On interpolation functions. II

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The purpose of this note is to indicate a certain simplification of the results of [2]. (We use throughout the terminology of that note.)

If $h=h(u)$ is a positive Borel measurable function on $(0, \infty)$ we agree to say that h is *pseudo-concave* if it is equivalent to a concave function, i.e. if there exist a concave function h_0 and a constant C such that $C^{-1}h_0(u) \leqq h(u) \leqq Ch_0(u)$ for all u.

Lemma. *h is pseudo-concave if and only if for some C holds*

(1) $$h(v) \leqq C \max(1, v/u)\, h(u) \quad \text{for all} \quad u, v.$$

Proof (necessity). It suffices to show that (1) holds for h_0. But from the concavity it follows that $h_0(u)$ is increasing but $\frac{h_0(u)}{u}$ is decreasing. Therefore (1) follows with $C=1$.

Proof (sufficiency). Let $\alpha_i \geqq 0$, $\sum \alpha_i = 1$, $u = \sum \alpha_i u_i$ (finite sums). Using (1) we obtain

$$\Sigma \alpha_i h(u_i) \leqq C \Sigma \alpha_i \max\left(1, \frac{u_i}{u}\right) h(u) \leqq C\left(\Sigma \alpha_i + \Sigma \frac{\alpha_i u_i}{u}\right) h(u) \leqq 2Ch(u).$$

This establishes the pseudo-concavity because we may now take

$$h_0(u) = \sup \sum \alpha_i h(u_i),$$

which is by the way the least concave majorant of h.

Corollary. *If $h(u)$ is pseudo-concave so is $k(u)=(h(u^r))^{\frac{1}{r}}$ where r is any real number $\neq 0$.*

Proof. Obvious. — This corollary was obtained in [2] as a by-product of the main result there which we now reformulate as follows.

Theorem. *A function $H=H(z_0, z_1)$ homogeneous of degree 1 and satisfying condition (2) in [2] is an interpolation function (of any power p) if and only if it is of the form $H(z_0, z_1) = z_0 h(z_1/z_0)$ with h pseudo-concave.*

Proof (necessity). Consider the space $X=\{x, y\}$ provided with the measure μ such that each of the two points x and y carries the mass 1. Take

$$\zeta_0(x)=1, \quad \zeta_0(y)=1; \quad \zeta_1(x)=u, \quad \zeta_1(y)=v$$

and define a linear mapping π by $\pi a(x)=0$, $\pi a(y)=a(x)$. The corresponding operator norms of π (considered as a mapping from $L^p_{\zeta_0}$ into $L^p_{\zeta_0}$ etc.) are

$$M_0=1, \qquad M_1=\frac{v}{u}, \qquad M=\frac{h(v)}{h(u)}.$$

By the inequality (*cf.* inequality (1) in [2]) $M\leqq C\max(M_0, M_1)$ we now get

$$\frac{h(v)}{h(u)} \leqq C\max\left(1, \frac{v}{u}\right)$$

which establishes (1).

Proof (sufficiency). By the lemma H is equivalent to a function of the form (4) in [2]. Therefore we may use the corresponding part of the proof of theorem 1 in [2] (*cf.* in particular pp. 168—169).

We conclude by illustrating our new result in a concrete case (*cf.* [1]).

Let $X=(0, \infty)$, $\mu=$ Haar measure, $\zeta_0(x)=1$, $\zeta_1(x)=e^{x^\alpha}$ $(\alpha>0)$. With

$$h(u) = e^{(\log u)^\lambda}, \quad \lambda = \frac{\beta}{\alpha} \qquad (\alpha > \beta > 0)$$

we then have

$$e^{x^\beta} = h(e^{x^\alpha}) = \zeta_0 h\left(\frac{\zeta_1}{\zeta_0}\right).$$

We claim that h is pseudo-concave. It is clear that $h(u)$ is increasing. It suffices therefore to show that $\frac{h(u)}{u}$ is decreasing if u is large, since the values $u<1$ do not interfere. To this end we consider the corresponding logarithmic derivative; we have

$$\frac{d}{du}\log(h(u)/u) = (\lambda(\log u)^{\lambda-1}-1)/u$$

which, since $\lambda<1$, eventually becomes <0. Applying now our theorem we thus get the following interpolation theorem: *If π is a continuous linear mapping from L^p_0 into L^p_0 and from L^p_α into L^p_α it is also a continuous linear mapping from L^p_β into L^p_β $(\alpha>\beta>0)$.* Here we have set $L^p_\gamma=L^p_{(e^{x^\gamma})}$ with $\gamma=0, \alpha, \beta$.

References

[1] C. Goulaouic, Interpolation entre les espaces L^p avec poids, *C. R. Acad. Sci. Paris*, **262** (1966), 333—336.

[2] J. Peetre, On interpolation functions, *Acta Sci. Math.*, **27** (1966), 167—171.

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