A note on the preceding paper by J. B. Miller

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1. Introduction

Let $\mathfrak{A}$ be an associative algebra over a field K of zero characteristic, $\mathfrak{A}_k[t]$ the algebra, over $\mathfrak{A}$, of polynomials of degree $\leqq k$ in a commutative indeterminate t with the usual multiplication modulo the principal ideal (t^{k+1}). We consider (algebra) homomorphisms of $\mathfrak{A}$ into $\mathfrak{A}_k[t]$. Much of the definitions and notation of [1] will be used, without further explanation.

Suppose H is a homomorphism of $\mathfrak{A}$ into $\mathfrak{A}_k[t]$ so that, if $a \in \mathfrak{A}$,

$$H(a) = a_0 + ta_1 + t^2 a_2 + \ldots + t^k a_k.$$

Writing $a_0 = \varphi(a)$, $a_i = F_i(a)$ $(i = 1, 2, \ldots, k)$ it is clear that the maps φ, F_i are linear transformations over $\mathfrak{A}$. Furthermore, since H is a homomorphism it follows, if $a, b \in \mathfrak{A}$, that

(i) $$\varphi(ab) = \varphi(a)\varphi(b),$$

(ii) $$F_i(ab) = \sum_{j=0}^{i} F_j(a) F_{i-j}(b),$$

where $F_0 = \varphi$.

The problem is to obtain a representation for H in terms of transformations on $\mathfrak{A}$ of some given type. This has been done in [1] under the supposition that φ is the identity endomorphism, I, on $\mathfrak{A}$. For completely general endomorphisms the problem appears intractible but under suitable restrictions a solution can be obtained.

To be more specific, let φ be an endomorphism in $\mathfrak{B}(\mathfrak{A})$. A homomorphism H of $\mathfrak{A}$ into $\mathfrak{A}_k[t]$ will be called a φ-homomorphism if

(a) φ is the endomorphism determined from H by $a_0 = \varphi(a)$.

(b) $\varphi(F_n(a)) = F_n(\varphi(a))$ $(n = 1, 2, \ldots, k)$.

Thus the ε-homomorphisms of [1] are I-homomorphisms in this nomenclature.

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An operator $D \in \mathfrak{B}(\mathfrak{A})$ will be called a φ-derivation if it commutes with φ and satisfies

$$D(ab) = D(a)\varphi(b) + \varphi(a)D(b) \qquad (a, b \in \mathfrak{A}).$$

Thus F_1 of (ii) is a φ-derivation if H is a φ-homomorphism.

Finally, the endomorphism φ will be called averaging if

$$\varphi(a\varphi(b)) = \varphi(a)\varphi(b) = \varphi(\varphi(a)b) \qquad (a, b \in \mathfrak{A}).$$

Clearly any idempotent endomorphism is averaging and, conversely, if $\mathfrak{A}$ has an identity, or if the range of φ contains an element which is not a left (or right) divisor of zero, then φ is idempotent.

The results of [1] are extended to φ-homomorphisms for φ an idempotent endomorphism and for φ an averaging endomorphism.

The author wishes to express his thanks to Professor J. B. Miller for suggesting the possibility of extending the results of [1] and for his constant encouragement and guidance in preparation of this paper.

2. Representations of φ-homomorphisms

Lemma. *Let $\varphi \in \mathfrak{B}(\mathfrak{A})$ be an averaging endomorphism and $D \in \mathfrak{B}(\mathfrak{A})$ a φ-derivation. Then for $a, b \in \mathfrak{A}$, $n = 1, 2, \ldots$,*

$$\varphi(D^n(ab)) = \varphi\left\{\sum_{i=0}^{n} \binom{n}{i} D^i(a)D^{n-i}(b)\right\}.$$

Proof. The result is clear for $n=1$. For $n=2$ we have, if $a, b \in \mathfrak{A}$,

$$\varphi(D^2(ab)) = \varphi(D(D(a)\varphi(b) + \varphi(a)D(b))) =$$

$$= \varphi(D^2(a)\varphi^2(b) + \varphi(D(a))\varphi(D(b)) + \varphi(D(a))\varphi(D(b)) + \varphi^2(a)D^2(b)) =$$

$$= \varphi(D^2(a)b + 2D(a)D(b) + aD^2(b))$$

since

$$\varphi(\varphi(x)y) = \varphi(x\varphi(y)) = \varphi(x)\varphi(y) = \varphi(xy)$$

if $x, y \in \mathfrak{A}$. An inductive argument gives the general case.

Corollary. *With φ, D as in the lemma define $\varphi \exp D$ as the (formal) sum $\sum_{n=0}^{\infty} \frac{1}{n!} \varphi D^n$. If $\varphi \exp D$ defines an operator in $\mathfrak{B}(\mathfrak{A})$, in particular if D is nilpotent, then this operator is an endomorphism.*

Proof. Follows from the lemma in the obvious manner.

Theorem. *Let $\varphi \in \mathfrak{B}(\mathfrak{A})$ be a given averaging endomorphism and $D_1, D_2, \ldots, \ldots, D_k \in \mathfrak{B}(\mathfrak{A})$ given φ-derivations. Let $\boldsymbol{\varphi}$ be the operator in $\mathfrak{T}_{k+1}(\mathfrak{B})$ with φ along the leading diagonal and zero elsewhere. Then $\boldsymbol{\varphi} \exp(\boldsymbol{D}_{k+1})$, as defined above, is the matrix in $\mathfrak{T}_{k+1}(\mathfrak{B})$ of a φ-homomorphism of $\mathfrak{A}$ into $\mathfrak{A}_k[t]$ with $F_1 = \varphi D_1$.*

Conversely, given a φ-homomorphism H of $\mathfrak{A}$ into $\mathfrak{A}_k[t]$, with φ idempotent, its matrix $\boldsymbol{F}_{k+1}$ satisfies

$$\boldsymbol{\varphi F}_{k+1} = \boldsymbol{\varphi} \exp(\boldsymbol{D}_{k+1})$$

where $D_1, D_2, \ldots, D_k \in \mathfrak{B}(\mathfrak{A})$ are φ-derivations and D_1 can be taken as F_1. Furthermore the φD_i are uniquely determined.

Proof. The proof of the first part is exactly as in [1]. For the partial converse, note that if H is a φ-homomorphism of $\mathfrak{A}$ into $\mathfrak{A}_k[t]$, with φ idempotent, then $H\varphi$ is a I-homomorphism of $\varphi(\mathfrak{A})$ into $\varphi(\mathfrak{A})_k[t]$. Thus by [1] the matrix $\boldsymbol{F}_{k+1}\boldsymbol{\varphi}$ of $H\varphi$ satisfies

$$\text{(iii)} \qquad \boldsymbol{F}_{k+1}\boldsymbol{\varphi} = \exp(\boldsymbol{\Delta}_{k+1})$$

where $\boldsymbol{F}_{k+1}$ is the matrix of H, and the operators $\Delta_1, \Delta_2, \ldots, \Delta_k$ of $\boldsymbol{\Delta}_{k+1}$ are derivations on $\varphi(\mathfrak{A})$. For $i=1, 2, \ldots, k$ define D_i by

$$D_i(a) = \begin{cases} 0 & \text{if } \ a \in \ker(\varphi), \\ \Delta_i(a) & \text{if } \ a \in \operatorname{im}(\varphi) \end{cases}$$

and extend D_i linearly to the whole of $\mathfrak{A}$. The resulting operator is well defined since $\mathfrak{A} = \ker(\varphi) \oplus \operatorname{im}(\varphi)$ (see § 3 below). But then if $a_j = x_j + y_j$, $x_j \in \ker(\varphi)$, $y_j \in \operatorname{im}(\varphi)$ for $j=1, 2$,

$$D_i(a_1 a_2) = D_i(x_1 x_2 + y_1 x_2 + x_1 y_2 + y_1 y_2) = D_i(y_1 y_2)$$

since $\ker(\varphi)$ is a two-sided ideal. Since $\operatorname{im}(\varphi)$ is a subalgebra

$$D_i(a_1 a_2) = \Delta_i(y_1 y_2) =$$

$$= \Delta_i(y_1) y_2 + y_1 \Delta_i(y_2) = D_i(a_1)\varphi(a_2) + \varphi(a_1) D_i(y_2);$$

moreover, $D_i = D_i\varphi = \varphi D_i$ and so D_i is a φ-derivation on $\mathfrak{A}$, $i=1, 2, \ldots, k$.

Also, if $a \in \mathfrak{A}$, $D_i(a) = \Delta_i(\varphi(a))$, so $\Delta_i\varphi = D_i$. By (iii) then, since $\boldsymbol{\varphi}^2 = \boldsymbol{\varphi}$ and $\Delta_i\varphi = \varphi\Delta_i\varphi$ for each i,

$$\boldsymbol{F}_{k+1}\boldsymbol{\varphi} = \exp(\boldsymbol{\Delta}_{k+1})\boldsymbol{\varphi} = \boldsymbol{\varphi} \exp(\boldsymbol{\Delta}_{k+1}\boldsymbol{\varphi}) = \boldsymbol{\varphi} \exp(\boldsymbol{D}_{k+1})$$

and the result follows.

Remarks. 1. The above result remains valid if the base field D has nonzero characteristic p, provided $k < p$. This restriction ensures that $\boldsymbol{\varphi} \exp \boldsymbol{D}_{k+1}$ is defined.

2. If it is supposed that $\varphi F_n = F_n \varphi = F_n$ $(n=1, 2, \ldots, k)$ then 'the process P' of [1] can be applied to give results analogous to those in [1] with φ-derivations in place of derivations.

3. The result for $k = \infty$ generalizes in the same manner.

3. Existence of φ-derivations

Lemma. *An algebra $\mathfrak{A}$ admits an idempotent endomorphism if and only if it admits a (vector space) direct sum representation $\mathfrak{A}=\mathfrak{J}\oplus\mathfrak{B}$ where $\mathfrak{J}$ is a two-sided ideal and $\mathfrak{B}$ is a subalgebra.*

Proof. If φ is an idempotent endomorphism, let $\mathfrak{J}=\ker(\varphi)$, $\mathfrak{B}=\operatorname{im}(\varphi)$. Then $\mathfrak{J}$ is a two sided ideal and $\mathfrak{B}$ is a subalgebra. If $a\in\mathfrak{A}$, $a=(a-\varphi(a))+\varphi(a)$ so $\mathfrak{A}=\mathfrak{J}+\mathfrak{B}$. Since φ is zero on $\mathfrak{J}$ and is the identity on $\mathfrak{B}$ it follows that the sum is direct.

Conversely, if $\mathfrak{A}=\mathfrak{J}\oplus\mathfrak{B}$ for some two-sided ideal $\mathfrak{J}$ and subalgebra $\mathfrak{B}$, define a transformation φ on $\mathfrak{A}$ as follows. If $a=x+y$, $x\in\mathfrak{J}$, $y\in\mathfrak{B}$, set $\varphi(a)=y$. Then φ is clearly idempotent and linear. Also, if $a_1=x_1+y_1$, $a_2=x_2+y_2$

$$\varphi(a_1a_2) = \varphi(x_1x_2+y_1x_2+x_1y_2+y_1y_2) =$$
$$= \varphi(y_1y_2) = y_1y_2 = \varphi(a_1)\varphi(a_2).$$

Thus φ is an idempotent endomorphism.

Theorem. *Let $\mathfrak{A}$ be an associative algebra which has a direct sum representation $\mathfrak{A}=\mathfrak{J}\oplus\mathfrak{B}$ as in the lemma, with $\mathfrak{B}$ non-commutative. Then $\mathfrak{A}$ admits a non-zero φ-derivation D such that $\varphi D=D\varphi=D$, φ being defined as in the lemma.*

Proof. Let $a\in\mathfrak{A}$ be any element such that $\varphi(a)$ is not in the centre of $\mathfrak{B}$. Straightforward calculation shows that the operator D defined by

$$D(x)=\varphi(ax-xa), \qquad x\in\mathfrak{A},$$

has the desired properties.

Corollary. *Let $\Delta_1, \Delta_2, \dots$ be inner derivations of $\mathfrak{A}$, $D_1, D_2, \dots$ the corresponding φ-derivations as defined in the preceding theorem, that is, $D_i=\varphi\Delta_i$, $i=1, 2, \dots$. Then $\boldsymbol{\varphi}\exp(\boldsymbol{D})=\boldsymbol{\varphi}\Gamma$ where Γ is an inner endomorphism in $\mathfrak{T}_\infty(\mathfrak{A})$ determined by the $\Delta_1, \Delta_2, \dots$ as in* [1].

Proof. By § 4 of [1],

$$\exp(\boldsymbol{\Delta})\boldsymbol{A}=\exp(\boldsymbol{C})\,\boldsymbol{A}\exp(-\boldsymbol{C}), \qquad \boldsymbol{A}\in\mathfrak{T}_\infty(\mathfrak{A}),$$

whence

$$(\boldsymbol{\varphi}\exp(\boldsymbol{\Delta}))\boldsymbol{A}=(\boldsymbol{\varphi}\Gamma)\boldsymbol{A}.$$

But

$$\boldsymbol{\varphi}\exp(\boldsymbol{D})=\boldsymbol{\varphi}\exp(\boldsymbol{\varphi\Delta})=\boldsymbol{\varphi}\exp\boldsymbol{\Delta},$$

and the result follows.

Bibliography

[1] J. B. Miller, Homomorphisms, higher derivations, and derivations of associative algebras, *Acta Sci. Math.* **28** (1967), 221—232.

(Received September 21, 1966)