Homomorphisms of partially ordered semigroups onto groups

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In a recent paper [1] L. FUCHS considered the order preserving homomorphisms of a partially ordered semigroup S with identity e onto a partially ordered group G. Assuming that the partial order in G is determined in a natural way by that in S, that the congruence classes are convex, and that e is greater than or equal to any element of S whose class is less than or equal to that of e, FUCHS determined all such homomorphisms. He showed that whenever S is generalized residuated (see below), the solution is a generalization of ARTIN's equivalence, which provided the answer for a commutative, residuated, semilattice semigroup with identity. (See [2], [3], [4]). The purpose of the present paper is to show that similar results may be obtained even if S has no identity, and even if S is not generalized residuated.

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Let S be a *partially ordered semigroup*. That is, S is a semigroup on which defined a partial order $\leqq$ with the property that for all $a, b, c \in S$, $a \leqq b$ implies $ac \leqq bc$ and $ca \leqq cb$. For $a, b \in S$ define the *generalized left residual* of a by b to be the set $\langle a \cdot . b \rangle = [x \in S \mid xb \leqq a]$, and the *generalized right residual* of a by b to be the set $\langle a . \cdot b \rangle = [x \in S \mid bx \leqq a]$. Using $\emptyset$ to denote the empty set, call S *generalized left (right) residuated* if $\langle a \cdot . b \rangle \neq \emptyset$ ($\langle a . \cdot b \rangle \neq \emptyset$) for all $a, b \in S$. If S is both right and left generalized residuated, call S generalized residuated.

If for $a, b \in S$ the set $\langle a \cdot . b \rangle$ ($\langle a . \cdot b \rangle$) is not empty and contains a maximum element, this element is called the *left (right) residual of a by b*, and is written $a \cdot . b$ ($a . \cdot b$); if $a \cdot . b$ ($a . \cdot b$) exists for all $a, b \in S$ then S is called *left (right) residuated.*

Define a *congruence relation* θ on S to be an equivalence relation satisfying, for all $a, b \in S$:

1. *$a \equiv b(\theta)$ implies $ac \equiv bc(\theta)$ and $ca \equiv cb(\theta)$.*

The set of congruence classes $\theta(a)$, $a \in S$, forms a semigroup S/θ if we define $\theta(a)\theta(b) = \theta(ab)$, but if we define an order relation $\leqq'$ on S/θ by setting:

$$\theta(a) \leqq' \theta(b) \quad \textit{if and only if there exist} \quad a' \in \theta(a), \quad b' \in \theta(b) \quad \textit{such that} \quad a' \leqq b',$$

then in general $\leqq'$ is not a partial order on S/θ. However, $\leqq'$ is compatible with the multiplication defined above, in the sense that for any $\theta(a)$, $\theta(b)$, $\theta(c) \in S/\theta$, $\theta(a) \leqq' \theta(b)$ implies $\theta(a)\theta(c) \leqq' \theta(b)\theta(c)$ and $\theta(c)\theta(a) \leqq' \theta(c)\theta(b)$; for if $a \in \theta(a)$, $b \in \theta(b)$,

$c\in\theta(c)$ with $a\leqq b$, then $ac\leqq bc$ implies $\theta(a)\theta(c)\leqq'\theta(b)\theta(c)$, and similarly for multiplication on the left.

If θ is a congruence relation on S which satisfies

2. $a\leqq c\leqq b$ and $a\equiv b(\theta)$ *imply* $a\equiv c(\theta)$,

we shall call θ a convex congruence relation on S.

Whenever θ is a convex congruence relation on S, with the properties that S/θ is a group and the identity class of S/θ contains an element f such that θ satisfies also:

3. $\theta(a)\leqq'\theta(f)$ *implies* $a\leqq f$,

it is true that $\leqq'$ is a partial order (so that S/θ is then a partially ordered group). For if now $\theta(a)\leqq'\theta(b)\leqq'\theta(a)$, choose $a'\in S$ such that $aa'\equiv f(\theta)$. Then since $\leqq'$ is compatible with multiplication in S/θ, $\theta(f)=\theta(aa')\leqq'\theta(ba')\leqq'\theta(aa')$, and so there exist $f'\in\theta(f)$ and $ba'\in\theta(ba')$ such that $f'\leqq ba'\leqq f$. By 2. it follows that $ba'\equiv f$, whence $\theta(b)=\theta(a)$. Thus $\leqq'$ is antisymmetric, and, similarly, transitive.

Note that we do not require that f be the identity of S, nor even that S have identity; in fact, f may not be idempotent.

Lemma 1. *The identity class of the group S/θ contains an element f satisfying* 3. *if and only if S contains an element $f=f^{\cdot}.f=f.{}^{\cdot}f$ satisfying* 3.

Proof. If $\theta(f)$ is the identity of S/θ, then $f^2\in\theta(f)\theta(f)=\theta(f)$ implies $f^2\leqq f$, while if $fx\leqq f$ then $\theta(fx)=\theta(f)\theta(x)=\theta(x)\leqq'\theta(f)$ implies that $x\leqq f$, so $f=f.{}^{\cdot}f$; similarly, $f=f^{\cdot}.f$. Conversely, if $f=f.{}^{\cdot}f=f^{\cdot}.f\in S$ satisfies 3., let $\theta(a)$ be the identity of S/θ. Then $\theta(f)=\theta(a)\theta(f)=\theta(af)$ implies $af\leqq f$, using 3. once more, and so $a\leqq f^{\cdot}.f=f$. Hence $\theta(a)\leqq'\theta(f)$. Now $f=f.{}^{\cdot}f$ satisfies $f^2\leqq f$, whence $\theta(f)\theta(f)\leqq'\theta(f)$ and $\theta(f)\leqq'\theta(a)$ in the group S/θ. As above, it follows that $\theta(a)=\theta(f)$.

If now $a\in S$ and $\langle f.{}^{\cdot}a\rangle\neq\emptyset$ ($\langle f^{\cdot}.a\rangle\neq\emptyset$), call $a'\in\langle f.{}^{\cdot}a\rangle$ ($\langle f^{\cdot}.a\rangle$) right (left) multiplicatively maximal in $\langle f.{}^{\cdot}a\rangle$ ($\langle f^{\cdot}.a\rangle$) if $x\in S$ and $a'x\in\langle f.{}^{\cdot}a\rangle$ ($xa'\in\langle f^{\cdot}.a\rangle$) imply $x\leqq f$. (See [1].)

Theorem 1. *Let S be a partially ordered semigroup containing an element $f=f.{}^{\cdot}f=f^{\cdot}.f$, and let θ be an equivalence relation on S satisfying* 1. *and* 2. *If S/θ is a group and θ satisfies* 3. *then:*

(i) *for any* $a\in S$, $\langle f.{}^{\cdot}a\rangle\neq\emptyset$, $\langle f^{\cdot}.a\rangle\neq\emptyset$, $\langle f.{}^{\cdot}a\rangle=\langle f^{\cdot}.a\rangle$,

(ii) *for any* $a\in S$, $\langle f^{\cdot}.a\rangle$ *contains left multiplicatively maximal elements.*

(iii) $\langle f^{\cdot}.a\rangle=\langle f^{\cdot}.b\rangle$ *if and only if* $a\equiv b(\theta)$.

Using Lemma 1, the proof is almost the same as that of Theorem 1 of [1], and we omit it. See also [5], p. 107.

Under the present hypotheses one cannot prove that S is generalized residuated; in general there may exist $a, b\in S$ such that $\langle a.{}^{\cdot}b\rangle=\emptyset$.

Theorem 2. *For any $a\in S$, $a.{}^{\cdot}a=f$ if and only if $af\leqq a$.*

Proof. For $a\in S$, the elements $a'\in S$ such that $aa'\equiv f(\theta)$ are right multiplicatively maximal in $\langle f.{}^{\cdot}a\rangle$, because $aa'\leqq f$ by 3., while if $a'x\in\langle f.{}^{\cdot}a\rangle$ for some $x\in S$ then $aa'x\leqq f$ implies $\theta(x)=\theta(aa')\theta(x)\leqq'\theta(f)$, so by 3. again, $x\leqq f$. Since $aa'\equiv f(\theta)$ is equivalent to $a'a\equiv f(\theta)$, and since $\langle f.{}^{\cdot}a\rangle=\langle f^{\cdot}.a\rangle$, a' is also left multiplicatively maximal in $\langle f.{}^{\cdot}a\rangle$. Now $a\in\langle f.{}^{\cdot}a'\rangle$, and if $ax\in\langle f.{}^{\cdot}a'\rangle=\langle f^{\cdot}.a'\rangle$ then $axa'\leqq f$, so $xa'\in\langle f.{}^{\cdot}a\rangle=\langle f^{\cdot}.a\rangle$. Since a' is left multiplicatively maximal in $\langle f^{\cdot}.a\rangle$

it follows that $x \leqq f$, and that a is then right, and similarly left, multiplicatively maximal in $\langle f .\cdot a' \rangle$.

Now suppose that $ax \leqq a$. For a' as above, $axa' \leqq aa' \leqq f$, so $xa' \in \langle f .\cdot a \rangle$; but a' is left multiplicatively maximal in this set, and therefore $x \leqq f$. Hence if $af \leqq a$ then the residual $a .\cdot a$ exists, and $a .\cdot a = f$; conversely, if $a .\cdot a = f$ then $af \leqq a$. It may happen that the only $a \in S$ for which $af \leqq a$ is $a = f$.

For the case considered in [1], where f is the identity of S, $a^{\cdot} . a = a .\cdot a = f$ for every $a \in S$; I am indebted to Mr. J. E. L'Heureux for this remark. Mme Dubreil-Jacotin points out [5], Lemma 5, that since $aff \leqq af$, one has $af .\cdot af = f$ for every $a \in S$.

Theorem 3. *Let S be a partially ordered semigroup containing an element $f = f^{\cdot} . f = f .\cdot f$, and let* (i) *and* (ii) *hold. Define a relation θ on S by* (iii). *Then θ satisfies* 1., 2. *and* 3., *and S/θ is a group.*

Proof. From the obvious properties of generalized residuals, θ satisfies 1 and 2. For 3., let $\theta(a) \leqq' \theta(f)$; there exist $a' \in \theta(a), f' \in \theta(f)$ such that $a' \leqq f'$, and then $f \in \langle f^{\cdot} . f \rangle = \langle f^{\cdot} . f' \rangle \subseteqq \langle f^{\cdot} . a' \rangle = \langle f^{\cdot} . a \rangle$, so $fa \leqq f$, $a \leqq f .\cdot f = f$. To show that S/θ is a group, note first that $\theta(f)$ is the identity of S/θ, for the following are equivalent: $x \in \langle f^{\cdot} . a \rangle$, $xa \leqq f$, $xaf \leqq f$, $x \in \langle f^{\cdot} . af \rangle$; that is, $\theta(a) = \theta(af) = \theta(a)\theta(f)$. Using $\langle f .\cdot a \rangle = \langle f^{\cdot} . a \rangle$, $\theta(f)$ is also a left identity for S/θ. Now let $a' \in S$ be left multiplicatively maximal in $\langle f^{\cdot} . a \rangle$, and let $x \in \langle f^{\cdot} . a'a \rangle$. Then $xa'a \leqq f$ implies $xa' \in \langle f^{\cdot} . a \rangle$, so $x \leqq f$ and $xf \leqq f^2 \leqq f$; that is, $x \in \langle f^{\cdot} . f \rangle$. Conversely, if $x \in \langle f^{\cdot} . f \rangle$ then $xf \leqq f$, $x \leqq f^{\cdot} . f = f$, $xa'a \leqq f^2 \leqq f$, $x \in \langle f^{\cdot} . a'a \rangle$. Hence $a'a \equiv f(\theta)$ and S/θ is a group, completing the proof.

A subset X of a semigroup S is said to be *reflective* if $ab \in X$ implies $ba \in X$. When S is partially ordered, an element $x \in S$ is called *reflective* if $ab \leqq x$ implies $ba \leqq x$. Mme. Dubreil-Jacotin proves [5], Theorem 7, that under the present hypotheses, f is reflective, and conversely, [5], Lemma 8, that if f is reflective, then $\langle f .\cdot a \rangle = \langle f^{\cdot} . a \rangle$ for any $a \in S$.

Let $H = [x \in S \mid x \leqq f]$. Clearly H is a subsemigroup of S, $x \in H$ and $y \leqq x$ imply $y \in H$, H is reflective, and for any $a \in S$ there exists $a' \in S$ such that $aa'x \in H$ implies $x \in H$. Thus H satisfies the conditions of Theorem 1 of [6], and so $\theta(a) = \theta(b)$ if and only if $H : a = H : b$, where $H : a = [x \in S \mid ax \in H] = [x \in S \mid xa \in H]$.

Recalling that the identity of a partially ordered group G is the maximum $x \in G$ satisfying $x^2 \leqq x$ in G, we note that with the present hypotheses the element f above is the maximum element of S which satisfies $x^2 \leqq x$ in S. For since $f = f .\cdot f$, certainly $f^2 \leqq f$; while if $x^2 \leqq x$ in S then $\theta(x)\theta(x) \leqq' \theta(x)$ in the group S/θ, so $\theta(x) \leqq' \theta(f)$, and by 3., $x \leqq f$. If S has identity e, then of course $e \equiv f(\theta)$, and also $e \leqq f = f^2$. Mme. Dubreil-Jacotin notes [5], Theorem 5, that f is maximum in each of the sets $U\langle a .\cdot a \rangle$, $U\langle a^{\cdot} . a \rangle$, where the unions are over all those $a \in S$ for which $\langle a .\cdot a \rangle \neq \emptyset$, $\langle a^{\cdot} . a \rangle \neq \emptyset$.

The following example may illustrate the situation. Let T be the set of points (p, i) in the plane, where $-\infty < p < 0$ and i is an integer, together with the point $(0, 0)$. Write a_{pi} for the point (p, i). Partially order T by setting $a_{pi} \leqq a_{qj}$ if and only if $p \leqq q$ and $i \leqq j$.

Define a multiplication $(\cdot)$ on T by setting

$$a_{pi} \cdot a_{qj} = a_{qj} \cdot a_{pi} = a_{\min(p,q), i+j}.$$

Then $S=T(\cdot)$ is a commutative generalized residuated semigroup with identity $f=a_{00}$, where

$$\langle a_{pi}\cdot . a_{qj}\rangle=[a_{rk}\in S \mid k\leqq i-j \text{ and (i) } -\infty<r\leqq 0 \text{ if } q\leqq p, \text{ (ii) } r\leqq p \text{ if } q>p].$$

For $i\neq j$ and $q\leqq p$, $\langle a_{pi}\cdot . a_{qj}\rangle$ does not contain a maximum element, so S is not residuated; yet for any $a_{pi}\in S$, $a_{pi}\cdot . a_{pi}=a_{pi} . \cdot a_{pi}=f$. If we define θ on S by (iii), then $a_{pi}\theta a_{qj}$ if and only if $i=j$, and θ satisfies 1., 2. and 3. The θ-classes are lines parallel to the x-axis, and S/θ is isomorphic to the additive, linearly ordered group of the integers. The multiplicatively maximal elements in $\langle a_{00}\cdot . a_{pi}\rangle$ are the elements $a_{q,-i}$, where $-\infty<q\leqq 0$, and these are exactly the elements $a'\in S$ such that $a' a_{pi}\equiv f(\theta)$. Except when $i=0$, the set $[a_{q,-i} \mid -\infty<q\leqq 0]$ has no maximum element.

Now define $(*)$ on T by: $a_{pi}*a_{qj}=a_{qj}*a_{pi}=a_{-1,i+j}$.

Then $S'=T(*)$ is a commutative partially ordered semigroup without identity. Write $f=a_{00}$. We have

$$\langle a_{pi}\cdot . a_{qj}\rangle=\emptyset \quad \text{if} \quad p<-1=$$
$$=[a_{rk}\in S \mid k\leqq i-j,\ -\infty<r\leqq 0 \quad \text{if} \quad p\geqq -1].$$

Clearly S' is not generalized residuated, but $f=f\cdot . f=f . \cdot f\in S'$ and $\langle f\cdot . a_{pi}\rangle$ is non-empty for $a_{pi}\in S'$. If θ is defined by (iii), conditions 1., 2. and 3. are satisfied, the θ-classes are the same as those in S, and S'/θ is isomorphic to the integers. The multiplicatively maximal elements in $\langle a_{00}\cdot . a_{pi}\rangle$ are as in S. For $p\geqq -1$, $fa_{pi}\leqq a_{pi}$ and $a_{pi}\cdot . a_{pi}=f$, but for $p<-1$, $\langle a_{pi}\cdot . a_{pi}\rangle=\emptyset$. Finally, $f^2=a_{-1,0}<f$.

References

[1] L. Fuchs, On group homomorphic images of partially ordered semigroups, *Acta Sci. Math.*, **25** (1964), 139—142.

[2] ——— *Partially ordered algebraic systems* (Oxford—London—New York—Paris, 1963).

[3] M. L. Dubreil-Jacotin, L. Lesieur et R. Croisot, *Théorie des treillis, des structures algébriques ordonnées et des treillis géométriques* (Paris, 1953).

[4] I. Molinaro, Demi-groupes résidutifs. I, *J. math. pures appl.*, (9) **39** (1960), 319—356.

[5] Mme. Dubreil-Jacotin, Sur les images homomorphes d'un demi-groupe ordonné, *Bull. Soc. Math. France*, **92** (1964), 101—115.

[6] A. Bigard, Sur les images homomorphes d'un demi-groupe ordonné, *C. R. Acad. Sc. Paris*, **260** (1965), 5987—5988.

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