The inner function in Rota's model

By J. W. MOELLER in Chicago (Illinois, U. S. A.)

Let K be a Hilbert space of dimension $\aleph_0$ with inner product $(\cdot,\cdot)$, and let $H^2(K)$ denote the Hardy class of vector-valued functions

$$k(z) = \sum_{n=0}^{\infty} k_n z^n \qquad \left(k_n \in K;\ \sum_{n=0}^{\infty} (k_n, k_n) < \infty\right).$$

An inner product for these Hardy functions can be defined by setting

$$\langle k(z), h(z) \rangle = \sum_{n=0}^{\infty} (k_n, h_n),$$

and $H^2(K)$ becomes a Hilbert space in its own right under this new inner product. It is well known from the work of BEURLING, LAX, and HALMOS (see [2], pp. 115—116) that every closed subspace of $H^2(K)$ which is invariant under multiplication by z has a representation of the form $G\,H^2(K)$, where

$$G(z) = \sum_{n=0}^{\infty} G_n z^n$$

and the Taylor coefficients of G are linear operators from K into K. In addition, the operator norm of G is bounded from above by one, and the radial limits of G on the boundary of the unit disc are equal almost everywhere to partial isometries. (Such functions are commonly called "inner functions" in the literature.) If S^* designates the operation of multiplication by z in $H^2(K)$, a straightforward calculation reveals that the adjoint of this operation is given by

$$(Sh)(z) = z^{-1}(h(z) - h(0)).$$

Henceforth we will call a closed subspace of $H^2(K)$ which is invariant under S a *left translation invariant subspace*, and a closed subspace which is invariant under S^* a *right translation invariant subspace*. It is not difficult to show that the orthogonal complement of a right translation invariant subspace is a left translation invariant subspace, and conversely.

In [3], G. C. ROTA established the following interesting result.

Theorem. *Let $A\colon K\to K$ be a bounded linear operator whose spectrum is contained in the interior of the unit disc. Then the set*

$$L_A=\{(I-zA)^{-1}k \mid k\in K\}$$

is a left translation invariant subspace of $H^2(K)$ and S acting on L_A is similar to A acting on K.

According to the Beurling—Lax theorem, we may write

$$L_A=H^2(K)\ominus G\,H^2(K)$$

for some G whose Taylor coefficients depend only on A. Whenever $\|A\|<1$, Helson proved ([1], pp. 104—106) that G is always equal to a unitary operator on the rim of the unit disc, and he further derived the explicit formula

$$G(z)=G_0+z(I-zA)^{-1}G_1.$$

He did not, however, relate the operators G_1 and G_0 to A in any way. Our aim here is to determine this relationship.

We begin by computing the orthogonal projection of the constant functions in $H^2(K)$ onto L_A in two different ways. If $k\in K$, we have

$$\langle (I-G(z)G^*(0))k,\, z^nG(z)G^*(0)k\rangle = \langle k,\, z^nG(z)G^*(0)k\rangle - \langle G(z)G^*(0)k,\, z^nG(z)G^*(0)k\rangle$$
$$=\langle k,\, z^nG(z)G^*(0)k\rangle - \langle G^*(0)k,\, z^nG^*(0)k\rangle = 0 \quad \text{for} \quad n=0,1,\dots .$$

From this relation and the simple identity

$$k=(I-G(t)G^*(0))k+G(z)G^*(0)k$$

we quickly deduce that

$$Pk=(I-G(z)G^*(0))k,$$

where P denotes the orthogonal projection onto L_A.

To complete the last part of our task, we will express P in terms of A alone. If we set

$$\tilde{A}=\sum_{n=0}^{\infty} A^{*n}A^n,$$

it follows immediately that

$$\langle k,\, (I-zA)^{-1}f\rangle \langle (I-zA)^{-1}f,\, (I-zA)^{-1}f\rangle^{-\frac{1}{2}} = (k,f)(\tilde{A}f,f)^{-\frac{1}{2}}.$$

For fixed k, the right hand side of the preceding expression assumes its maximum when $f=\tilde{A}^{-1}k$, so we conclude that

$$Pk=(I-zA)^{-1}\tilde{A}^{-1}k.$$

After identifying the Taylor coefficients in HELSON's formula, we find

(1) $$I - G_0 G_0^* = \tilde{A}^{-1} \quad \text{and} \quad G_1 G_0^* = -A\tilde{A}^{-1}.$$

Since $G(z)$ is a unitary operator on the boundary of the unit disc,

(2) $$G^*(e^{i\theta})G(e^{i\theta}) = I \quad \text{and} \quad G(e^{i\theta})G^*(e^{i\theta}) = I.$$

Setting $\theta = 0$ in the last identity gives

$$(G_0 + (I-A)^{-1}G_1)(G_0^* + G_1^*(I-A^*)^{-1}) = I,$$

which, together with (1), implies

$$G_1 G_1^* = (I-A)\tilde{A}^{-1}(I-A^*) + (I-A)\tilde{A}^{-1}A^* + A\tilde{A}^{-1}(I-A^*)^{-1}.$$

Thus we finally have

(3) $$G_1 G_1^* = \tilde{A}^{-1} - A\tilde{A}^{-1}A^*.$$

The first equation in (2) may be rewritten in the form

$$I = G_0^* G_0 + G_1^* \tilde{A} G_1,$$

and premultiplication by G_0 gives

$$G_0 = (G_0 G_0^*)G_0 + (G_0 G_1^*)\tilde{A} G_1.$$

In other words, $(I - G_0 G_0^*)G_0 = (G_0 G_1^*)\tilde{A} G_1$, so

(4) $$G_0 = -A^* \tilde{A} G_1.$$

Hence, $I = G_1^*(\tilde{A}AA^*\tilde{A} + \tilde{A})G_1$ and we infer that the operator

(5) $$U = (\tilde{A}AA^*\tilde{A} + \tilde{A})^{\frac{1}{2}} G_1$$

is an isometry.

An easy application of the identity $\tilde{A} = I + A^*\tilde{A}A$ reveals that

$$(\tilde{A}^{-1} - A\tilde{A}^{-1}A^*)(\tilde{A}AA^*\tilde{A} + \tilde{A}) = I \quad \text{and} \quad (\tilde{A}AA^*\tilde{A} + \tilde{A})(\tilde{A}^{-1} - A\tilde{A}^{-1}A^*) = I.$$

We now see from (3) and (5) that U is actually a unitary operator and

(6) $$G_1 = (\tilde{A}^{-1} - A\tilde{A}^{-1}A^*)^{\frac{1}{2}} U.$$

Equations (4) and (6) thus determine the inner function associated with L_A up to multiplication on the right by a constant unitary factor.

Bibliography

[1] H. HELSON, *Lectures on Invariant Subspaces* (New York, 1964).
[2] K. HOFFMAN, *Banach Spaces of Analytic Functions* (New Jersey, 1962).
[3] G. C. ROTA, On Models for Linear Operators, *Comm. Pure Appl. Math.*, **13** (1960), 469—472.

(Received June 9, 1967)