Operators of the form C^*C in indefinite inner product spaces

By J. BOGNÁR and A. KRÁMLI in Budapest

To professor M. G. Krein on the occasion of his 60th anniversary

1. Introduction

Let A denote a continuous self-adjoint operator in a J-space H (for definitions see Sections 2 and 3 below). In the present paper we give a necessary and sufficient condition (Theorem 1) for the existence of a continuous linear operator C in H such that

$$A = C^*C. \tag{1}$$

In the special case when the space H is of type H_k we obtain (in Theorems 2 and 3) the solution of a problem proposed in [1]. Partial answers to other questions contained in [1] are to be found in the communications [2]—[5].

We mention that Theorem 2 is equivalent to an early result of POTAPOV ([6], Chapter 2, Theorem 2).

In J-spaces whose positive and negative components are of equal infinite dimension it turns out that the representation (1) is always possible (Theorem 4).

In some J-spaces property (1) is known [1] to be less restrictive than the existence of a self-adjoint square root:

$$A = B^2 \qquad (B^* = B). \tag{2}$$

Therefore we do not hope that our conditions would have a significance similar to that of positivity in Hilbert space. However, they are so simple comparatively to the criteria for (2), contained in [2] that it seems desirable to use the factorization (1) instead of (2) as far as possible.

Lemmas 1—3 are known; Lemmas 4—7 slightly generalize some results of GINZBURG, JOHVIDOV and WITTSTOCK. It should be noted that Lemmas 5 and 6 show the invariant character of some of the notions applied, but actually they are not used in the following.

2. Basic facts concerning J-spaces

We consider a complex vector space H and a hermitian form $(\cdot, \cdot)$ defined on $H \times H$. The corresponding quadratic form is not assumed to be positive definite. We shall say that (x, y) is the *inner product* of the elements $x, y \in H$.

Two elements, x and y are *orthogonal* to each other if $(x, y)=0$. Two sets $F, G \subset H$ are said to be orthogonal if any element of F is orthogonal to any element of G.

An element $x \in H$ is called *positive* if $(x, x)>0$, *neutral* if $(x, x)=0$, and *negative* if $(x, x)<0$. A subspace (linear manifold) $L \subset H$ is said to be positive (neutral, negative) if all its elements except 0 are positive (neutral, negative).

The positive (negative) subspace L is *intrinsically complete* if it is complete with respect to the *intrinsic norm*

$$|x|_L = |(x, x)|^{\frac{1}{2}} \qquad (x \in L). \tag{3}$$

In the following we assume that H is a *J-space* i. e. H is an orthogonal direct sum

$$H = H^+ \oplus H^- \tag{4}$$

of an intrinsically complete positive subspace H^+ and an intrinsically complete negative subspace H^-.

In the special case $\dim H^- = k < \infty$ we say that H is a *space of type H_k*. Spaces with $\dim H^+ < \infty$ have essentially the same properties.

In a J-space H we put

$$[x, y] = (x^+, y^+) - (x^-, y^-) \qquad (x, y \in H) \tag{5}$$

where $x = x^+ + x^-$, $y = y^+ + y^-$ denote the decompositions of x and y corresponding to (4).

It is evident that $[x, x] > 0$ if $x \neq 0$. Therefore the hermitian form (5) may be called the *definite inner product* belonging to the decomposition (4). By definition, the J-space is a Hilbert space with respect to this definite inner product. The functional

$$\|x\| = [x, x]^{\frac{1}{2}} \qquad (x \in H) \tag{6}$$

is called the *norm* belonging to (4).

The norm (6) defines a topology in H. In the following the words "*closed*", "*continuous*", etc. will always refer to this topology.

Lemma 1. *For any $x, y \in H$ we have* $|(x, y)| \leqq \|x\| \, \|y\|$.

Proof. We shall use the same notations as in (5). By the orthogonality of H^+ and H^- we have

$$(x^+, y^-) = (x^-, y^+) = 0.$$

On the other hand, it follows from (5) that

$$[x^+, x^-] = [y^+, y^-] = 0.$$

Thus we can write

$$|(x, y)| \leqq |(x^+, y^+)| + |(x^- y^-)| = |[x^+, y^+]| + |[x^-, y^-]| \leqq \|x^+\| \|y^+\| + \|x^-\| \|y^-\| \leqq$$
$$\leqq (\|x^+\|^2 + \|x^-\|^2)^{\frac{1}{2}} (\|y^+\|^2 + \|y^-\|^2)^{\frac{1}{2}} = \|x\| \|y\|.$$

Let T be a continuous (everywhere defined) linear operator in the J-space H By virtue of Lemma 1 (Tx, y) is a continuous linear form in x and the Riesz representation theorem assures the existence of an element $y_* \in H$ such that

$$(Tx, y) = [x, y_*] \qquad (x \in H).$$

Setting

$$T^* y = y_*^+ - y_*^- \qquad (y \in H)$$

where $y_* = y_*^+ + y_*^-$ $(y_*^+ \in H^+, y_*^- \in H^-)$ we obtain

$$(Tx, y) = (x, T^* y) \qquad (x, y \in H).$$

One verifies easily that T^* is a single-valued continuous linear operator. We call T^* the *adjoint* of T. The operator T is said to be *self-adjoint* provided $T^* = T$.

For a continuous self-adjoint operator A one can define the *A-inner product* by

(7) $$(x, y)_A = (Ax, y) \qquad (x, y \in H).$$

The form $(\cdot, \cdot)_A$ is hermitian and continuous (Lemma 1). In the special case $A = I$ it turns into the original inner product $(\cdot, \cdot)$.

Using the A-inner product the notions of *A-orthogonality*, *A-positivity*, *intrinsic A-completeness* etc. can be introduced in the same way as orthogonality, positivity, intrinsic completeness have been defined with the help of the original inner product. The *intrinsic A-norm* on an A-positive or A-negative subspace L has the form

(8) $$|x|_{A, L} = |(x, x)_A|^{\frac{1}{2}} \qquad (x \in L).$$

An *A-fundamental decomposition* is a representation of H as the A-orthogonal direct sum of an A-neutral subspace H_A^0, an A-positive subspace H_A^+ and an A-negative subspace H_A^-:

(9) $$H = H_A^0 \dotplus H_A^+ \dotplus H_A^-.$$

In the case $A = I$ we speak of a *fundamental decomposition.* E. g. the decomposition (4) appearing in the definition of a J-space is a fundamental one.

An A-fundamental decomposition (9) is *regular* if $H_A^+ \dotplus H_A^-$ is closed.

Lemma 2. (See e.g. [7], § 3, section 2.) *Let A be a continuous self-adjoint operator in the J-space H. Then H admits at least one regular A-fundamental decomposition.*

Lemma 3. *The A-neutral component H_A^0 of an arbitrary A-fundamental decomposition* (9) *consists of all elements in H that are A-orthogonal to H:*

$$H_A^0 = \{x : (x, y)_A = 0 \text{ for every } y \in H\}. \tag{10}$$

Proof. If $x \in H_A^0$ then x is A-orthogonal to H_A^+ and H_A^- by the definition of the A-fundamental decomposition. Furthermore, $(\cdot, \cdot)_A$ is a semi-definite form on H_A^0, hence the Schwarz inequality $|(x, y)_A|^2 \leqq (x, x)_A(y, y)_A$ $(x, y \in H_A^0)$ is valid. It follows that x is A-orthogonal to H_A^0.

If, conversely, the element

$$x = x_A^0 + x_A^+ + x_A^- \qquad (x_A^0 \in H_A^0, \quad x_A^+ \in H_A^+, \quad x_A^- \in H_A^-) \tag{11}$$

is A-orthogonal to H then $(x, x_A^+)_A = (x_A^+, x_A^+)_A = 0$ and $(x, x_A^-)_A = (x_A^-, x_A^-)_A = 0$ But H_A^+ is A-positive and H_A^- is A-negative. Therefore x_A^+ and x_A^- must be 0.

As a corollary we obtain that every fundamental decomposition of a J-space is of the form (4).

Lemma 4. *Each component of a regular A-fundamental decomposition* (9) *is closed.*

Proof. H_A^0 is closed by Lemma 3 and the continuity of the A-inner product (cf. Lemma 1).

Denote by $\bar{H}_A^+$ the relative closure of H_A^+ in $H_A^+ \dotplus H_A^-$. If $\bar{H}_A^+ \neq H_A^+$ then $\bar{H}_A^+$ has a non-trivial intersection with the A-negative subspace H_A^-. But this is impossible, since it follows from the continuity of the A-inner product that $\bar{H}_A^+$ is A-non-negative. Hence H_A^+ is closed in $H_A^+ \dotplus H_A^-$ and, the decomposition (9) being regular, in H as well.

For H_A^- the argument is similar.

3. Invariant properties. Intrinsic A-dimension

We consider an A-fundamental decomposition (9) and define

$$[x, y]_A = (x_A^+, y_A^+)_A - (x_A^-, y_A^-)_A \qquad (x, y \in H). \tag{12}$$

Here

$$x = x_A^0 + x_A^+ + x_A^-, \quad y = y_A^0 + y_A^+ + y_A^-$$

are the decompositions corresponding to (9).

It is clear that $[x, x]_A \geqq 0$ for every $x \in H$. We call $[\cdot, \cdot]_A$ the *semi-definite A-inner product* belonging to (9). The corresponding *A-semi-norm* is

$$\|x\|_A = [x, x]_A^{\frac{1}{2}} \qquad (x \in H). \tag{13}$$

Lemma 5. *The A-semi-norms belonging to any two A-fundamental decompositions are topologically equivalent.*

Proof. Let H_A^0 be defined by (10), and put

$$H_A^{(1)} = \{x : [x, y] = 0 \text{ for every } y \in H_A^0\}.$$

Then $H_A^{(1)}$ is a closed subspace and we have

$$H = H_A^0 \dotplus H_A^{(1)}. \tag{14}$$

We consider an arbitrary A-fundamental decomposition (9), and set

$$Vx = x_A^0 + x_A^{(1)} \qquad (x \in H) \tag{15}$$

where x_A^0 is defined by (11), and $x_A^{(1)}$ is the component of x in $H_A^{(1)}$ corresponding to the decomposition (14):

$$x = x_A^{(0)} + x_A^{(1)} \qquad (x_A^{(0)} \in H_A^0, x_A^{(1)} \in H_A^{(1)}). \tag{16}$$

It is evident that V is a one-to-one linear mapping of H onto H which leaves the elements of H_A^0 fixed, and carries $H_A^+ \dotplus H_A^-$ into $H_A^{(1)}$. Moreover, according to (15), (16) and (10) the identity

$$(Vx, Vy)_A = (x, y)_A \qquad (x, y \in H) \tag{17}$$

holds. Introducing the notations

$$VH_A^+ = H_A^{(+)}, \quad VH_A^- = H_A^{(-)} \tag{18}$$

we obtain an A-fundamental decomposition

$$H = H_A^0 \dotplus H_A^{(+)} \dotplus H_A^{(-)} \tag{19}$$

where

$$H_A^{(+)} \dotplus H_A^{(-)} = H_A^{(1)}. \tag{20}$$

Let

$$Vx_A^+ = x_A^{(+)}, \quad Vx_A^- = x_A^{(-)}. \tag{21}$$

Then in virtue of (18) we have $x_A^{(+)} \in H_A^{(+)}$, $x_A^{(-)} \in H_A^{(-)}$ and the relations (11), (21) imply that

$$Vx = x_A^0 + x_A^{(+)} + x_A^{(-)}. \tag{22}$$

Comparing (22) with (15) we obtain:

$$x_A^{(+)} + x_A^{(-)} = x_A^{(1)}. \tag{23}$$

The equalities (23) and (16) yield:

$$x = x_A^{(0)} + x_A^{(+)} + x_A^{(-)} \qquad (x_A^{(0)} \in H_A^0, x_A^{(+)} \in H_A^{(+)}, x_A^{(-)} \in H_A^{(-)}). \tag{24}$$

The A-semi-norm belonging to (19) is

$$\|x\|_A^{(\ )} = \left((x_A^{(+)}, x_A^{(+)})_A - (x_A^{(-)}, x_A^{(-)})_A\right)^{\frac{1}{2}} \qquad (x \in H). \tag{25}$$

Taking the analogous definition (13), (12) of $\|x\|_A$ and the relations (21), (17) into account we see that

$$\|x\|_A = \|x\|_A^{(\ )} \qquad (x \in H).$$

But, according to (23), (24) and (25),

$$\|x\|_A^{(\ )} = \|x_A^{(1)}\|_A^{(\ )} \qquad (x \in H).$$

Hence

$$\|x\|_A = \|x_A^{(1)}\|_A^{(\ )} \qquad (x \in H).$$

Consequently, it is sufficient to show that for any two A-fundamental decompositions, which are of the form (19) and satisfy (20), the corresponding A-semi-norms (25) are topologically equivalent on the closed subspace $H_A^{(1)}$. As, by virtue of (10) and (14), the A-inner product is non-degenerate on $H_A^{(1)}$, i.e. for $x \in H_A^{(1)}$ $(x \neq 0)$ there is an element $y \in H_A^{(1)}$ such that $(x, y)_A \neq 0$, the statement of our lemma follows from WITTSTOCK's theorem ([8], Theorem 15; cf. also [9]).

In the special case $A = I$ Lemma 5 (or WITTSTOCK's theorem itself) asserts that the topology of H does not depend on the choice of the fundamental decomposition (4).

Lemma 6. *If the component $H_A^{(+)}$ $(H_A^{(-)})$ of an A-fundamental decomposition* (19) *is intrinsically A-complete then the respective component H_A^+ (H_A^-) of any other A-fundamental decomposition* (9) *is also intrinsically A-complete.*

Proof. We denote by P the projection operator belonging to the subspace H_A^+ and the decomposition (9), i. e.

$$Px = x_A^+ \qquad (x \in H)$$

where x_A^+ is defined by (11).

According to (12) and (13) we have

$$\|x_A^+\|_A^2 + \|x_A^-\|_A^2 = \|x\|_A^2 \qquad (x \in H). \tag{26}$$

On the other hand, $\|x_A^+\|_A^2 - \|x_A^-\|_A^2 = (x, x)_A \geqq 0$ for $x \in H_A^{(+)}$, so that

$$0 \leqq \|x_A^-\|_A^2 \leqq \|x_A^+\|_A^2 \qquad (x \in H_A^{(+)}). \tag{27}$$

Using (27) we obtain from (26)

$$\|x_A^+\|_A \leqq \|x\|_A \leqq \sqrt{2}\, \|x_A^+\|_A \qquad (x \in H_A^{(+)}). \tag{28}$$

Since $\|\,.\,\|_A$ is a norm on both of the A-positive subspaces $H_A^{(+)}$, H_A^+, the relations (28) and the definition of P imply that with respect to $\|\,.\,\|_A$ the operator

P induces a topological isomorphism (a linear, one-to-one, bicontinuous mapping) between $H_A^{(+)}$ and the subspace $PH_A^{(+)} \subset H_A^+$ (cf. [7]).

If $H_A^{(+)}$ is intrinsically A-complete then it is complete in the norm

$$|x|_{A, H_A^{(+)}} = \|x\|_A^{(\ )} \qquad (x \in H_A^{(+)})$$

(see (8) and (25)) and, as a consequence of Lemma 5, in the norm $\| \, . \, \|_A$. Therefore the image $PH_A^{(+)}$ is also complete in the norm

$$\|x\|_A = |x|_{A, H_A^+} = |x|_{A, PH_A^{(+)}} \quad (x \in PH_A^{(+)}).$$

In other words, the subspace $PH_A^{(+)}$ is intrinsically A-complete.

We shall show that $PH_A^{(+)} = H_A^+$. Assuming the contrary, the intrinsic A-completeness of $PH_A^{(+)}$ would imply the existence of an element $x_0 \in H_A^+$ $(x_0 \neq 0)$ which is A-orthogonal to $PH_A^{(+)}$. Then x_0 is A-orthogonal to $H_A^{(+)}$, so that the span of x_0 and $H_A^{(+)}$ is an A-positive extension of $H_A^{(+)}$. But this is impossible because, in virtue of (19), any subspace properly containing $H_A^{(+)}$ has a non-trivial intersection with the A-non-positive subspace $H_A^0 \dot{+} H_A^{(-)}$.

For an intrinsically A-complete $H_A^{(-)}$ the proof is similar.

In the special case $A = I$ we obtain that the components of any fundamental decomposition of a J-space are intrinsically complete.

Consider an A-positive or A-negative subspace $L \subset H$. The dimension of the completion of L with respect to the intrinsic A-norm (8) will be called the *intrinsic A-dimension* of L. It is equal to the minimal power of those systems in L which are complete in L with respect to (8). The equivalence of the two definitions follows essentially by the same argument as the separability of the subsets of a separable metric space (see [10], Section 33).

Ju. L. Šmul'jan called our attention to the fact that for a closed A-positive or A-negative subspace L the intrinsic A-dimension coincides with the usual Hilbert dimension. This can be seen as follows.

L is a Hilbert space with respect to the definite inner product (5). Let L be A-positive. Then (Ax, y) is a continuous positive form on L, and there exists a continuous positive operator B acting in the Hilbert space L such that

$$(Ax, y) = [Bx, y] \qquad (x, y \in L).$$

Taking the positive square root $B^{\frac{1}{2}}$ we have

$$(Ax, y)) = [B^{\frac{1}{2}}x, B^{\frac{1}{2}}y] \qquad (x, y \in L).$$

Therefore if a system $\{e_\gamma\}_{\gamma \in \Gamma}$ is complete in L with respect to the A-inner product then $\{B^{\frac{1}{2}}e_\gamma\}_{\gamma \in \Gamma}$ is complete in $B^{\frac{1}{2}}L$ with respect to the definite inner product. As $B^{\frac{1}{2}}L$ is dense in L we obtain that the Hilbert dimension of L is not greater than the intrinsic A-dimension of L. The converse inequality is trivial.

Instead of "intrinsic I-dimension" we shall use the term *intrinsic dimension.*

Lemma 7. *Let $L^+(L^-)$ be an A-positive (A-negative) subspace of the J-space H, and let* (9) *denote any A-fundamental decomposition of H. Denote the intrinsic A-dimensions of L^+ and H_A^+ (L^- and H_A^-) by d^+ and k_A^+ (d^- and k_A^-) respectively. Then $d^+ \leqq k_A^+$ ($d^- \leqq k_A^-$). In particular, the cardinal numbers k_A^+, k_A^- do not depend on the choice of the A-fundamental decomposition.*

Proof. In the same way as it has been done in the first half of the preceding proof one can show that, with respect to the A-seminorm $\|\,.\,\|_A$ belonging to the decomposition (9), L^+ is topologically isomorphic to a subspace of H_A^+. Observing that

$$\|x\|_A = |x|_{A, H_A^+} \qquad (x \in H_A^+)$$

we obtain the inequality $d_1^+ \leqq k_A^+$ where d_{1A}^+ stands for the dimension of the completion of L^+ with respect to $\|\,.\,\|_A$, i. e. for the minimal power of systems in L^+ which are complete in L^+ with respect to $\|\,.\,\|_A$.

On the other hand, for $x \in L^+$ we have

$$|x|_{A, L^+}^2 = (x, x)_A = (x_A^+, x_A^+)_A + (x_A^-, x_A^-)_A \leqq (x_A^+, x_A^+)_A - (x_A^-, x_A^-)_A = \|x\|_A^2.$$

Therefore $d^+ \leqq d_1^+$.

We have proved that $d^+ \leqq k_A^+$. The inequality $d^- \leqq k_A^-$ can be verified similarly.

4. The representation $A = C^*C$

Theorem 1. *Consider a continuous self-adjoint operator A in the J-space H. Denote by k^+ and k^- the intrinsic dimension of the positive resp. negative component of a fundamental decomposition, and by k_A^+ and k_A^- the intrinsic A-dimension of the A-positive resp. A-negative component of an A-fundamental decomposition of H. Then A admits a representation* (1) *with a continuous linear operator C if and onlv if*

(29) $$k_A^+ \leqq k^+$$

and

(30) $$k_A^- \leqq k^-.$$

Proof. First we remark that (1) is equivalent to the identity

(31) $$(Ax, y) = (Cx, Cy) \qquad (x, y \in H).$$

Now we assume that A and C satisfy (31). Applying Lemma 2 we choose some A-fundamental decomposition (9) and put $CH_A^+ = R^+$. Then, in virtue of (31), R^+ is a positive subspace, and C is a linear one-to-one mapping of H_A^+ onto R^+.

Applying the relation (31) once again we obtain that the intrinsic dimension r^+ of R^+ is equal to the intrinsic A-dimension k_A^+ of H_A^+. But, according to Lemma 7, $r^+ \leqq k^+$. Therefore (29) is valid. The inequality (30) can be proved in the same way.

Conversely, let the operator A satisfy the relations (29), (30). Consider a regular A-canonical decomposition (9) (Lemma 2) and a fundamental decomposition (4) of the space H.

The completion $\tilde{H}_A^+$ of H_A^+ with respect to the A-inner product is a Hilbert space of dimension k_A^+ (Lemma 7). On the other hand, H^+ is a Hilbert space of dimension k^+ with respect to the original inner product (Lemmas 6 and 7). It follows from (29) that there exists a linear isometric imbedding of $\tilde{H}_A^+$ into H^+. Restricting the imbedding operator to H_A^+ we obtain a linear operator C^+ such that

$$(Ax, y) = (C^+x, C^+y) \qquad (x, y \in H_A^+). \tag{32}$$

Analogously, one can find a linear operator C^-, which maps H_A^- into H^- and has the property

$$(Ax, y) = (C^-x, C^-y) \qquad (x, y \in H_A^-). \tag{33}$$

For an arbitrary element (11) we define

$$Cx = C^+x_A^+ + C^-x_A^- \qquad (x \in H). \tag{34}$$

C is a linear operator of H to itself. Moreover, as a consequence of (32), (33), the orthogonality of the decomposition (4), the A-orthogonality of the decomposition (9), and the A-orthogonality of H_A^0 to H (Lemma 3), C fulfils the relation (31).

It remains to prove that C is continuous. For this purpose we apply the norm (6) which belongs to the fundamental decomposition (4) occuring in the above construction (cf. Lemma 5). We have

$$\|Cx\|^2 = (Cx_A^+, Cx_A^+) - (Cx_A^-, Cx_A^-) = (Ax_A^+, x_A^+) - (Ax_A^-, x_A^-).$$

Therefore, by Lemma 1, one obtains

$$\|Cx\|^2 \leqq \|A\| (\|x_A^+\|^2 + \|x_A^-\|^2). \tag{35}$$

As we are considering a regular A-fundamental decomposition, H_A^0 and $H_A^+ \dotplus H_A^-$ are closed subspaces of the complete space H (Lemma 4). Thus, according to a well-known corollary to BANACH's theorem, $x_A^+ \dotplus x_A^-$ depends continuously on x. On the other hand, H_A^+ and H_A^- are closed subspaces of H (Lemma 4) and, consequently, they are closed subspaces of $H_A^+ \dotplus H_A^-$. A second application of the Banach theorem yields that x_A^+ and x_A^- depend continuously on $x_A^+ + x_A^-$. As a result, x_A^+ and x_A^- are continuous functions of the element x. This fact together with the relation (35) implies the continuity of the operator C.

The theorem is proved.

In the following we consider some consequences of Theorem 1.

Theorem 2. *Let A denote a self-adjoint operator in an n-dimensional space H of type H_k $(0 \leqq k \leqq n < \infty)$. Then A admits a representation* (1) *with a linear C if and only if the following two conditions are fulfilled:*

α) H contains an A-non-positive subspace of dimension k;

β) H does not contain any A-negative subspace of dimension $k+1$.

Proof. With the notations of Theorem 1 we have:

$$k^- = k, \quad k^+ = n - k. \tag{36}$$

Consider an A-fundamental decomposition (9) and put $\dim H_A^0 = k_A^0$. If L is a subspace and $\dim L > k_A^0 + k_A^-$, then L has a non-trivial intersection with the A-positive subspace H_A^+. Analogously, if $\dim L > k_A^-$, then L has a non-trivial intersection with the A-non-negative subspace $H_A^0 \dotplus H_A^+$. Therefore $k_A^0 + k_A^-$ (k_A^-) is equal to the maximal dimension of A-non-positive (resp. A-negative) subspaces.

It follows from the foregoing that the conditions α), β) can be written in the form

$$k_A^0 + k_A^- \geqq k, \tag{37}$$

resp.

$$k_A^- \leqq k. \tag{38}$$

By virtue of (36) the relations (37), (38) are equivalent to the conditions (29), (30) in Theorem 1.

Theorem 3. *Consider an infinite-dimensional space H of type H_k, and a continuous self-adjoint operator A in H. The representation* (1), *where C denotes a continuous linear operator, is possible if and only if A satisfies condition β) of Theorem* 2.

Proof. One of the statements of the preceding proof remains valid for the present situation in the modified form that k_A^- is equal to the maximal dimension of A-negative subspaces, provided that one of these numbers is finite. Consequently, β) is equivalent to (38) or, what is the same, to (30) even now.

On the other hand, the ordinary dimension (i. e. the dimension with respect to a norm (6)) of H is $k^+ + k^- = k^+ + k = k^+$. As the continuity of the A-inner product (see Lemma 1) implies that the intrinsic A-dimension of the component H_A^+ of an A-fundamental decomposition (9) is not greater than the ordinary dimension of H_A^+, in our case the inequality (29) holds for every A.

Now the conclusion of our theorem follows from Theorem 1.

Theorem 4. *If the J-space H has infinite dimension, and the cardinal numbers k^+ and k^- defined in Theorem* 1 *are equal to each other, then any continuous self-adjoint operator A in H admits the representation* (1) *with a continuous linear C.*

Proof. In the present case both of the relations (29), (30) are always satisfied, since the ordinary dimension of H is equal to $k^+ + k^- = k^+ = k^-$, and any ordinary dimension in H, a fortiori (see Lemma 1) any intrinsic A-dimension in H, does not exceed this common value. Therefore our theorem is a consequence of Theorem 1.

References

[1] J. Bognár, Non-negativity properties of operators in spaces with indefinite metric, *Ann. Acad. Sci. Fenn.*, A. I, no. 336/10 (1963).

[2] J. Bognár, О существовании квадратного корня из оператора, самосопряженного относительно индефинитной метрики, *MTA Mat. Kut. Int. Közl.*, **6** (1961), 351—363.

[3] J. Bognár, О некоторых соотношениях между свойствами неотрицательности операторов в пространствах с индефинитной метрикой, *MTA Mat. Kut. Int. Közl.*, **8** (1963), 201—212.

[4] ——— II, *Studia Sci. Math. Hung.*, **1** (1966), 97—102.

[5] ——— III, *Studia Sci. Math. Hung.*, **1** (1966), 419—426.

[6] В. П. Потапов, Мультипликативная структура *J*-нерастягивающих матриц-функций, *Труды Московского матем. общества*, **4** (1955), 125—236.

[7] Ю. П. Гинзбург—И. С. Иохвидов, Исследования по геометрии бесконечномерных пространств с билинейной метрикой, *Успехи матем. наук*, **17** (1962), No. 4, 3—56.

[8] G. Wittstock, Über Zerlegungsmajoranten indefiniter Metriken, *Math. Zeitschr.*, **91**, (1966) 421—430.

[9] J. Bognár, On decomposition majorants of an indefinite metric, *Math. Zeitschr.*, **101** (1967), 65—67.

[10] F. Riesz—B. Sz.-Nagy, *Leçons d'analyse fonctionnelle* (3rd edition, Paris—Budapest, 1955).

(Received April 20, 1967)