On the endomorphism ring of direct sums of groups

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§ 1

In this paper we investigate the commutativity of endomorphism rings $E(G)$ of groups G and apply the results on the rings R, which can be defined on G. A ring R is said to be defined on G, in case the additive group of R, denoted by R^+, is G. In the special case that G is a discrete direct sum of groups we obtain conditions for the uniqueness of the holomorphs of rings R, defined on G.

In [5] SZELE—SZENDREI have completely solved the problem of the commutativity of $E(G)$, in case G is a torsion group. For the case of mixed groups they have got some partial results. We consider a group G, which is a discrete direct sum of groups G_λ and obtain necessary and sufficient conditions that $E(G)$ be commutative (Theorem 2 and 2a). As a special case we have the torsion-free completely decomposable groups $G = \sum_\lambda A_\lambda$, where the A_λ are torsion-free groups of rank 1, i. e. subgroups of the additive group of all rationals $\mathfrak{R}$ (Theorem 3, Corollaries 3 and 4). Then we apply our result to torsion groups and obtain Theorem 4, which occurs as Theorem 1 in [5]. We also investigate the finite and finitely generated groups. A finite or a finitely generated group G has a commutative $E(G)$ if and only if G is a cyclic group (Corollaries 5 and 6). For mixed groups we have Theorem 5, due to SZELE—SZENDREI [5], and, in a special case, Corollary 7.

As to the holomorphs of a ring, we first prove a theorem for rings R, which are the ring-theoretic discrete direct sum of rings R_λ ($\lambda \in \Lambda$). In Theorem 1 we give a necessary and sufficient condition that such a ring R have one holomorph. For the definition of holomorph we refer to our paper [3]. From Theorem 1 a result of WEINERT—EILHAUER is easily obtained [6] (Corollary 1) and likewise our Theorem 1 in [3], (Corollary 2). In Theorem 6 we consider a ring R which is defined on a group $G = \sum_\lambda G_\lambda$ (discrete direct sum), where the G_λ are fully invariant subgroups of G. The ring R is the direct sum of its ideals G_λ (as rings). Now the uniqueness of the holomorph of R depends only on the same property for the direct summands G_λ of R. In the special case that the G_λ are rational groups, each G_λ (as a ring) has one holomorph $P(G_\lambda)$, which is isomorphic to the direct sum $G_\lambda \oplus G_\lambda$ (G_λ as a ring) (Theorem 7).

The groups, used in this paper, are all abelian groups, the rings are associative rings. For the definition of group-theoretic notions such as type of an element of a torsion-free group, divisible group, etc. we refer to the book of L. FUCHS [2].

§ 2

Theorem 1. *A ring* $R=\sum_{\lambda\in\Lambda} R_\lambda$ *(ring-theoretic discrete direct sum) has one holomorph if and only if each* R_λ $(\lambda\in\Lambda)$ *has one holomorph and each* R_λ *is invariant for the components of double homothetisms of* R.

Proof. First suppose that R has one holomorph. Consider the projection $\eta_\lambda: R\to R_\lambda$ of R $(r\to r_\lambda)$. It is easily seen that $(\eta_\lambda, \eta_\lambda)$ is a double homothetism of R. Now suppose that (α_1, α_2) is an arbitrary double homothetism of R. As $(\alpha_1, \alpha_2)\sim(\eta_\lambda, \eta_\lambda)$ (R has one holomorph) we have $\alpha_1\eta_\lambda=\eta_\lambda\alpha_1$ or $\alpha_1\eta_\lambda(r)=\eta_\lambda\alpha_1(r)$ or $\alpha_1(r_\lambda)=\eta_\lambda\{\alpha_1(r)\}\in R_\lambda$ for every $r_\lambda\in R_\lambda$. This shows that R_λ is invariant for the components of double homothetisms of R. Then take two arbitrary double homothetisms (α_1^*, α_2^*) and (β_1^*, β_2^*) of R_λ. Then we define $\alpha_1(r)=\alpha_1^*(r_\lambda)$ and $\alpha_2(r)=\alpha_2^*(r_\lambda)$, $\beta_1(r)=\beta_1^*(r_\lambda)$ and $\beta_2(r)=\beta_2^*(r_\lambda)$, for $r\in R$ and r_λ is the projection of r (λ is fixed). Now one proves easily, that (α_1, α_2) and (β_1, β_2) are double homothetisms of R. As R has one holomorph, $\alpha_1\beta_2(r)=\beta_2\alpha_1(r)$ and $\alpha_2\beta_1(r)=\beta_1\alpha_2(r)$ for all $r\in R$. Or $\alpha_1\beta_2^*(r_\lambda)=\beta_2\alpha_1^*(r_\lambda)$ and $\alpha_2\beta_1^*(r_\lambda)=\beta_1\alpha_2^*(r_\lambda)$ or $\alpha_1^*\beta_2^*(r_\lambda)=\beta_2^*\alpha_1^*(r_\lambda)$ and $\alpha_2^*\beta_1^*(r_\lambda)=\beta_1^*\alpha_2^*(r_\lambda)$. This proves $(\alpha_1^*, \alpha_2^*)\sim(\beta_1^*, \beta_2^*)$ and R_λ has one holomorph.

Conversely, let us suppose that each R_λ $(\lambda\in\Lambda)$ has one holomorph and is invariant for the components of double homothetisms of R. We take two arbitrary double homothetisms (α_1, α_2) and (β_1, β_2) of R. Then $\alpha_1(\sum_\lambda r_\lambda)=\sum_\lambda \alpha_1 r_\lambda$ and $\alpha_1 r_\lambda\in R_\lambda$ for each λ, $\beta_2(\sum_\lambda r_\lambda)=\sum_\lambda \beta_2 r_\lambda$ and $\beta_2 r_\lambda\in R_\lambda$ for each λ. And $(\alpha_1\beta_2-\beta_2\alpha_1)(\sum_\lambda r_\lambda)=\sum_\lambda(\alpha_1\beta_2-\beta_2\alpha_1)r_\lambda$, where $(\alpha_1\beta_2-\beta_2\alpha_1)r_\lambda\in R_\lambda$ for each λ. Consider a fixed direct summand R_λ of R and define $\alpha_1^*(r_\lambda)=\alpha_1(r_\lambda)$ and $\alpha_2^*(r_\lambda)=\alpha_2(r_\lambda)$ for each $r_\lambda\in R_\lambda$. Then (α_1^*, α_2^*) is a double homothetism of R_λ. Likewise (β_1^*, β_2^*) is a double homothetism of R_λ, if we define $\beta_1^*(r_\lambda)=\beta_1(r_\lambda)$, $\beta_2^*(r_\lambda)=\beta_2(r_\lambda)$ for each $r_\lambda\in R_\lambda$. As R_λ has one holomorph, one gets $(\alpha_1^*, \alpha_2^*)\sim(\beta_1^*, \beta_2^*)$, which means $\alpha_1^*\beta_2^*=\beta_2^*\alpha_1^*$. Therefore $(\alpha_1^*\beta_2^*-\beta_2^*\alpha_1^*)\,(r_\lambda) = (\alpha_1\beta_2-\beta_2\alpha_1)\,(r_\lambda)=0$ for each r_λ in R_λ. As this is the case for each R_λ, we obtain that $(\alpha_1\beta_2-\beta_2\alpha_1)\,(\sum_\lambda r_\lambda)=0$. Likewise $(\alpha_2\beta_1-\beta_1\alpha_2)(\sum_\lambda r_\lambda)=0$. Therefore $(\alpha_1, \alpha_2)\sim(\beta_1, \beta_2)$, i. e. R has one holomorph.

Corollary 1. *If* $R = R^2\oplus n_R$ *(direct sum of the ideal generated by all products in R and the annullator in R), then R has one holomorph if and only if the endomorphism ring of* n_R^+ *is commutative* (see Weinert—Eilhauer [6], Theorem 4).

It is clear that both R^2 and n_R are invariant for components of double homothetisms of R. From $R = R^2\oplus n_R$ and n_R has one holomorph it follows that R^2 has one holomorph. Therefore n_R has one holomorph is a necessary and sufficient condition for the uniqueness of the holomorph of R. As n_R is a zero-ring this is the case if and only if $E(n_R^+)$ is commutative (see Rédei [4]).

In the special case that $R=\sum_{\lambda\in\Lambda} R_\lambda$ and $\operatorname{Hom}(R_{\lambda_i}^+, R_{\lambda_j}^+)=0$ for $i\neq j$ we have that $E(R^+) = \sum_{\lambda\in\Lambda} E(R_\lambda^+)$ (direct sum) and each R_λ^+ is a fully invariant subgroup of R^+. Particularly, the R_λ are invariant for the components of double homothetisms of R. So we get:

Corollary 2. $R=\sum_{\lambda} R_\lambda$ *with* Hom $(R^+_{\lambda_i}, R^+_{\lambda_j})=0$ *for* $i\neq j$ *has one holomorph if and only if each of the* R_λ *has one holomorph.*

Moreover the holomorph of R is the direct sum of the holomorphs of the R_λ, (cf. Theorem 6). Again specializing we have that a finite ring R is the direct sum of its p-components R_p and the holomorph of R is the direct sum of the holomorphs of the R_p (cf. Theorem 1, [3]), if each of the R_p has one holomorph.

§ 3

In order to get further information about the holomorphs of direct sums of rings, we have to investigate the commutativity of the endomorphism rings of direct sums of groups.

Theorem 2. *The endomorphism ring of a discrete direct sum* $G=\sum_{\lambda} G_\lambda$ *of groups* G_λ *is commutative if and only if each of the summands* G_λ *has a commutative* $E(G_\lambda)$ *and none of* G_λ *can be mapped homomorphically onto a non-zero subgroup of another* G_λ.

Proof. *Necessity.* As $E(G)$ is commutative, it follows that every endomorphic image of G is fully invariant (Lemma 1, [5]). As every direct summand is an endomorphic image, it follows that the G_λ are fully invariant subgroups of G $(\lambda\in\Lambda)$. Suppose now that G_{λ_i} is mapped homomorphically onto a subgroup $(\neq 0)$ of G_{λ_j} by the homomorphism $\vartheta\in\mathrm{Hom}\,(G_{\lambda_i}, G_{\lambda_j})$ $(\lambda_i\neq\lambda_j)$. We define the mapping ϑ' of G into itself by: $\vartheta' g_\lambda=0$ if $g_\lambda\in G_\lambda$ with $\lambda\neq\lambda_i$; $\vartheta' g_{\lambda_i}=\vartheta g_{\lambda_i}$ if $g_{\lambda_i}\in G_{\lambda_i}$. Then ϑ' is an endomorphism of G or $\vartheta'\in E(G)$. But $\vartheta' G_{\lambda_i}\nsubseteq G_{\lambda_i}$, since ϑ' coincides with ϑ on G_{λ_i}. Therefore G_{λ_i} is not fully invariant, which is a contradiction. We conclude that none of G_λ can be mapped homomorphically onto a non-zero subgroup of another G_λ. Now let σ_λ, ϱ_λ be two arbitrary endomorphisms of G_λ (λ is fixed). G_λ is an endomorphic image of G and let η_λ be the projection of G onto G_λ. Then we can extend the endomorphisms σ_λ resp. ϱ_λ of G_λ to endomorphisms σ resp. ϱ of G defining $\sigma(\sum g_\mu)=\sum_{\mu}\sigma g_\mu$ and $\sigma g_\mu=0$ if $g_\mu\in G_\mu$ with $\mu\neq\lambda$, $\sigma g_\lambda=\sigma_\lambda g_\lambda$ if $g_\lambda\in G_\lambda$ and likewise for ϱ with respect to ϱ_λ. Then $\sigma\varrho(\eta_\lambda g)=\varrho\sigma(\eta_\lambda g)$ $(g\in G)$, as $E(G)$ is commutative, or $\sigma\varrho_\lambda(g_\lambda)=\varrho\sigma_\lambda(g_\lambda)$, $g_\lambda\in G_\lambda$, or $\sigma_\lambda\varrho_\lambda(g_\lambda)=\varrho_\lambda\sigma_\lambda(g_\lambda)$ for every $g_\lambda\in G_\lambda$. This means $\sigma_\lambda\varrho_\lambda=\varrho_\lambda\sigma_\lambda$ or $E(G_\lambda)$ is commutative.

Sufficiency. Let α be an arbitrary endomorphism of G. Then $\alpha(\sum_{\lambda} g_\lambda)=\sum_{\lambda}\alpha g_\lambda$. Take a fixed G_λ. Now $\alpha g_\lambda=\sum_{\mu} g_{\lambda\mu}\,(g_{\lambda\mu}\in G_\mu)$ is a finite sum and if we put $\alpha_{\lambda\mu}g_\lambda=g_{\lambda\mu}$, then $\alpha_{\lambda\mu}$ clearly belongs to Hom (G_λ, G_μ). Therefore $\alpha_{\lambda\mu}=0$ for $\lambda\neq\mu$, and $g_{\lambda\mu}=0$ for $\lambda\neq\mu$. Then $\alpha g_\lambda=\sum_{\mu} g_{\lambda\mu}=g_{\lambda\lambda}\in G_\lambda$, which means that G_λ is a fully invariant subgroup of G. $E(G)=\sum_{\lambda\in\Lambda} E(G_\lambda)$ (direct sum) and as each G_λ has a commutative $E(G_\lambda)$, it follows that $E(G)$ is commutative.

From the proof above we see that Theorem 2 also may be read as:

Theorem 2a. *Let* $G=\sum_{\lambda} G_\lambda$ *be a discrete direct sum of groups* G_λ. *Then* $E(G)$

is commutative if and only if each G_λ has a commutative $E(G_\lambda)$ and is a fully invariant subgroup of G.

Theorem 3. *A completely decomposable torsion-free group $G=\sum_\lambda A_\lambda$, where the A_λ are torsion-free groups of rank 1 and G is their direct sum, has a commutative $E(G)$ if and only if the types of the components A_λ are pairwise incomparable.*

Proof. First we remark that, if A_{λ_i} and A_{λ_j} are two torsion-free groups of rank 1, of type $\mathfrak{a}$ and $\mathfrak{b}$ respectively, then A_{λ_i} is isomorphic to a subgroup of A_{λ_j} if and only if $\mathfrak{a}\leqq\mathfrak{b}$. Now suppose that the conditions of the theorem are satisfied. Then we show, that none of the groups A_λ can be mapped homomorphically onto a non-zero subgroup of another A_λ. For let $A_{\lambda_i}, A_{\lambda_j}$ be torsion-free groups of rank 1 $(\lambda_i\neq\lambda_j)$ and let φ be a homomorphism of A_{λ_i} onto a subgroup $(\neq 0)$ of A_{λ_j}. Then it is easy to see, that $\mathrm{Ker}(\varphi)=0$ or φ is a monomorphism (isomorphism into). This means A_{λ_i} is isomorphic to a subgroup of A_{λ_j} i. e. $\varphi(A_{\lambda_i})$, but this is impossible by the remark above as the types of A_{λ_i} and A_{λ_j} are incomparable. As the A_λ are rational groups, they have commutative endomorphism rings, and $E(G)$ is commutative by Theorem 2.

Conversely, if $E(G)$ is commutative, then again none of the A_λ is isomorphic to a subgroup of another A_λ by theorem 2. This means, the types of the components A_λ are pairwise incomparable.

The class of completely decomposable groups comprises all groups of rank 1, all free abelian groups as well as all divisible torsion-free abelian groups. Thus we have the corollaries:

Corollary 3. *A free abelian group G has a commutative $E(G)$ if and only if $G\cong C(\infty)$ (infinite cyclic group).*

Corollary 4. *A divisible torsion-free abelian G has a commutative $E(G)$ if and only if $G\cong\mathfrak{R}$, where $\mathfrak{R}$ is the additive group of all rational numbers.*

§ 4

a) *Torsion groups.* Every torsion group may be represented as a direct sum of p-groups G_p belonging to different primes p. The G_p, uniquely determined by G, are called the *p-components* of G. They are fully invariant subgroups of G. Therefore by Theorem 2a, $G=\sum_p G_p$ has a commutative $E(G)$ if and only if each G_p has a commutative $E(G_p)$. Then we have to characterize the p-groups with commutative endomorphism ring. Now let p be a fixed prime and consider the p-component G_p of G. The center of $E(G_p)$ is the ring $\mathfrak{P}$ of p-adic integers or the residue class ring $I/(p^k)$ of the integers mod p^k, where I is the ring of rational integers ([2], Theorem 56. 3). Therefore, $E(G_p)$ is commutative if and only if $E(G_p)$ is either the ring $\mathfrak{P}$ of p-adic integers or the ring $I/(p^k)$ of integers mod p^k. We now use: if A is a group $C(p^k)$ $(k=1, 2, \ldots, \infty)$, and B is a p-group such that $E(B)\cong E(A)$, then $B\cong A$, (see [2], p. 215). Incase $E(G_p)\cong\mathfrak{P}=E(C(p^\infty))$, we have $G_p\cong C(p^\infty)$. In case $E(G_p)\cong$ $\cong I/(p^k)=E(C(p^k))$, we have $G_p\cong C(p^k)$. Thus a p-component G_p of G has a commutative $E(G_p)$ if and only if G_p is either $C(p^\infty)$ or $C(p^k)$. Then $G=\sum_p G_p$ has a commutative $E(G)$ if and only if G is a direct sum of groups $C(p^k)$ $(k=1, 2, \dots, \infty)$ for different primes p.

Theorem 4. *An abelian torsion group G has a commutative $E(G)$ if and only if G is a subgroup of C, where C is the additive group of rational numbers* mod 1 (cf. [5], § 4, Theorem 1).

If G is a finite abelian group, then components $C(p^\infty)$ do not occur in a direct decomposition of $G = \sum_p G_p$ in p-components. But then G is a direct sum of a finite number of cyclic groups $C(p^k)$ for different primes p, that means, G is cyclic. So we get:

Corollary 5. *A finite abelian group G has a commutative $E(G)$ if and only if G is a cyclic group.*

More generally, a *finitely generated* group G is a direct sum of a finite number of cyclic groups of infinite and/or prime power order, say $G = \sum_m C(\infty) + \sum_p C(p^k)$. Let G have a commutative $E(G)$. If G is torsion-free, then $G = C(\infty)$ (Corollary 3). If G is a torsion group, then $G = \sum_p C(p^k)$ for different primes p, or G is a cyclic group (Corollary 5). If G is a mixed group, then the torsion-free component of G is $C(\infty)$, as none of the direct summands can be mapped homomorphically onto another one. The maximal torsion subgroup of G is $\sum_p C(p^k)$ and as $E(G)$ is commutative, $\sum_p C(p^k)$ has a commutative endomorphism ring (Theorem 2). Then $\sum_p C(p^k)$ is a subgroup of C (Theorem 4); in this case, as G is finitely generated, $\sum_p C(p^k)$ is a cyclic group $C(n)$ (Corollary 5). Now $G = C(\infty) + C(n)$ is impossible, as $\mathrm{Hom}\,(C(\infty), C(n)) \cong C(n)$ and this contradicts the commutativity of $E(G)$. Therefore a mixed group G, which is finitely generated and has commutative $E(G)$, is impossible. We have proved:

Corollary a) 6. *A finitely generated abelian group G has a commutative $E(G)$ if and only if G is a cyclic (infinite or finite) group.*

Remark. a) For a *torsion group G*, SZELE—SZENDREI [5] have proved that G has a commutative $E(G)$ if and only if G has this property *locally*, i. e. every finitely generated subgroup of G has a commutative $E(G)$. By Corollary 6, this means, every finitely generated subgroup of G is cyclic or G is locally cyclic. Now a torsion group G is locally cyclic if and only if it is a subgroup of C, which is again Theorem 4.

b) For a *torsion-free group G* it is clear that if every finitely generated subgroup F of G has a commutative $E(F)$, then G has a commutative $E(G)$. For, according to Corollary 6, this means that every finitely generated subgroup is $C(\infty)$, or G is locally cyclic. But a locally cyclic torsion-free group G is a rational group or a subgroup of $\mathfrak{R}$, the additive group of all rationals. Therefore G has a commutative $E(G)$. The converse does not hold. A counter-example is: let $p_1, p_2, \dots$ be an infinite sequence of different prime numbers and let R_{p_n} be the additive group of those rationals, whose denominator is relatively prime to p_n. Then the complete direct sum $G = \sum_{p_n}^{*} R_{p_n}$ has a commutative $E(G)$ (SZELE—SZENDREI [5]), but G is not locally cyclic.

c) *Mixed groups.* Let G be an arbitrary (mixed) group and p be an arbitrary prime number. If the group G contains an element of order p, then p is called *relevant* for G. Let $G = T+J$ be a *splitting* mixed group, i. e. G decomposes into a direct sum of a torsion group T and a torsion-free group J. Here we have the following theorem, due to SZELE—SZENDREI [5]:

Theorem 5. *Let $G = T+J$ be a splitting mixed group, where T is the torsion subgroup of G. Then $E(G)$ is commutative if and only if T is a locally cyclic group containing no subgroup of type $C(p^\infty)$ and J has a commutative $E(J)$ and $pJ=J$ holds for all primes p relevant for G.*

Remark. As a special case of Theorem 5 we consider the mixed groups G with bounded maximal torsion subgroup. Let G be a mixed group with bounded maximal torsion subgroup T ($nT=0$). Then G is a splitting mixed group: $G = T+J$ ([2], Corollary 50. 4). Now suppose that G has a commutative $E(G)$. By Theorem 5, T is a locally cyclic group containing no subgroup of type $C(p^\infty)$. From $nT=0$ we infer that only those cyclic components $C(p^k)$ can occur in T, for which $p|n$. As n has only a finite number of prime divisors, it follows that T has a finite number of direct summands, i. e. T is a cyclic group and a subgroup of $C(n)$. We may assume, without loss of generality, that n is the least positive integer such that $nT=0$. Then we get $T=C(n)$. Evidently we also have $T=G[n]$, where $G[n]$ is the set of all $g\in G$ with $ng=0$. Now it is clear that $J \cong G/T = G/G[n] \cong nG$, i. e. the set of all ng with $g\in G$, hence $E(nG)$ is commutative by Theorem 5. As $T=C(n)$, the prime divisors p_i of n are relevant for G. From Theorem 5 it follows that $p_iJ=J$ for all $p_i|n$. Hence $nJ=J$ or $nG=J$, as $nG=nJ$. Conversely, if G is a mixed group with bounded maximal torsion subgroup $T=C(n)$, then again $T=G[n]$. If nG is the torsion-free component of G, then we have the direct decomposition $G=G[n]+nG$. Both $G[n]$ and nG are fully invariant subgroups of G. Moreover T as a cyclic group has a commutative $E(T)$. By Theorem 2a the commutativity of $E(nG)$ is sufficient now for the commutativity of $E(G)$. Thus we get:

Corollary 7. *Let G be a mixed group with bounded maximal torsion subgroup T such that $nT=0$ and n is the least positive integer with this property. Then $E(G)$ is commutative if and only if $T=C(n)$ and nG is the torsion-free component of G and has a commutative $E(nG)$.*

Now we want to apply these results to the investigation of rings which can be defined on direct sums of groups. Let G be an arbitrary (abelian) group. An (associative) ring R *on* G is a ring R, such that $R^+=G$. Such a ring R has one holomorph if the endomorphism ring $E(R^+)=E(G)$ is commutative [6]. If G is a discrete direct sum of groups, and every direct summand is a fully invariant subgroup of G, the structure of the holomorph of a ring R on G can be described.

Theorem 6. *Let $G = \sum_{\lambda\in\Lambda} G_\lambda$ be a discrete direct sum of groups G_λ, such that each G_λ is a fully invariant subgroup of G. Then in each ring R on G the G_λ are ideals and R is their direct sum in ring-theoretic sense. A ring R on G has one holomorph* *if and only if each of the* G_λ *(as a ring) has one holomorph. If R has one holomorph* $P(R)$, *then* $P(R)$ *is an interdirect sum of the holomorphs* $P(G_\lambda)$ $(\lambda \in \Lambda)$.

Proof. Let g be a fixed element of G. Then multiplication of the elements of G from the left by g in a ring R on G induces an endomorphism of G. As G_λ is fully invariant in G, we get $gg_\lambda \in G_\lambda$ for each $g_\lambda \in G_\lambda$. Likewise we find that g, operating on the right side on the elements of G, induces an endomorphism of G and therefore $g_\lambda g \in G_\lambda$ for each $g_\lambda \in G_\lambda$. G_λ is a two-sided ideal in G. Moreover $g_\lambda g_\mu \in G_\lambda \cap G_\mu = (0)$ for $\lambda \neq \mu$ or $G_\lambda G_\mu = (0)$. As G is a direct sum of groups G_λ, we infer that R is a direct sum of its ideals G_λ in ring-theoretic sense. Then, each G_λ is fully invariant in G implies in particular that each G_λ is invariant for the components of double homothetisms of R. By Theorem 1, R has one holomorph if and only if each of the G_λ (as a ring) has one holomorph. Finally we have to prove that the holomorph $P(R)$ of R is an interdirect sum of the holomorps $P(G_\lambda)$, $(\lambda \in \Lambda)$. Let D resp. D_λ be the maximal ring of related double homothetisms of R resp. G_λ $(\lambda \in \Lambda)$. The elements of the holomorph $P(R)$ are the pairs (α, a), $\alpha \in D$, $a \in R$ and sum and product are obtained as follows: $(\alpha, a)+(\beta, b) = (\alpha+\beta, a+b)$, $(\alpha, a)(\beta, b) = (\alpha\beta, \beta_2 a+\alpha_1 b+ab)$ with $\alpha=(\alpha_1, \alpha_2)$, $\beta=(\beta_1, \beta_2)$. As $G=\sum_\lambda G_\lambda$ is the discrete direct sum of its fully invariant subgroups G_λ, it is clear that $E(G)$ is the complete direct sum of the groups $E(G_\lambda)$. Likewise D is the complete direct sum of the rings D_λ. Any $\alpha \in D$ induces a well-defined double homothetism α_λ of D_λ for every λ. If $\alpha=(\alpha_1, \alpha_2) \in D$ and α_1 induces $\alpha_{1\lambda}$ in D_λ, α_2 induces $\alpha_{2\lambda}$ in D_λ, then $\alpha_\lambda=(\alpha_{1\lambda}, \alpha_{2\lambda})$ is a double homothetism of D_λ. Every double homothetism $\alpha_\lambda \in D_\lambda$ (λ fixed) may be obtained as the "λ^{th} component" of a double homothetism $\alpha \in D$. The mapping $(\alpha, a) \to \langle \dots, (\alpha_\lambda, a_\lambda), \dots \rangle$ is a homomorphism of $P(R)=D \circ R$ into the complete direct sum of the $P(G_\lambda) = D_\lambda \circ G_\lambda$. Moreover, this homomorphism is an isomorphism, because if $(\alpha_\lambda, a_\lambda) = (0, 0)$ holds for all $\lambda \in \Lambda$, then $(\alpha, a)=(0, 0)$. Then $P(R)$ is isomorphic to a subring of the complete direct sum of the rings $P(G_\lambda)=D_\lambda \circ G_\lambda$ i. e. an interdirect sum of the rings $P(G_\lambda)$ $(\lambda \in \Lambda)$. This completes the proof of Theorem 6.

Now we will give examples of groups, which satisfy the requirements of Theorem 6. In the *torsion* case, we have that every torsion group G may be represented as a direct sum of its p-components G_p. These p-components G_p are fully invariant subgroups of G. Therefore Theorem 6 may be applied to torsion groups. If G is a finite group, say of order n, then, if $n=p_1^{k_1}\dots p_r^{k_r}$, G is the direct sum of r subgroups G_i of order $p_i^{k_i}$ $(i=1, \dots, r)$. Every ring R on G is a finite ring and the ring-theoretic direct sum of finite p_i-rings R_{p_i}, which are rings on G_i $(i=1, \dots, r)$ and annihilate each other for different primes p_i. The ring R has one holomorph if and only if each of the R_{p_i} has one holomorph. Moreover $P(R)$ is the direct sum of the $P(R_{p_i})$. This establishes Theorem 1 of my paper [3], (cf. also Corollary 2 of this paper).

In the *torsion-free* case, we consider a torsion-free group G which is the direct sum of homogeneous groups such that the types of the components G are pairwise incomparable. By a *homogeneous* group we mean a torsion-free group all of whose elements $\neq 0$ are of one and the same type $\mathfrak{a}$. We denote by $G(\mathfrak{a})$ the set of all elements a in G for which $T(a) \geqq \mathfrak{a}$. Now let G_λ be a fixed homogeneous component of G of type $\mathfrak{a}_\lambda$. As the types of the components G_λ are pairwise incomparable, we get $G(\mathfrak{a}_\lambda)=G_\lambda$. Now the subgroups $G(\mathfrak{a})$ are, for any type $\mathfrak{a}$, fully invariant in G. Therefore G_λ is a fully invariant subgroup of G for every λ. We do not know, however, whether a homogeneous group G_λ has a commutative $E(G_\lambda)$. If the homogeneous components G_λ are torsion-free groups of rank 1 or rational groups the group $G=\sum_\lambda G_\lambda$ is completely decomposable. If now the types of the rational groups are pairwise incomparable, then the G_λ are fully invariant in G. A ring R on G is the direct sum of its ideals G_λ. In this case, any ring R on G has one holomorph, as each of the G_λ (as a ring) has one holomorph. The last result is due to the fact, that each of the G_λ (as a rational group) has a commutative $E(G_\lambda)$ and this is a sufficient condition for the uniqueness of the holomorph $P(G_\lambda)$. The uniqueness of the holomorph of R is also an easy consequence of Theorem 3, as the ring $E(R^+)=E(G)$ is commutative. By Theorem 6, $P(R)$ is an interdirect sum of the holomorphs $P(G_\lambda)$ $(\lambda\in\Lambda)$.

If G_λ is a rational group, then any ring R_λ on G_λ is a subring of the rational number field or a zero-ring [1]. Now we have the theorem:

Theorem 7. *Let G_λ denote a subgroup of the additive group $\mathfrak{R}$ of all rationals and assume that $1\in G_\lambda$. Let R_λ be a non-zero ring on G_λ and let $1\times 1=1$ in R_λ. Then the holomorph $P(R_\lambda)$ of R_λ is isomorphic to $R_\lambda\oplus R_\lambda$ (ring-theoretic direct sum).*

Proof. Any $\eta\in E(R_\lambda^+)=E(G_\lambda)$ maps 1 upon a rational r and this r characterizes η. A double endomorphism (α_1,α_2) of R_λ^+, $\alpha_1\in E(R_\lambda^+)$, $\alpha_2\in E^\circ(R_\lambda^+)$ is a double homothetism of R_λ if the following conditions are satisfied: $\alpha_1(ab)=(\alpha_1 a)b$, $\alpha_2(ab)=a(\alpha_2 b)$, $(\alpha_2 a)b=a(\alpha_1 b)$ and $\alpha_2(\alpha_1 a)=\alpha_1(\alpha_2 a)$ for all $a, b\in R_\lambda$. As $1\times 1=1$ in R_λ, the multiplication in R_λ is the usual one of rational numbers. Now, if $\alpha_1 1=r_1$ and $\alpha_2 1=r_2$ $(r_1, r_2\in R_\lambda)$, it is clear that $\alpha_1 a=r_1 a$, $\alpha_2 a=r_2 a$ for all $a\in R_\lambda$. This means, that $\alpha_1(ab)=(\alpha_1 a)b$, $\alpha_2(ab)=a(\alpha_2 b)$ and $\alpha_2(\alpha_1 a)=\alpha_1(\alpha_2 a)$ for all $a, b\in R_\lambda$. From $(\alpha_2 a)b=a(\alpha_1 b)$ it follows that $r_2(ab)=r_1(ab)$ for all $a, b\in R_\lambda$. As R_λ has no zero-divisors (R_λ is a subring of the rational number field), we get $r_1=r_2$ or $\alpha_1=\alpha_2$. The double homothetisms of R_λ have the form (α,α), where $\alpha\in E(R_\lambda^+)$. Now R_λ has one maximal ring D_λ of related double homothetisms, as all double homothetisms are pairwise related. The mapping $(\eta,\eta)\to\eta$ provides an isomorphism of D_λ onto $E(R_\lambda^+)$. Now every double homothetism $(\eta,\eta)\in D_\lambda$ is an inner one, i. e. every (η,η) is induced by a rational number $r\in R$ such that $\eta a=ra$ for all $a\in R_\lambda$. Therefore $D_\lambda=D_{o_\lambda}=$ ring of all inner double homothetisms of R_λ. It is known, that $R_\lambda/n_{R_\lambda}\cong D_{o_\lambda}$, where n_{R_λ} is the annullator of R_λ (RÉDEI [4]). But $n_{R_\lambda}=(0)$, therefore $R_\lambda\cong D_{o_\lambda}=D_\lambda$. The elements of $P(R_\lambda)$ are pairs (η, a), $\eta=(\eta,\eta)\in D_\lambda$, $a\in R_\lambda$. We write these elements as (a, b), $a, b\in R_\lambda$, as $R_\lambda\cong D_\lambda$. Addition and multiplication are defined by

$$(a, b)+(c, d) = (a+c, b+d), \qquad (a, b)(c, d) = (ac, bc+ad+bd).$$

In this case $P(R_\lambda) = D_\lambda\circ R_\lambda$ is a direct sum. For let $(a, b)\to\pi(a, b) = (a, a+b)$ be a permutation of the elements of R_λ. Then we define: $(a, b)\dot{+}(c, d) = \pi(\pi^{-1}(a, b)+\pi^{-1}(c, d))$ and $(a, b)\dot{\times}(c, d) = \pi(\pi^{-1}(a, b)\pi^{-1}(c, d))$, and it turns out that $(a, b)\dot{+}(c, d) = (a+c, b+d)$ and $(a, b)\dot{\times}(c, d) = (ac, bd)$. Then $P(R_\lambda) = D_\lambda\circ R_\lambda\cong D_\lambda\oplus R_\lambda\cong R_\lambda\oplus R_\lambda$. Finally, let $G = T+J$ be a splitting *mixed* group, where T is the torsion subgroup of G and both T and J satisfy the conditions of Theorem 5. T is the maximal torsion subgroup of G and therefore T is a fully invariant subgroup of G. As $pJ=J$ for all primes relevant for G, it is clear that the equation $p^n x=a$ $(a\in J)$ is solvable in J for every natural number n and every prime p relevant for G. Then J is a fully invariant subgroup of G. Thus $G = T+J$ is the direct sum of its fully invariant subgroups T and J and we may apply Theorem 6.

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