$

$$\sum\nolimits_2 \ll x^{1+\varepsilon} \sum_{K \geqq x^{3/4}} \frac{\varrho(K)}{K} \ll \operatorname{li} x.$$

Since for $K \geqq x$

$$B_K(x) \ll \varrho(K) \ll x^{\varepsilon},$$

and the number of the square-full number in the interval $[1, x^{7/4}]$ is majorized by $x^{7/8+\varepsilon}$, so

$$\sum\nolimits_3 \ll \mathrm{li}\, x.$$

Finally, let $K \geqq x^{7/4}$. Let L^2 denote the greatest square divisor of K. Since K is a square-full number, so $L^2 \geqq K^{2/3}$ $(\geqq x^{7/6})$.

It is obvious, that

$$\sum\nolimits_4 \ll x^{\varepsilon} \sum_{K \geqq x^{7/4}} \sum_{\substack{f(n) \equiv 0 (\mathrm{mod}\, K) \\ n \leqq x}} 1 \ll x^{\varepsilon} \sum_{L^2 \geqq x^{7/6}} \sum_{\substack{f(n) = hL^2 \\ n \leqq x}} 1.$$

Since the degree of $f(n)$ is 2, so $h \ll x^{5/6}$. Changing the order of summation and applying Lemma 8, we have

$$\sum\nolimits_4 \ll x^{\varepsilon} \sum_{h \leqq cx^{5/6}} \sum_{\substack{f(n) = hL^2 \\ n \leqq x}} 1 \ll \mathrm{li}\, x.$$

Consequently

$$\sum_K S_K = \sum_{K \leqq \xi} S_K + O(\mathrm{li}\, x).$$

Taking into account that

$$\sum_{K \leqq \xi} |a(K)|\, \bar{B}_K(x) \ll \mathrm{li}\, x \sum_{K \leqq \xi} \frac{|a(K)|}{\varphi(K)} \ll \mathrm{li}\, x,$$

we have

$$\sum_K S_K = \sum_{K \leqq \xi} k_3 S_{K,A} + O(\mathrm{li}\, x).$$

By Lemma 3 we obtain that

$$\Big|\sum_{K \leqq \xi} k_3 S_{K,A} - x_2 \sum_{K \leqq \xi} k_3 \bar{B}_K(x)\Big| \ll \Big(\sum_{K \leqq \xi} k_3^2 \bar{B}_k(x)\Big)^{1/2} \Big(\sum_{p \leqq x} \big(U(f(p)) - x_2\big)^2\Big)^{1/2} \ll$$

$$\ll (\mathrm{li}\, x)^{1/2} (\mathrm{li}\, x \cdot x_2 \cdot x_3)^{1/2} \ll \mathrm{li}\, x \cdot \sqrt{x_2 x_3}.$$

Consequently for the proof of Theorem 2 it is enough to prove that

$$\sum_{K \leqq \xi} k_3 \bar{B}_K(x) = d\, \mathrm{li}\, x \cdot x_2 + O\big(\mathrm{li}\, x \cdot \sqrt{x_2 x_3}\big). \tag{5. 1}$$

The proof of (5. 1) is very similar to that of (4. 2) and so it can be omitted.

References

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