On a generalization of completely 0-simple semigroups

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§ 1. Introduction

The well-known theorem of REES characterizes the completely 0-simple semigroups with the help of matrix semigroups over a group with zero. In this paper we generalize this theorem by giving a class of semigroups that are characterized as matrix semigroups over a semigroup with zero and identity.

In § 2 we introduce the notion of left (right) S-translation between two left (right) ideals of a semigroup S with 0. This notion is a generalization of right (left) translation of S in the sense of CLIFFORD—PRESTON [1]. Two left (right) ideals $\mathfrak{a}_1, \mathfrak{a}_2$ of S are called left (right) S-similar if there exists a one-to-one left (right) S-translation from $\mathfrak{a}_1$ onto $\mathfrak{a}_2$. In Proposition 2. 1 a necessary and sufficient condition is given in order that the left ideals Se_1, Se_2 ($e_i^2=e_i$; $i=1, 2$) of S be S-similar.

In § 3 we show that all 0-minimal left (right) ideals of a completely 0-simple semigroup are left (right) S-similar. Proposition 3. 4 gives the following characterization of the completely 0-simple semigroups: a semigroup S with zero is completely 0-simple if and only if S has the form $S=\bigcup_{\lambda\in\Lambda} Se_\lambda$ with idempotents e_λ where Se_λ are 0-minimal, left S-similar left ideals of S. In view of this result we define the following generalization of the completely 0-simple semigroups. Let S be a semigroup with 0 such that

$$S=\bigcup_{\lambda\in\Lambda} Se_\lambda=\bigcup_{i\in I} e_iS \qquad (e_\lambda^2=e_\lambda,\quad e_i^2=e_i;\quad 1\in I\cap\Lambda),$$

where Se_λ (e_iS) are left (right) S-similar left (right) ideals of S with $Se_\mu\cap Se_\nu=0$ ($\mu, \nu\in\Lambda; \mu\neq\nu$) and $e_jS\cap e_kS=0$ ($j, k\in I; j\neq k$). These semigroups are called S-similarly decomposable.

The theorem of REES states that a semigroup is completely 0-simple if and only if it is isomorphic to a regular Rees matrix semigroup over a group with zero. In order to give an analogous characterization of the S-similarly decomposable semigroups, we introduce the notion of the locally regular Rees matrix semigroup $M^\circ(H; I, \Lambda; P)$ over a semigroup H with zero and identity. (See at the end of § 3.) The regular Rees matrix semigroups are locally regular. Then we have: a semigroup S with zero is S-similarly decomposable if and only if it is isomorphic to a locally regular Rees matrix semigroup over a semigroup with zero and identity. (See Theorem 4. 1.) We intend to deal with the homomorphisms of a locally regular Rees matrix semigroup in another paper.

It is known that the Brandt semigroups are just the completely 0-simple inverse semigroups, therefore they have representations by special regular Rees matrix semigroups. (See Theorem 3. 9 in [1].) In § 5 we define the special S-similarly decomposable semigroups and we prove an analogous theorem concerning them. (See Theorem 5. 1.) It is interesting that these semigroups have an application in the theory of codes and finite-state transducers.

§ 2. On the translations

Let S be a semigroup with zero and $\mathfrak{l}_1, \mathfrak{l}_2$ left ideals of S. By a *left S-translation of* $\mathfrak{l}_1$ *into* $\mathfrak{l}_2$ we mean a single valued mapping φ of $\mathfrak{l}_1$ into $\mathfrak{l}_2$ such that

(2. 1) $$x\varphi \in \mathfrak{l}_2, \; s(x\varphi) = (sx)\varphi \qquad (\text{for all } x \in \mathfrak{l}_1 \text{ and } s \in S).$$

If ω is a left S-translation such that for every element x of $\mathfrak{l}_1$

$$x\omega = 0 \qquad (x \in \mathfrak{l}_1)$$

holds, then ω is called *the zero left S-translation of* $\mathfrak{l}_1$ *into* $\mathfrak{l}_2$.

Let a_2 be a fixed element of $\mathfrak{l}_2$. Then the mapping

(2. 2) $$x \to xa_2 \qquad (x \in \mathfrak{l}_1;\; a_2 \in \mathfrak{l}_2)$$

is a left S-translation of $\mathfrak{l}_1$ into $\mathfrak{l}_2$.

In the case $\mathfrak{l}_1 = \mathfrak{l}_2 = S$ the *left* S-translation of S into itself and the *right* translation of S in the sense of CLIFFORD—PRESTON [1] are the same notions.

Analogously, one can define the *right S-translation of the right ideal* $\mathfrak{r}_1$ *into the right ideal* $\mathfrak{r}_2$ *of S.*

We say that the left ideals $\mathfrak{l}_1, \mathfrak{l}_2$ of S are *left S-similar* [1]) if there exists a one-to-one left S-translation φ of $\mathfrak{l}_1$ onto $\mathfrak{l}_2$. It is easy to see that this notion defines an equivalence relation among the left ideals of S.

One can define dually the *right S-similarity* of right ideals.

Proposition 2. 1. *Let S be a semigroup with* 0 *and $e_1 \neq 0$, $e_2 \neq 0$ idempotents in S. Then the left ideals Se_1 and Se_2 are left S-similar if and only if there exist elements q_{12} and q_{21} in S such that*

(2. 3) $$e_1 q_{12} e_2 = q_{12}, \qquad e_2 q_{21} e_1 = q_{21},$$

(2. 4) $$q_{12} q_{21} = e_1, \qquad q_{21} q_{12} = e_2.$$

Proof. Let Se_1 and Se_2 be left S-similar and φ a one-to-one left S-translation of Se_1 onto Se_2. Set $e_1\varphi = q_{12} (\in Se_2)$, $e_2\varphi^{-1} = q_{21} (\in Se_1)$. Then in view of (2. 1) and $e_1^2 = e_1$, $e_2^2 = e_2$ the relations (2. 3) hold. Furthermore,

$$e_1 = (e_1\varphi)\varphi^{-1} = q_{12}\varphi^{-1} = (q_{12}e_2)\varphi^{-1} = q_{12}(e_2\varphi^{-1}) = q_{12}q_{21}.$$

Similarly $q_{21}q_{12} = e_2$.

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[1]) In his paper [2], H.-J. HOEHNKE defines a more general, analogous notion for the S-systems.

Conversely, assume that some q_{12} and q_{21} in S satisfy the relations (2. 3), (2. 4). Let φ be the mapping of Se_1 into Se_2 satisfying $(se_1)\varphi = se_1 q_{12}$ $(se_1 \in Se_1)$. Then $(se_1)\varphi = (te_1)\varphi$ (se_1 and $te_1 \in Se_1$) and (2. 4_1) imply

$$se_1 q_{12} = te_1 q_{12} \Rightarrow se_1 q_{12} q_{21} = te_1 q_{12} q_{21} \Rightarrow se_1 = te_1.$$

If ue_2 is an arbitrary element of Se_2 then because of (2. 4_2) $(uq_{21}e_1)\varphi = uq_{21}e_1 q_{12} = uq_{21}q_{12} = ue_2$. Thus φ is a one-to-one mapping of Se_1 onto Se_2 with property (2. 1), i.e. Se_1 and Se_2 are left S-similar.

A dual proposition holds on the right S-similar right ideals e_1S, e_2S of S.

Remark 1. It is easy to show that the conditions (2. 4) alone are sufficient to assure the left S-similarity of Se_1 and Se_2.

Since the conditions on e_1 and e_2 of Proposition 2. 1 are left-right symmetric, it is clear that we have the following

Corollary 2. 2. *Let S be a semigroup with zero and $e_1 \neq 0$, $e_2 \neq 0$ idempotents in S. Then the left ideals Se_1 and Se_2 are left S-similar if and only if the right ideals e_1S and e_2S are right S-similar.*

Proposition 2. 1 and Corollary 2. 2 are analogous to Proposition III. 7. 4 and its Corollary in JACOBSON [3].

An other consequence of Proposition 2. 1 is the following

Corollary 2. 3 (Cf. STEINFELD [6] Theorem 5. 4). *If the left ideals Se_1, Se_2 $(e_1^2 = e_1 \neq 0,\ e_2^2 = e_2 \neq 0)$ of a semigroup S with zero are left S-similar, then the subsemigroups e_1Se_1 and e_2Se_2 of S are isomorphic.*

Proof. Since the left ideals Se_1, Se_2 are left S-similar, elements q_{12} and q_{21} with properties (2. 3), (2. 4) exist. We shall show that

$$(2.\ 5)\qquad e_1se_1 \to q_{21}sq_{12} \qquad (e_1se_1 \in e_1Se_1)$$

is an isomorphism of e_1Se_1 onto e_2Se_2. For, let e_1se_1 and $e_1te_1 \in e_1Se_1$; then in view of (2. 5) and (2. 4_1)

$$e_1se_1 \cdot e_1te_1 \to q_{21}se_1tq_{12} = q_{21}sq_{12} \cdot q_{21}tq_{12}.$$

So (2. 5) is a homomorphism. Furthermore, if the images $q_{21}sq_{12}$ and $q_{21}tq_{12}$ of e_1se_1 and e_1te_1 are equal, then

$$(2.\ 6)\qquad e_1se_1 = q_{12} \cdot q_{21}sq_{12} \cdot q_{21} = q_{12} \cdot q_{21}tq_{12} \cdot q_{21} = e_1te_1.$$

Finally, let $e_2ue_2 \in e_2Se_2$. In view of (2. 4_2) the element $e_1q_{12}e_2ue_2q_{21}e_1$ of e_1Se_1 is mapped by (2. 5) upon the element $q_{21} \cdot q_{12}uq_{21} \cdot q_{12} = e_2ue_2$. Thus (2. 5) is an isomorphic mapping of e_1Se_1 onto e_2Se_2, indeed.

§ 3. On the completely 0-simple semigroups

Now we need the following

Proposition 3. 1 (STEINFELD [5] Satz 6). *Let* $\mathfrak{l}$ *be a* 0*-minimal left ideal of a semigroup S with zero and* $e \neq 0$ *an idempotent) in* $\mathfrak{l}$. *Then* $e\mathfrak{l}$ *is a group with zero.*

Let Se_1, Se_2 $(e_1^2 = e_1;\ e_2^2 = e_2)$ be two 0-minimal left ideals of a semigroup S with 0, and $a \in S$. By the 0-minimality of Se_2 either $Se_1ae_2 = Se_2$ or $Se_1ae_2 = 0$ holds.

The first possibility implies the existence of an element $be_1 \in Se_1$ such that $be_1ae_2 = e_2$. Hence $e_2be_1 \cdot e_1ae_2 = e_2^2 = e_2$. From this we get

$$e_1ae_2 \cdot e_2be_1 \cdot e_1ae_2 \cdot e_2be_1 = e_1ae_2 \cdot e_2 \cdot e_2be_1 = e_1ae_2 \cdot e_2be_1,$$

that is $e_1ae_2 \cdot e_2be_1 \in e_1Se_1$ is an idempotent. Since e_1Se_1 is a group with zero and $e_1ae_2 \cdot e_2be_1 \neq 0$, we obtain $e_1ae_2 \cdot e_2be_1 = e_1$. By Proposition 2. 1 and the properties of the elements e_1ae_2, e_2be_1, the left ideals Se_1 and Se_2 are left S-similar.

The second possibility implies that the mapping

$$se_1 \to se_1 \cdot e_1ae_2 = 0 \qquad (se_1 \in Se_1)$$

is the zero left S-translation of Se_1 into Se_2.

Thus we have: if an element ae_2 $(\in Se_2)$ exists such that $Se_1ae_2 = Se_2$, then Se_1 and Se_2 are left S-similar; if such an element does not exist, then the only left S-translation between Se_1 and Se_2 is the zero S-translation. Therefore:

Proposition 3. 2. *Let* Se_1, Se_2 $(e_1^2 = e_1;\ e_2^2 = e_2)$ *be* 0*-minimal left ideals of a semigroup S with zero. Then either* Se_1, Se_2 *are left S-similar or the only left S-translation between* Se_1 *and* Se_2 *is the zero left S-translation.*

These imply

Corollary 3. 3. *All* 0*-minimal left (right) ideals of a completely* 0*-simple semigroup S are left (right) S-similar.*

Proof. Let $\mathfrak{l}_1$, $\mathfrak{l}_2$ two 0-minimal left ideals of the completely 0-simple semigroup S. It is known that $\mathfrak{l}_i$ has the form $\mathfrak{l}_i = Se_i$ $(e_i^2 = e_i;\ i = 1, 2)$. In view of the 0-minimality of Se_2 the product $Se_1 \cdot ae_2 (a \in S)$ is either 0 or Se_2. As S is a 0-simple semigroup $Se_1S = S$ holds. Thus at least one element ae_2 $(\in Se_2)$ exists with $Se_1ae_2 = Se_2$. This and Proposition 3. 2 imply our assertion.

We shall prove the following characterization of completely 0-simple semigroups.

Proposition 3. 4 (cf. STEINFELD [7] Theorem 15). *A semigroup S with zero is completely* 0*-simple if and only if S has the form*

$$S = \bigcup_{\lambda \in \Lambda} Se_\lambda \qquad (e_\lambda^2 = e_\lambda) \tag{3. 1}$$

where Se_λ *are pairwise left S-similar* 0*-minimal left ideals of S.*

Proof. By Corollary 2.49 of [1], a completely 0-simple semigroup S is the union of its 0-minimal left ideals $\mathfrak{l}_\lambda$ $(\lambda \in \Lambda)$. As S is a regular semigroup we can write $\mathfrak{l}_\lambda = Se_\lambda$ $(e_\lambda^2 = e_\lambda;\ \lambda \in \Lambda)$. Thus, by Corollary 3.3, the necessity of the stated condition follows.

Conversely, let S be a semigroup with the stated properties. In view of Exercise 12 for § 2.7 of [1] it is enough to prove that S is 0-simple. As S has at least one non-zero idempotent, we have $S^2 \neq 0$. By (3.1), any ideal $\mathfrak{a}(\neq 0)$ of S has a non-zero element of the form $ae_\mu \in \mathfrak{a}(\mu \in \Lambda)$. Hence

$$0 \neq ae_\mu \in \mathfrak{a}Se_\mu \qquad (e_\mu^2 = e_\mu).$$

Because of the 0-minimality of Se_μ, this implies $Se_\mu = \mathfrak{a}Se_\mu \subseteqq \mathfrak{a}$. As Se_μ and every Se_λ $(\lambda \in \Lambda)$ are left S-similar, 0-minimal left ideals of S in view of Proposition 2.1

$$Se_\lambda = Se_\mu \cdot Se_\lambda \subseteqq \mathfrak{a} \cdot Se_\lambda \subseteqq \mathfrak{a} \qquad (\lambda \in \Lambda)$$

holds. This and (3.1) imply

$$S = \bigcup_{\lambda \in \Lambda} Se_\lambda \subseteqq \mathfrak{a}$$

establishing the 0-simplicity of S.

The dual characterization of the completely 0-simple semigroup S holds by the right S-similar, 0-minimal right ideals e_iS $(e_i^2 = e_i;\ i \in I)$ of S.

It is easy to show that the left ideal Se $(e^2 = e \neq 0)$ of the completely 0-simple semigroup S is 0-minimal if and only if eS is a 0-minimal right ideal of S, therefore one can suppose that in the decompositions

$$S = \bigcup_{\lambda \in \Lambda} Se_\lambda = \bigcup_{i \in I} e_iS$$

$1 \in I \cap \Lambda$ holds. Naturally the 0-minimal left ideals Se_λ $(\lambda \in \Lambda)$ in (3.1) are different, therefore $Se_\mu \cap Se_\nu = 0$ if $\mu \neq \nu$ and $\mu, \nu \in \Lambda$.

We now generalize the notion of completely 0-simple semigroups.

Let S be a semigroup with 0 such that

$$(3.2) \qquad S = \bigcup_{\lambda \in \Lambda} Se_\lambda = \bigcup_{i \in I} e_iS \qquad (e_\lambda^2 = e_\lambda;\quad e_i^2 = e_i;\quad 1 \in I \cap \Lambda)$$

where Se_λ $(\lambda \in \Lambda)[e_iS(i \in I)]$ are left [right] 0-similar left [right] ideals of S such that $Se_\mu \cap Se_\nu = 0$ $(\mu, \nu \in \Lambda;\ \mu \neq \nu)$ and $e_jS \cap e_kS = 0$ $(j, k \in I;\ j \neq k)$. We call a semigroup with these properties *S-similarly decomposable.*

By Proposition 3.4 and its dual, the completely 0-simple semigroups are S-similarly decomposable.

The well-known theorem of Rees (Clifford—Preston [1], Theorem 3.5) characterizes the completely 0-simple semigroups by the regular Rees matrix semigroups over a group with 0. In the next § we wish to give an analogous characterization of the S-similarly decomposable semigroups. For this characterization we need to generalize the notion of the regular Rees matrix semigroup.

Let H be a semigroup with 0 and with the identity element e. Let $M^\circ(H;\ I,\ \Lambda;\ P)$ denote the Rees matrix semigroup over H with a sandwich matrix $P = (p_{\lambda i})$ $(\lambda \in \Lambda;\ i \in I;\ p_{\lambda i} \in H)$. Denote the elements of M° by $(a)_{i\lambda}$ with a in H, i in I and λ in Λ. The product of the matrices $(a)_{i\lambda}$, $(b)_{j\mu}$ is defined by

$$(3.3) \qquad (a)_{i\lambda} \circ (b)_{j\mu} = (ap_{\lambda j}b)_{i\mu} \qquad (a, b \in H;\ i, j \in I;\ \lambda, \mu \in \Lambda).$$

We say that $M^\circ(H; I, \Lambda; P)$ is *locally regular* if $P=(p_{\lambda i})$ has the following properties:

1) in every row λ of P there exists an element $p_{\lambda j(\lambda)}$ $(j(\lambda)\in I)$ which has a right inverse $p'_{\lambda j(\lambda)}$ in H, that is

(3. 4) $$p_{\lambda j(\lambda)}\, p'_{\lambda j(\lambda)}=e;$$

2) in every column i of P there exists an element $p_{\mu(i)i}$ $(\mu(i)\in\Lambda)$ which has a left inverse $p''_{\mu(i)i}$ in H, that is

(3. 5) $$p''_{\mu(i)i}\, p_{\mu(i)i}=e;$$

3) there exists at least one element $p_{\lambda i}$ in P which has a (right and left) inverse in H.

A regular Rees matrix semigroup $M^\circ(G; I, \Lambda; P)$ over a group G with zero is locally regular.

§ 4. A generalization of the Rees theorem

Theorem 4. 1. *A semigroup S with zero is S-similarly decomposable if and only if it is isomorphic to a locally regular Rees matrix semigroup over a semigroup with zero and identity.*

Proof. Let S be a S-similarly decomposable semigroup. Then S has a decomposition (3. 2). In view of (3. 2) an arbitrary element $a\neq 0$ of S belongs to exactly one right ideal e_iS $(i\in I)$ and to exactly one left ideal Se_λ $(\lambda\in\Lambda)$. Hence

(4. 1) $$a=e_iae_\lambda \qquad (i\in I;\ \lambda\in\Lambda).$$

As every left ideal Se_λ $(\lambda\in\Lambda)$ is left S-similar to Se_1 $(1\in I\cap\Lambda)$ and every right ideal e_iS $(i\in I)$ is right S-similar to e_1S $(1\in I\cap\Lambda)$, by Proposition 2. 1 there exist elements $q_{1\lambda}$ $(\in e_1Se_\lambda)$, $q_{\lambda 1}(\in e_\lambda Se_1)$ and $r_{1i}(\in e_1Se_i)$, $r_{i1}(\in e_iSe_1)$ such that

(4. 2) $$q_{1\lambda}q_{\lambda 1}=e_1, \qquad q_{\lambda 1}q_{1\lambda}=e_\lambda$$

and

(4. 3) $$r_{1i}r_{i1}=e_1, \qquad r_{i1}r_{1i}=e_i.$$

Let $M^\circ(e_1Se_1; I, \Lambda; P)$ denote the Rees matrix semigroup over the semigroup e_1Se_1 with the sandwich matrix $P=(p_{\lambda i})=(q_{1\lambda}r_{i1})$. We shall prove that the mapping

(4. 4) $$\varphi\colon a=e_iae_\lambda\to(r_{1i}aq_{\lambda 1})_{i\lambda} \qquad (a\in S;\ i\in I;\ \lambda\in\Lambda)$$

is an isomorphism of S onto $M^\circ=M^\circ(e_1Se_1; I, \Lambda; P=(q_{1\lambda}r_{i1}))$. First we show that φ is one-to-one. If the images $(r_{1i}aq_{\lambda 1})_{i\lambda}$ and $(r_{1j}bq_{\mu 1})_{j\mu}$ of the elements $a=e_iae_\lambda$ and $b=e_jbe_\mu$ $(i,j\in I;\ \lambda,\mu\in\Lambda)$ are equal, then $i=j$; $\lambda=\mu$ and $r_{1i}aq_{\lambda 1}=r_{1i}bq_{\lambda 1}$. Hence, by (4. 2) and (4. 3),

$$a=e_iae_\lambda=r_{i1}\cdot r_{1i}aq_{\lambda 1}\cdot q_{1\lambda}=r_{i1}\cdot r_{1i}bq_{\lambda 1}\cdot q_{1\lambda}=e_ibe_\lambda=b.$$

φ is a homomorphism. For, let $a=e_iae_\lambda$ and $c=e_jce_\mu$ $(i,j\in I;\ \lambda,\mu\in\Lambda)$ be two elements of S. By (4. 4), (4. 2), (4. 3) and (3. 3) we get

$$ac=e_iae_\lambda e_jce_\mu\to(r_{1i}ae_\lambda e_jcq_{\mu 1})_{i\mu}=$$
$$=(r_{1i}aq_{\lambda 1}q_{1\lambda}r_{j1}r_{1j}cq_{\mu 1})_{i\mu}=(r_{1i}aq_{\lambda 1})_{i\lambda}\circ(r_{1j}cq_{\mu 1})_{j\mu}.$$

φ maps S onto M°. For, an arbitrary element $(e_1ue_1)_{i\lambda}$ of M° is the image of the element $r_{i1}uq_{1\lambda}$ of S:

$$(r_{1i}r_{i1}uq_{1\lambda}q_{\lambda 1})_{i\lambda}=(e_1ue_1)_{i\lambda}.$$

We still have to prove that the Rees matrix semigroup is locally regular, that is, the sandwich matrix $P=(p_{\lambda i})=(q_{1\lambda}r_{i1})$ fulfils the properties 1), 2) and 3). Let us consider the idempotent $e_\lambda\in Se_\lambda$ $(\lambda\in\Lambda)$. In view of (3. 2) there exists a $j=j(\lambda)\in I$ such that $e_\lambda\in e_jS$. Hence

$$e_\lambda=e_je_\lambda \qquad (j\in I;\ \lambda\in\Lambda). \tag{4. 5}$$

The element $p_{\lambda j}=q_{1\lambda}r_{j1}\in e_1Se_1$ in the λ-th row of the matrix P has a right inverse element $p'_{\lambda j}=r_{1j}q_{\lambda 1}\in e_1Se_1$, since because of (4. 2), (4. 3) and (4. 5)

$$p_{\lambda j}p'_{\lambda j}=q_{1\lambda}r_{j1}\cdot r_{1j}q_{\lambda 1}=q_{1\lambda}e_jq_{\lambda 1}=q_{1\lambda}e_je_\lambda q_{\lambda 1}=q_{1\lambda}e_\lambda q_{\lambda 1}=q_{1\lambda}q_{\lambda 1}=e_1.$$

Similarly, in the i-th column $(i\in I)$ of the matrix P the element $p_{\mu i}=q_{1\mu}r_{i1}$ $(\mu=\mu(i)\in\Lambda)$ has a left inverse $p''_{\mu i}=r_{1i}q_{\mu 1}\in e_1Se_1$.

By the assumption $1\in I\cap\Lambda$, the matrix $P=(p_{\lambda i})=(q_{1\lambda}r_{i1})$ has the entry $p_{11}=q_{11}r_{11}$. From (4. 2) and (4. 3) it follows that $p^*_{11}=r_{11}q_{11}$ satisfies $p_{11}p^*_{11}=$ $=q_{11}r_{11}\cdot r_{11}q_{11}=e_1$ and $p^*_{11}p_{11}=r_{11}q_{11}\cdot q_{11}r_{11}=e_1$. Thus $p_{11}=q_{11}r_{11}\in e_1Se_1$ has an inverse. Consequently, for the sandwich matrix $P=(p_{\lambda i})=(q_{1\lambda}r_{i1})$ conditions 1), 2) and 3) are fulfilled.

Conversely, let S be isomorphic to the locally regular Rees matrix semigroup $M^\circ(H;I,\Lambda;P)$ over the semigroup H with 0 and with identity e. Denote the elements of M° by $(a)_{i\lambda}$ $(a\in H;\ i\in I;\ \lambda\in\Lambda)$. Let $\mathfrak{l}_\lambda$ $(\lambda\in\Lambda)$ be the set of the matrices $(a)_{i\lambda}$ for all $a\in H$ and $i\in I$. From (3. 3) it follows that $\mathfrak{l}_\lambda$ is a left ideal of M°. The decomposition

$$M^\circ=\bigcup_{\lambda\in\Lambda}\mathfrak{l}_\lambda \qquad (\mathfrak{l}_\lambda\cap\mathfrak{l}_\mu=0 \quad \text{if} \quad \lambda\neq\mu) \tag{4. 6}$$

trivially holds. Because of the local regularity of $M^\circ=M^\circ(H;I,\Lambda;P)$ the sandwich matrix $P=(p_{\lambda i})$ $(\lambda\in\Lambda;\ i\in I)$ has in every row λ an element $p_{\lambda j(\lambda)}$ $(j(\lambda)\in I)$ such that for a suitable $p'_{\lambda j(\lambda)}\in H$

$$p_{\lambda j(\lambda)}p'_{\lambda j(\lambda)}=e. \tag{4. 7}$$

For every $\lambda\in\Lambda$ let a pair of elements $p_{\lambda j(\lambda)}, p'_{\lambda j(\lambda)}\in H$ with the property (4. 7) be chosen.

By the property 3) of the local regularity of M° we can assume that for some $\nu(\in\Lambda)$, $p'_{\nu j(\nu)}$ is an inverse of $p_{\nu j(\nu)}$.

We shall prove that the elements $(p'_{\lambda j(\lambda)})_{j(\lambda)\lambda}=E_\lambda$ of M° are idempotent and the left ideals $\mathfrak{l}_\lambda$ have the form $\mathfrak{l}_\lambda=M^\circ E_\lambda$. In view of (3. 3) and (4. 7) we have

$$E_\lambda\circ E_\lambda=(p'_{\lambda j(\lambda)})_{j(\lambda)\lambda}\circ(p'_{\lambda j(\lambda)})_{j(\lambda)\lambda}=$$

$$=(p'_{\lambda j(\lambda)}p_{\lambda j(\lambda)}p'_{\lambda j(\lambda)})_{j(\lambda)\lambda}=(p'_{\lambda j(\lambda)})_{j(\lambda)\lambda}=E_\lambda.$$

From the definition of $\mathfrak{l}_\lambda$ and E_λ it follows that $E_\lambda\in\mathfrak{l}_\lambda$ whence $M^\circ E_\lambda\subseteqq\mathfrak{l}_\lambda$. On the other hand, if $(a)_{i\lambda}$ is an element of $\mathfrak{l}_\lambda$, then by (3. 3) and (4. 7) we get

$$(a)_{i\lambda}\circ E_\lambda=(a)_{i\lambda}\circ(p'_{\lambda j(\lambda)})_{j(\lambda)\lambda}=(ap_{\lambda j(\lambda)}p'_{\lambda j(\lambda)})_{i\lambda}=(a)_{i\lambda},$$

that is, $(a)_{i\lambda} \in M^\circ E_\lambda$, and thus $I_\lambda = M^\circ E_\lambda$. This and (4. 6) imply

$$(4.8)\qquad M^\circ = \bigcup_{\lambda\in\Lambda} M^\circ E_\lambda \qquad (M^\circ E_\lambda \cap M^\circ E_\mu = 0 \quad \text{if} \quad \lambda \neq \mu).$$

In order to show that the left ideals $M^\circ E_\lambda$ and $M^\circ E_\mu$ $(\lambda, \mu \in \Lambda)$ are left M°-similar, let $E_\lambda = (p'_{\lambda j(\lambda)})_{j(\lambda)\lambda}$ and $E_\mu = (p'_{\mu k(\mu)})_{k(\mu)\mu}$ $(j(\lambda), k(\mu) \in I)$. In view of Proposition 2. 1 and Remark 1 the mentioned similarity follows from the existence of the elements

$$(p'_{\lambda j(\lambda)})_{j(\lambda)\mu} \in M^\circ E_\mu \quad \text{and} \quad (p'_{\mu k(\mu)})_{k(\mu)\lambda} \in M^\circ E_\lambda$$

satisfying

$$(p'_{\mu k(\mu)})_{k(\mu)\lambda} \circ (p'_{\lambda j(\lambda)})_{j(\lambda)\mu} = (p'_{\mu k(\mu)} p_{\lambda j(\lambda)} p'_{\lambda j(\lambda)})_{k(\mu)\mu} = (p'_{\mu k(\mu)})_{k(\mu)\mu} = E_\mu$$

and

$$(p'_{\lambda j(\lambda)})_{j(\lambda)\mu} \circ (p'_{\mu k(\mu)})_{k(\mu)\lambda} = (p'_{\lambda j(\lambda)} p_{\mu k(\mu)} p'_{\mu k(\mu)})_{j(\lambda)\lambda} = (p'_{\lambda j(\lambda)})_{j(\lambda)\lambda} = E_\lambda .$$

Analogously, one can show that M° has a dual decomposition

$$(4.8')\qquad M^\circ = \bigcup_{i\in I} E_i M^\circ \qquad (E_i M^\circ \cap E_j M^\circ = 0 \quad \text{if} \quad i \neq j)$$

where E_i $(i \in I)$ are idempotents and $E_i M^\circ$ are right M°-similar right ideals of M°.

Finally, one can assume by a suitable ordering of the indices that for some $\nu \in \Lambda$ and $j(\nu) \in I$

$$\nu = j(\nu) = 1 \in I \cap \Lambda.$$

From Theorem 4. 1 we get the Rees theorem as a special case:

Theorem 4. 2 (Rees). *A semigroup is completely 0-simple if and only if it is isomorphic to a regular Rees matrix semigroup over a group with zero.*

It is possible to prove the Rees theorem with the help of Theorem 4. 1, but this proof is more complicated than the direct one.

§ 5. A generalization of Brandt semigroups

In Theorem 3. 9 of [1] the Brandt semigroups are characterized by special regular Rees matrix semigroups. We shall give a generalization of this result.

A semigroup S with 0 having the following property: if a, b, c are elements of S such that $ac = bc \neq 0$ or $ca = cb \neq 0$, then $a = b$, is called *0-cancellative.*

A *generalized Brandt semigroup* is a semigroup S with 0 satisfying the following conditions:

(α) S is 0-cancellative;

(β) to each element a of S there corresponds an element e of S such that $ae = a$ and an element f of S such that $fa = a$;

(γ) if e_i and e_j are idempotents of S then $e_i e_j = e_j e_i$;

(δ) for all pairs e_i, e_j of non-zero idempotents of S there exist elements q_{ij}, q_{ji} in S such that

$$q_{ij} q_{ji} = e_i \quad \text{and} \quad q_{ji} q_{ij} = e_j .$$

Later we shall show that Brandt semigroups are generalized Brandt semigroups.

By a *special S-similarly decomposable semigroup* we mean a semigroup S with 0 having the properties

(a) $$S = \bigcup_{i \in I} Se_i = \bigcup_{i \in I} e_i S \qquad (e_i^2 = e_i; \quad e_i e_j = 0 \quad \text{for} \quad i \neq j; \quad i, j \in I);$$

(b) Se_i $(i \in I)$ are left S-similar;
(c) There exists at least one idempotent e_k $(k \in I)$ such that the semigroup $e_k S e_k$ is 0-cancellative.

Remark 2. In his paper [4] A. E. LAEMMEL has shown that semigroups S having properties (a), (b) and (c) play an important role in the mathematical theory of codes and finite-state transducers.

It is easy to see that these semigroups are special cases of the S-similarly decomposable semigroups defined in the foregoing §.

Theorem 5. 1. *The following three conditions on a semigroup S with zero are equivalent:*
(i) *S is a generalized Brandt semigroup;*
(ii) *S is a special S-similarly decomposable semigroup;*
(iii) *S is isomorphic with a (locally regular) Rees $I \times I$ matrix semigroup $M^\circ(H; I, I; \Delta)$ over a 0-cancellative semigroup H with zero and identity and with the $I \times I$-identity matrix Δ as sandwich matrix.*

Proof. (i) *implies* (ii) [2]). Let $a(\neq 0)$ be an element of S. From (β) it follows the existence of an element $e(\in S)$ with $ae = a$. Hence $ae^2 = ae = a \neq 0$ and in view of (α) this implies $e^2 = e$.

Let e_i $(i \in I)$ denote the idempotent elements of S. Then $S = \bigcup_{i \in I} Se_i$ holds. From $xe_i = ye_j \neq 0$ $(x, y \in S;\ i, j \in I)$ it follows $xe_i e_i = xe_i = ye_j = ye_j e_j$, whence because of ($\alpha$) we get $e_i = e_j$. Thus $Se_i \cap Se_j = 0$ for $e_i \neq e_j$. This and (γ) imply $e_i e_j = e_j e_i = 0$ if $i \neq j$.

In view of Proposition 2. 1 and Remark 1 the condition (δ) implies that the left ideals Se_i $(i \in I)$ are left S-similar.

As condition (c) is an immediate consequence of (α) we have only to prove that $S = \bigcup_{i \in I} e_i S$. If $a(\neq 0)$ is an element of S then because of (β) there exists an element $f(\in S)$ such that $fa = a$. But we can show — as above — that f is idempotent, therefore $a \in e_i S$ for a suitable e_i.

(ii) *implies* (iii). Assume (ii). From the assumption (a) it follows that an arbitrary element a of S has the form

(5. 1) $$a = e_i a e_j \qquad (i, j \in I).$$

Let e_k $(k \in I)$ be a *fixed* idempotent of S with property (c). By (b) the left ideals Se_k and Se_i $(i \in I)$ are left S-similar, therefore by Proposition 2. 1 there exist elements q_{ki} $(\in e_k Se_i)$, q_{ik} $(\in e_i Se_k)$ such that

(5. 2) $$q_{ki} q_{ik} = e_k \quad \text{and} \quad q_{ik} q_{ki} = e_i \qquad (k, i \in I).$$

[2]) Cf. this part of the proof and Theorem 2 of LAEMMEL [4].

Let $M^\circ(e_k Se_k; I, I; P)$ denote the Rees matrix semigroup over the semigroup $e_k Se_k$ with the sandwich matrix $P=(p_{ij})=(q_{ki}q_{jk})$. $e_k Se_k$ is a semigroup with 0 and with the identity e_k; furthermore by (c), $e_k Se_k$ is 0-cancellative. As in the proof of Theorem 4. 1 we can show that the mapping

$$a=e_i ae_j \to (q_{ki}aq_{jk})_{ij} \qquad (a\in S;\ i,j\in I) \tag{5.3}$$

is an isomorphism of S onto $M^\circ=M^\circ(e_k Se_k; I, I; P=(q_{ki}q_{jk}))$. In view of (5. 2) and the assumption $e_i e_j=0$ for $i\neq j$ we get for q_{ki} $(\in e_k Se_i)$ and q_{jk} $(\in e_j Se_k)$

$$q_{ki}q_{jk}=\begin{cases} e_k & \text{if } i=j,\\ 0 & \text{if } i\neq j.\end{cases}$$

Hence

$$P=\begin{pmatrix} e_k & 0 & \cdots & & \\ 0 & e_k & & 0 & \\ \vdots & & \ddots & & \\ & & & e_k & \\ & 0 & & & \ddots \end{pmatrix}$$

is the identity matrix, indeed.

(iii) *implies* (i). Assume (iii) and let $S=M^\circ(H; I, I; \Delta)$. Denote the elements of S by $(a)_{ij}$ $(a\in H;\ i,j\in I)$. If e is the identity of H then $(e)_{ii}\circ(a)_{ij}=(ea)_{ij}=(a)_{ij}$ and $(a)_{ij}\circ(e)_{jj}=(ae)_{ij}=(a)_{ij}$ hold. Thus condition (β) is fulfilled.

To prove (α), let $(a)_{ij}$, $(b)_{kl}$, $(c)_{mn}$ be elements of S such that $(a)_{ij}\circ(b)_{kl}=$ $=(a)_{ij}\circ(c)_{mn}\neq(0)$. This holds if and only if $j=k=m$, $l=n$ and $ab=ac\neq 0$. As H is 0-cancellative, this implies $b=c$ whence $(b)_{kl}=(b)_{jl}=(c)_{jl}=(c)_{mn}$. Similarly from $(b)_{kl}\circ(a)_{ij}=(c)_{mn}\circ(a)_{ij}\neq(0)$ it follows $(b)_{kl}=(c)_{mn}$. So condition (α) is proved.

Let $(a)_{ij}$ be a non-zero idempotent of S. Then $(a)_{ij}\circ(a)_{ij}=(a)_{ij}\neq(0)$ if and only if $j=i$ and $a^2=a\neq 0$. Hence $(a)_{ij}\circ(a)_{ij}=(a)_{ii}\circ(a)_{ii}=(a)_{ii}=(e)_{ii}\circ(a)_{ii}$ whence by (α)

$$(a)_{ij}=(a)_{ii}=(e)_{ii}.$$

If $(e)_{jj}$ and $(e)_{kk}$ $(j, k\in I)$ are non-zero idempotents of S then

$$(e)_{jj}\circ(e)_{kk}=\begin{cases}(e)_{jj} & \text{if } j=k,\\ (0) & \text{if } j\neq k;\end{cases}$$

this proves condition (γ). Furthermore, from

$$(e)_{jk}\circ(e)_{kj}=(e)_{jj} \quad\text{and}\quad (e)_{kj}\circ(e)_{jk}=(e)_{kk}$$

(δ) follows.

The proof is finished.

In Theorem 3. 9 of [1], it is proved that the following three conditions on a semigroup S with zero are equivalent:

(i′) S is a Brandt semigroup;

(ii′) S is a completely 0-simple inverse semigroup;

(iii′) S is isomorphic to a (regular) Rees $I\times I$ matrix semigroup $M^\circ(G; I, I; \Delta)$ over a group with zero G and with the $I\times I$-identity matrix Δ as sandwich matrix.

We shall show that the conditions (i′) and (ii′) and (iii′) are special cases of conditions (i) and (ii) and (iii) of Theorem 5. 1, respectively.

It is trivial that (iii′) is a special case of (iii). In view of Theorem 5. 1 and Theorem 3. 9 of [1] this implies that (i′) [(ii′)] is a special case of (i) [(ii)].

We remark that it is possible to prove Theorem 3. 9 of [1] with the help of Theorem 5. 1, but this proof is more complicated than the original.

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