Complete sets of unitary invariants for compact and trace-class operators

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I. Introduction

A complete set of unitary invariants for operators in a family $\mathcal{F}$ of operators on a (complex) Hilbert space is an indexed collection $\{O_\gamma\}_{\gamma\in\Gamma}$ of objects attached to each operator in $\mathcal{F}$ such that if $A, B\in\mathcal{F}$, then A is unitarily equivalent to B if and only if $O_\gamma(A)=O_\gamma(B)$ for each $\gamma\in\Gamma$.

For several families of operators complete sets of unitary invariants are known. For example, probably the best known family is the family of normal operators, where the theory of spectral multiplicity provides such a complete set of unitary invariants (see [2]). However, no complete set of unitary invariants has been found for arbitrary operators. The object of this paper is to solve the problem for compact operators. RADJAVI [5] has recently given a completely different characterization of unitary equivalence for compact operators.

The first problem which one encounters in trying to obtain a complete set of unitary invariants for compact operators on a Hilbert space $\mathfrak{H}$ is that of obtaining a complete set of unitary invariants for $n\times n$ matrices, that is, of solving the problem in the special case that $\mathfrak{H}$ is finite dimensional. Such a set of invariants was provided by SPECHT [7], who obtained the following result: Let Ω denote the free multiplicative semigroup in the free variables x and y. Two $n\times n$ matrices A and B are unitarily equivalent if and only if $t[\omega(A, A^*)]=t[\omega(B, B^*)]$ for each $\omega(x, y)\in\Omega$, where $t(A)$ denotes the trace of A in the usual sense.

PEARCY has shown in [4] that for each n there is a finite subset Ω_n of Ω (containing at most 4^{n^2} members) such that two $n\times n$ matrices A and B are unitarily equivalent if and only if $t[\omega(A, A^*)]=t[\omega(B, B^*)]$ for each $\omega(x, y)\in\Omega_n$. We shall refer to the above two sets of invariants as the Specht and Specht—Pearcy invariants, respectively.

Throughout this paper we shall denote the null space of an operator A by $\mathfrak{N}(A)$, the closure of the range of A by $\mathfrak{R}(A)$, and the operator $(A^*A)^{\frac{1}{2}}$ by $[A]$.

Since compact operators on a Hilbert space can be uniformly approximated by operators of finite rank, which are essentially operators on finite dimensional

*) This paper is essentially a thesis presented to the Rice University in partial fulfillment of the requirements for the degree of Doctor of Philosophy. The author wishes to thank his thesis director, Professor ARLEN BROWN, and also Professor CARL PEARCY for their helpful suggestions and constructive criticism.

spaces, it is reasonable to expect the above sets of invariants to provide some sort of complete sets of unitary invariants for compact operators. This is, indeed, the case. In § III we show that if the appropriate approximates of two compact operators A and B are unitarily equivalent and if $\dim[\mathfrak{N}(A)\cap\mathfrak{N}(A^*)]=\dim[\mathfrak{N}(B)\cap\mathfrak{N}(B^*)]$, then A and B are unitarily equivalent. This, together with the choice of approximate canonical approximating sequences, yields complete sets of unitary invariants for compact operators.

In § IV we make use of a class of compact operators on Hilbert space having well defined numerical traces. This class, called the trace class, has been studied extensively by SCHATTEN (see [6]). We show that if f is a strictly increasing continuous real valued function on the non-negative reals such that $f(0)=0$, then

$$\{t[f([A])\omega(A, A^*)]:\omega(x,y)\in\Omega\} \quad \text{and} \quad \dim[\mathfrak{N}(A)\cap\mathfrak{N}(A^*)]$$

form a complete set of unitary invariants for operators A such that $f([A])$ is a member of the trace class. With each compact operator A we associate a function f_A such that $f_A([A])$ is in the trace class and such that $f_A=f_B$ if A and B are unitarily equivalent; this then extends SPECHT's theorem to compact operators.

SPECHT's theorem extends more directly to the trace class. For this class

$$\{t[\omega(A, A^*)]: \omega(x,y)\in\Omega\} \quad \text{and} \quad \dim[\mathfrak{N}(A)\cap\mathfrak{N}(A^*)]$$

form a complete set of unitary invariants. The same result holds for the Schmidt-class (the class of Hilbert—Schmidt operators), except that the words x and y must be omitted.

II. Preliminaries

We say, as usual, that two operators A and B on a Hilbert space $\mathfrak{H}$ are unitarily equivalent if there is a unitary operator U on $\mathfrak{H}$ such that $UAU^*=B$.

We denote by $\mathfrak{S}(A)$ the subspace $\mathfrak{H}\ominus[\mathfrak{N}(A)\cap\mathfrak{N}(A^*)]$; the subspace $\mathfrak{N}(A)\cap\mathfrak{N}(A^*)$ is the largest subspace which reduces A and on which A is the zero operator.

Definition. Two operators A and B are *isometrically equivalent* if there is a partial isometry U with initial space $\mathfrak{S}(A)$ and final space $\mathfrak{S}(B)$ such that $UAU^*=B$ (or, equivalently, $UA=BU$).

If A and B are unitarily equivalent, say via a unitary operator U, then U maps $\mathfrak{N}(A)\cap\mathfrak{N}(A^*)$ isometrically onto $\mathfrak{N}(B)\cap\mathfrak{N}(B^*)$ and $\mathfrak{S}(A)$ isometrically onto $\mathfrak{S}(B)$, so that A and B are also isometrically equivalent and $\dim[\mathfrak{N}(A)\cap\mathfrak{N}(A^*)]=\dim[\mathfrak{N}(B)\cap\mathfrak{N}(B^*)]$. Conversely, if A and B are isometrically equivalent and if $\dim[\mathfrak{N}(A)\cap\mathfrak{N}(A^*)]=\dim[\mathfrak{N}(B)\cap\mathfrak{N}(B^*)]$, then it is obvious that A and B are unitarily equivalent.

An operator A on $\mathfrak{H}$ is said to be of finite rank if $\dim\mathfrak{R}(A)<\infty$. If $\{\varphi_i\}$ is an orthonormal basis for $\mathfrak{H}$, we define the trace $t(A)$ of an operator A of finite rank to be $\Sigma_i(A\varphi_i,\varphi_i)$. The sum is finite and is independent of the basis chosen (§ IV). If $\mathfrak{H}_1$ is an m-dimensional subspace of $\mathfrak{H}$ containing $\mathfrak{S}(A)$, we can choose $\{\varphi_i\}$ such that $\varphi_1,\dots,\varphi_m$ is a basis for $\mathfrak{H}_1$. Then the trace of A is the trace of the restriction of A to $\mathfrak{H}_1$ as calculated for operators on finite dimensional spaces.

Let A and B be of finite rank and suppose that $t[\omega(A, A^*)] = t[\omega(B, B^*)]$ for each $\omega(x, y) \in \Omega$. Then, by SPECHT's theorem, there is a unitary operator U_1 defined on the subspace $\mathfrak{H}_1$ spanned by $\mathfrak{S}(A)$ and $\mathfrak{S}(B)$ which implements the unitary equivalence of the restrictions of A and B to $\mathfrak{H}_1$. The operator U which equals U_1 on $\mathfrak{H}_1$ and which equals the identity operator on $\mathfrak{H} \ominus \mathfrak{H}_1$ then implements the unitary equivalence of A and B.

If A and B are of finite rank and if $\dim \mathfrak{S}(A) = \dim \mathfrak{S}(B) = n$, there is a unitary operator V which maps $\mathfrak{S}(A)$ isometrically onto $\mathfrak{S}(B)$. If, in addition, the n-dimensional Specht—Pearcy invariants of A and B are equal, the restrictions of VAV^* and B to $\mathfrak{S}(B)$ are unitarily equivalent as operators on $\mathfrak{S}(B)$. Thus, as above, A is unitarily equivalent to B.

We summarize these results in the following

Lemma 2. 1. *Each of the following is a complete set of unitary invariants for operators A of finite rank:*

(1) $\{t\{\omega[A, A^*)]: \omega(x, y) \in \Omega\}$

(2) $\dim \mathfrak{S}(A)$ and $\{t[\omega(A, A^*)]: \omega(x, y) \in \Omega_{\dim \mathfrak{S}(A)}\}$.

III. Unitary equivalence of compact operators

In this section we establish a sort of "continuity" property for isometric equivalence and then use this result to obtain complete sets of unitary invariants for compact operators.

Lemma 3. 1. *Suppose that P and Q are projections of finite rank and that $\{P_n\}$ and $\{Q_n\}$ are sequences of projections converging uniformly to P and Q, respectively. Suppose also that for each n there is a partial isometry U_n whose initial space contains $\mathfrak{R}(P_n)$ and whose final space contains $\mathfrak{R}(Q_n)$ such that $U_n P_n = Q_n U_n$. Then there is a subsequence $\{U_{n_k}\}$ of $\{U_n\}$ such that the sequence of the restrictions of the U_{n_k}'s to $\mathfrak{R}(P)$ converges to a linear map sending $\mathfrak{R}(P)$ isometrically onto $\mathfrak{R}(Q)$.*

Proof. Let $x_1, \dots, x_p$ be an orthonormal basis of $\mathfrak{R}(P)$. It suffices to find a subsequence $\{U_{n_k}\}$ of $\{U_n\}$ such that $U_{n_k} x_i \to y_i$ strongly, $i = 1, \dots, p$, where $y_1, \dots, y_p$ is some orthonormal basis of $\mathfrak{R}(Q)$. Since $QU_n x_i \in \mathfrak{R}(Q)$, which is finite dimensional, and since

$$1 \geqq \|QU_n x_i\| = \|U_n P_n x_i - U_n P_n x_i + QU_n x_i\| = \|U_n P_n x_i - Q_n U_n x_i + QU_n x_i\| \geqq$$
$$\geqq \|U_n P_n x_i\| - \|Q_n U_n x_i - QU_n x_i\| = \|P_n x_i\| - \|(Q_n - Q)U_n x_i\| =$$
$$= \|Px_i + P_n x_i - Px_i\| - \|(Q_n - Q)U_n x_i\| \geqq 1 - (\|P_n - P\| + \|Q_n - Q\|) \to 1,$$

there is a subsequence $\{U_{n_k}\}$ of $\{U_n\}$ such that $QU_{n_k} x_i \to y_i$, $i = 1, \dots, p$, and $\|y_i\| = 1$. Moreover,

$$0 \leqq \|U_n x_i - QU_n x_i\| = \|U_n P x_i - U_n P_n x_i + Q_n U_n x_i - QU_n x_i\| \leqq$$
$$\leqq \|U_n(P - P_n)x_i\| + \|(Q_n - Q)U_n x_i\| \to 0,$$

so

$$U_{n_k} x_i \to y_i, \quad i = 1, \dots, p.$$

Also,

$$\| U_n P_n x_i - U_n x_i \| = \| U_n (P_n - P) x_i \| \to 0,$$

so

$$U_{n_k} P_{n_k} x_i \to y_i .$$

If $i \neq j$,

$$|(U_n P_n x_i, U_n P_n x_j)| = |(P_n x_i, P_n x_j)| =$$

$$= |(P_n x_i, x_j) - (x_i, x_j)| = |([P_n - P] x_i, x_j)| \leqq \| P_n - P \| \to 0,$$

and hence

$$(y_i, y_j) = 0.$$

Since, from [1], p. 73, if $\| P_n - P \| < 1$ and $\| Q_n - Q \| < 1$, then

$$\dim \mathfrak{R}(P) = \dim \mathfrak{R}(P_n) = \dim \mathfrak{R}(Q_n) = \dim \mathfrak{R}(Q),$$

it follows that $y_1, \ldots, y_p$ is a basis of $\mathfrak{R}(Q)$. This completes the proof of the lemma.

Lemma 3.2. *Suppose that $\{P_k\}$ and $\{Q_k\}$ are sequences of projections of finite rank and that, for each k, $\{P_{k,n}\}$ and $\{Q_{k,n}\}$ are sequences of projections converging in the uniform topology to P_k and Q_k, respectively. Suppose also that, for each n, there is a partial isometry U_n whose initial space contains $\mathfrak{R}(P_{k,n})$ and whose final space contains $\mathfrak{R}(Q_{k,n})$ such that, for each k, $U_n P_{k,n} = Q_{k,n} U_n$. Then there is a partial isometry U such that for each k the initial space of U contains $\mathfrak{R}(P_k)$ and the final space of U contains $\mathfrak{R}(Q_k)$ and such that $UP_k = Q_k U$.*

Proof. We first choose subsequences $\{U_n^{(r)}\}$ of $\{U_n\}$ inductively. Let $\{U_n^{(0)}\} = \{U_n\}$, and suppose that $\{U_n^{(0)}\}, \ldots, \{U_n^{(r)}\}$ have been chosen. By lemma 3.1, we may choose $\{U_n^{(r+1)}\}$ to be a subsequence of $\{U_n^{(r)}\}$ converging uniformly on $\mathfrak{R}(P_{r+1})$. The diagonal sequence $\{U_n^{(n)}\}$ converges on $\mathfrak{R}(P_k)$ to a map sending $\mathfrak{R}(P_k)$ isometrically onto $\mathfrak{R}(Q_k)$ for each k. Let $\mathfrak{M}$ be the submanifold spanned by $\{\mathfrak{R}(P_k)\}_{k=1}^{\infty}$ and let $x \in \mathfrak{M}$, say $x = x_1 + \ldots + x_r$, where $x_k \in \mathfrak{R}(P_k)$, $k = 1, \ldots, r$. Since the sequence of vectors $\{U_n^{(n)} x_k\}_{n=1}^{\infty}$ converges strongly for each $k = 1, \ldots, r$, and since $U_n^{(n)} x = U_n^{(n)} x_1 + \ldots + U_n^{(n)} x_r$, the sequence of operators $\{U_n^{(n)}\}$ converges strongly on $\mathfrak{M}$ to an operator U_0 (defined on $\mathfrak{M}$) such that $U_0 P_k = Q_k U_0$, $k = 1, 2, \ldots$. Also, setting

$$\varepsilon_n = \| P_1 - P_{1,n} \| \, \| x_1 \| + \ldots + \| P_r - P_{r,n} \| \, \| x_r \|,$$

we have

$$\| x \| \geqq \| U_n x \| = \| U_n x_1 + \ldots + U_n x_r \| =$$

$$= \| U_n [P_{1,n} x_1 + \ldots + P_{r,n} x_r] + U_n [(P_1 - P_{1,n}) x_1 + \ldots + (P_r - P_{r,n}) x_r] \| \geqq$$

$$\geqq \| U_n [P_{1,n} x_1 + \ldots + P_{r,n} x_r] \| - \varepsilon_n = \| P_{1,n} x_1 + \ldots + P_{r,n} x_r \| - \varepsilon_n =$$

$$= \| x_1 + \ldots + x_r + (P_{1,n} - P_1) x_1 + \ldots + (P_{r,n} - P_r) x_r \| - \varepsilon_n \geqq$$

$$\geqq \| x_1 + \ldots + x_r \| - 2\varepsilon_n \to \| x \|,$$

so $\| U_0 x \| = \| x \|$. The extension U of U_0 defined by continuity on the closure of $\mathfrak{M}$ and defined to be zero on $\mathfrak{H} \ominus \mathfrak{M}$ has the desired properties.

Theorem 1. *Let A and B be compact operators on a Hilbert space. If there exist sequences $\{A_n\}$ and $\{B_n\}$ of (not necessarily compact) operators converging uniformly to A and B, respectively, such that, for each n, A_n is isometrically equivalent to B_n, then A is isometrically equivalent to B.*

Proof. We denote by Σ_A the spectrum of A, by $\operatorname{Re} A$ the operator $(A+A^*)/2$, and by $\operatorname{Im} A$ the operator $(A-A^*)/2i$. If Δ is a Borel subset of the line, we denote by $E_n(\Delta)$, $E(\Delta)$, $F_n(\Delta)$, $F(\Delta)$, $G_n(\Delta)$, $G(\Delta)$, $H_n(\Delta)$ and $H(\Delta)$ the spectral projections of $\operatorname{Re} A_n$, $\operatorname{Re} A$, $\operatorname{Im} A_n$, $\operatorname{Im} A$, $\operatorname{Re} B_n$, $\operatorname{Re} B$, $\operatorname{Im} B_n$, and $\operatorname{Im} B$, respectively, associated with Δ. Since A_n is isometrically equivalent to B_n, there is a partial isometry U_n with initial space $\mathfrak{S}(A_n)$ and final space $\mathfrak{S}(B_n)$ such that $U_nA_n=B_nU_n$. If Δ is any Borel subset of the line not containing zero, $\mathfrak{N}[E_n(\Delta)]$ and $\mathfrak{N}[F_n(\Delta)]$ are contained in $\mathfrak{S}(A_n)$, and $\mathfrak{N}[G_n(\Delta)]$ and $\mathfrak{N}[H_n(\Delta)]$ are contained in $\mathfrak{S}(B_n)$. As in the case of unitary equivalence, $U_nE_n(\Delta)=G_n(\Delta)U_n$ and $U_nF_n(\Delta)=H_n(\Delta)U_n$. In order to show that A is isometrically equivalent to B, it suffices to show that there is a partial isometry U with initial space $\mathfrak{S}(A)$ and final space $\mathfrak{S}(B)$ such that $UE(\Delta)=G(\Delta)U$ and $UF(\Delta)=H(\Delta)U$ for all Borel subsets of the line not containing zero. In fact, since each non-zero member of $\Sigma_{\operatorname{Re} A}$ or $\Sigma_{\operatorname{Im} A}$ is isolated, it suffices to show that $\Sigma_{\operatorname{Re} A}=\Sigma_{\operatorname{Re} B}$, $\Sigma_{\operatorname{Im} A}=\Sigma_{\operatorname{Im} B}$, and that if $\lambda\neq 0$ then $UE[(\lambda-\varepsilon,\lambda+\varepsilon)]=G[(\lambda-\varepsilon,\lambda+\varepsilon)]U$ and $UF[(\lambda-\varepsilon,\lambda+\varepsilon)]=H[(\lambda-\varepsilon,\lambda+\varepsilon)]U$ for all sufficiently small $\varepsilon>0$.

We first show that if $\lambda\neq 0$, then $\lambda\in\Sigma_{\operatorname{Re} A}$ if and only if for each $\varepsilon>0$, $E_n[(\lambda-\varepsilon,\lambda+\varepsilon)]\neq 0$ for $n>n_0(\varepsilon)$. This and the analogous results for $\Sigma_{\operatorname{Im} A}$, $\Sigma_{\operatorname{Re} B}$ and $\Sigma_{\operatorname{Im} B}$ guarantee that $\Sigma_{\operatorname{Re} A}=\Sigma_{\operatorname{Re} B}$ and $\Sigma_{\operatorname{Im} A}=\Sigma_{\operatorname{Im} B}$.

If $\lambda\notin\Sigma_{\operatorname{Re} A}$, let ε be less than the distance d from λ to $\Sigma_{\operatorname{Re} A}$. Then $\|(\zeta-\operatorname{Re} A)^{-1}\|$ is bounded for $|\zeta-\lambda|<\varepsilon$, say by M. One can easily see by power series expansions that if $\|\operatorname{Re} A_n-\operatorname{Re} A\|<1/M$, then $(\zeta-\operatorname{Re} A_n)$ is invertible, so that the interval $(\lambda-\varepsilon,\lambda+\varepsilon)$ contains no points of $\Sigma_{\operatorname{Re} A_n}$.

If $\lambda\in\Sigma_{\operatorname{Re} A}$, $\lambda\neq 0$, let d be the distance from λ to $\Sigma_{\operatorname{Re} A}-\{\lambda\}$; d is positive since A is compact. We shall show that $E_n[(\lambda-\varepsilon,\lambda+\varepsilon)]\to E[(\lambda-\varepsilon,\lambda+\varepsilon)]$ uniformly, at least for $0<\varepsilon<d/3$. As above, the intervals $(\lambda-2d/3,\lambda-\varepsilon)$ and $(\lambda+\varepsilon,\lambda+2d/3)$ contain no points of $\Sigma_{\operatorname{Re} A_n}$ for n sufficiently large. Thus

$$E_n[(\lambda-\varepsilon,\lambda+\varepsilon)] = (1/2\pi i)\oint_C(\zeta-\operatorname{Re} A_n)^{-1}\,d\zeta$$

and

$$E[(\lambda-\varepsilon,\lambda+\varepsilon)] = (1/2\pi i)\oint_C(\zeta-\operatorname{Re} A)^{-1}\,d\zeta,$$

where C is the circle $|\zeta-\lambda|=d/2$. Since inversion is a continuous operation where it is defined,

$$\|E_n[(\lambda-\varepsilon,\lambda+\varepsilon)]-E[(\lambda-\varepsilon,\lambda+\varepsilon)]\| =$$

$$= (1/2\pi)\left\|\oint_C[(\zeta-\operatorname{Re} A_n)^{-1}-(\zeta-\operatorname{Re} A)^{-1}]\,d\zeta\right\| \leqq$$

$$\leqq (1/2\pi)\oint_C\|(\zeta-\operatorname{Re} A_n)^{-1}-(\zeta-\operatorname{Re} A)^{-1}\|\,|d\zeta|\to 0 \quad\text{as}\quad n\to\infty.$$

Now, let $\alpha_1,\alpha_2,\dots$ and $\beta_1,\beta_2,\dots$ be the distinct non-zero members of $\Sigma_{\operatorname{Re} A}$ and $\Sigma_{\operatorname{Im} A}$, respectively, $\Delta_k=(\alpha_k-d/3,\alpha_k+d/3)$ where d is the distance from

α_k to $\Sigma_{\mathrm{Re}\,A}-\{\alpha_k\}$, and $\Delta_k'=(\beta_k-d/3,\ \beta_k+d/3)$, where d is the distance from β_k to $\Sigma_{\mathrm{Im}\,A}-\{\beta_k\}$. Set

$$\begin{aligned} P_{2k-1}&=E(\Delta_k), & P_{2k-1,\,n}&=E_n(\Delta_k),\\ P_{2k}&=F(\Delta_k'), & P_{2k,\,n}&=F_n(\Delta_k'),\\ Q_{2k-1}&=G(\Delta_k), & Q_{2k-1,\,n}&=G_n(\Delta_k),\\ Q_{2k}&=H(\Delta_k'), & Q_{2k,\,n}&=H_n(\Delta_k'). \end{aligned}$$

An application of lemma 3.2 completes the proof.

We now apply the preceding results to the problem of obtaining complete sets of unitary invariants for compact operators on Hilbert space. For this purpose, let A and B be any two compact operators on a Hilbert space $\mathfrak{H}$. We order the distinct non-zero eigenvalues of $\mathrm{Re}\,A$, $\mathrm{Im}\,A$, $\mathrm{Re}\,B$ and $\mathrm{Im}\,B$ and denote these sequences by $\{\alpha_k\}$, $\{\beta_k\}$, $\{\gamma_k\}$ and $\{\delta_k\}$, respectively. We require of the orderings that $|\alpha_k|\geqq|\alpha_{k+1}|$, that $|\alpha_k|=|\alpha_{k+1}|$ implies that $\alpha_k>0$ and $\alpha_{k+1}<0$, and analogously for $\{\beta_k\}$, $\{\gamma_k\}$ and $\{\delta_k\}$. (This guarantees that if $\Sigma_{\mathrm{Re}\,A}=\Sigma_{\mathrm{Re}\,B}$, then the sequences $\{\alpha_k\}$ and $\{\gamma_k\}$ are identical, and similarly for $\Sigma_{\mathrm{Im}\,A}$ and $\Sigma_{\mathrm{Im}\,B}$.) If E_k, F_k, G_k, and H_k are the spectral projections of $\mathrm{Re}\,A$, $\mathrm{Im}\,A$, $\mathrm{Re}\,B$ and $\mathrm{Im}\,B$ corresponding to α_k, β_k, γ_k and δ_k, respectively, then A and B can be written $A=\sum_k\alpha_kE_k+i\sum_k\beta_kF_k$ and $B=\sum_k\gamma_kG_k+i\sum_k\delta_kH_k$. We write $A_n=\sum_{k=1}^n\alpha_kE_k+i\sum_{k=1}^n\beta_kF_k$ and $B_n=\sum_{k=1}^n\gamma_kG_k+i\sum_{k=1}^n\delta_kH_k$, with obvious modifications if any of the sequences are finite. Then $\{A_n\}$ and $\{B_n\}$ converge uniformly to A and B, respectively.

Now suppose that A is isometrically equivalent to B, say $UAU^*=B$. Then $B=\sum_k\alpha_kUE_kU^*+i\sum_k\beta_kUF_kU^*$. Thus, since the spectral representation of an operator is unique, $\alpha_k=\gamma_k$, $\beta_k=\delta_k$, $UE_kU^*=G_k$, and $UF_kU^*=H_k$ for each k. It follows that $UA_nU^*=B_n$, so for each n, A_n is unitarily equivalent to B_n. On the other hand, if, for each n, A_n is unitarily equivalent to B_n, then A is isometrically equivalent to B by theorem 1. We have thus proved

Theorem 2. *Let A be compact, let the sequence $\{A_n\}$ be obtained from the Cartesian decomposition of A as described above, and let I be either of the complete sets of unitary invariants for operators of finite rank described in lemma 2.1. Then $\{I(A_n)\}_{n=1}^\infty$ is a complete set of isometric invariants for A. The addition of* $\dim[\mathfrak{N}(A)\cap\mathfrak{N}(A^*)]$ *to the above collection of isometric invariants yields a complete set of unitary invariants for A.*

A different complete set of unitary invariants can be obtained by using the polar decomposition of a compact operator to obtain a canonical set of approximating operators of finite rank. Let $\{\mu_k\}$ be the non-zero eigenvalues of $[A]$, $\mu_1>\mu_2>\ldots$, and let E_k be the (finite dimensional) spectral projection of $[A]$ associated with μ_k. Then the series $\sum_k\mu_kE_k$ converges to $[A]$ in the uniform topology. Let $A=W[A]$ be the polar decomposition of A, and denote by W_k the partial isometry of finite rank WE_k. The series $\sum_k\mu_kW_k=\sum_k\mu_kWE_k=W\sum_k\mu_kE_k$ converges to A in the uniform topology. Let $B=\sum_k \nu_k V_k$ in a similar fashion, and suppose U implements the isometric equivalence of A and B, $UAU^*=B$. Let T_k be the partial isometry UW_kU^* and let F_k be the projection on the initial space of T_k. The series $\sum_k T_k$ converges in the strong operator topology to a partial isometry T and

$$B = U(\sum_k \mu_k W_k)U^* = \sum_k \mu_k UW_k U^* = \sum_k \mu_k T_k = \sum_k \mu_k T_k F_k = T\sum_k \mu_k F_k.$$

The operator $\sum_k \mu_k F_k$ is positive, so, by the unicity of the polar decomposition of an operator, $\nu_k=\mu_k$ and $V_k=T_k=UW_kU^*$. Thus, if $A_n=\sum_{k=1}^{n}\mu_k W_k$ and $B_n=\sum_{k=1}^{n}\nu_k V_k$, we have $UA_nU^*=B_n$. Conversely, $\{A_n\}$ and $\{B_n\}$ converge uniformly to A and B, respectively, so, by theorem 1, we have

Theorem 3. *Let A be compact, let $A_n=\sum_{k=1}^{n}\mu_k W_k$ be obtained from the polar decomposition of A as described above, and let I be either of the complete sets of unitary invariants for operators of finite rank described in lemma 2. 1. Then $\{I(A_n)\}_{n=1}^{\infty}$ is a complete set of isometric invariants for A. The addition of* $\dim[\mathfrak{N}(A)\cap\mathfrak{N}(A^*)]$ *to the above collection of isometric invariants yields a complete set of unitary invariants for A.*

IV. Unitary invariants involving traces

Before discussing the Schmidt- and trace-classes of operators we prove a lemma which will be useful in the proof of theorem 4.

Lemma 4. 1. *Suppose that $\{a_k\}$ and $\{b_k\}$ are sequences of complex numbers, that $\{\mu_k\}$ and $\{\nu_k\}$ are strictly decreasing sequences of real numbers converging to zero, and that $\sum_k|a_k|\mu_k^2<\infty$ and $\sum_k|b_k|\nu_k^2<\infty$. Suppose also that, for each positive integer p, $\sum_k a_k\mu_k^{2p}=\sum_k b_k\nu_k^{2p}$. Then:*

(1) *If, for each k, $a_k, b_k\neq 0$, then $a_k=b_k$ and $\mu_k=\nu_k$ for each k.*

(2) *If $\mu_{k_1}=\nu_{k_1}$, then $a_{k_1}=b_{k_1}$.*

Proof. The series $\sum_k a_k\mu_k^2/(z^2-\mu_k^2)$ converges uniformly in any domain in which z is uniformly bounded away from $\{\pm\mu_k\}$ to a function which we shall denote $f(z)$, and similarly for $g(z)=\sum_k b_k\nu_k^2/(z^2-\nu_k^2)$. $f(z)$ has a pole of order one and residue $\pm\frac{1}{2}a_k\mu_k$ at $z=\pm\mu_k$ for each k such that $a_k\neq 0$, a limit point of poles at $z=0$, and is holomorphic elsewhere; $g(z)$ has a pole of order one and residue $\pm\frac{1}{2}b_k\nu_k$ at $z=\pm\nu_k$ for each k such that $b_k\neq 0$, a limit point of poles at $z=0$, and is holomorphic elsewhere. For z in the domain $\{z:|z|>\mu_1\}$ we can expand $\mu_k^2/(z^2-\mu_k^2)$ about $z=\infty$ to obtain

$$f(z)=\sum_k a_k\sum_p(\mu_k/z)^{2p}.$$

In order to change the order of summation, we note that

$$\sum_k |a_k| \sum_p (\mu_k/|z|)^{2p} = \sum_k |a_k| \mu_k^2/(|z|^2-\mu_k^2) \leqq (\sum_k |a_k| \mu_k^2)/(|z|^2-\mu_1^2) < \infty;$$

thus

$$f(z) = \sum_p (\sum_k a_k \mu_k^{2p}) 1/z^{2p}.$$

Similarly

$$g(z) = \sum_p (\sum_k b_k \nu_k^{2p}) 1/z^{2p}$$

for $|z| > \nu_1$. Thus, by hypothesis, $f(z) = g(z)$ for $|z| > \max(\mu_1, \nu_1)$. By analytic continuation, $f(z)$ and $g(z)$ are identical, and the conclusion of the lemma follows.

The reader is referred to [6] for the proofs of the following and other interesting facts about the trace- and Schmidt-classes.

Let A be an operator on a Hilbert space $\mathfrak{H}$ and let $\{\varphi_i\}$ be an orthonormal basis of $\mathfrak{H}$. A is in the *Schmidt-class* (σc) if $\sum_i \|A\varphi_i\|^2 < \infty$; the sum is independent of the basis chosen. The Schmidt-class is a proper subset of the set of compact operators. If $\mathfrak{H}$ is L_2 of the unit interval, (σc) consists of all operators of the form

$$(Af)(x) = \int K(x, y) f(y)\, dy$$

where $K(x, y)$ is in L_2 of the unit square.

An operator A is in the *trace-class* (τc) if A is the product of two members of the Schmidt-class. The following are equivalent:

(1) $A \in (\tau c)$,

(2) $[A] \in (\tau c)$,

(3) $[A]^{\frac{1}{2}} \in (\sigma c)$,

(4) $\sum_i ([A]\varphi_i, \varphi_i) < \infty$ for some, and thus every, orthonormal basis $\{\varphi_i\}$ of $\mathfrak{H}$.

If A is in the trace class and $\{\varphi_i\}$ is an orthonormal basis of $\mathfrak{H}$, then $\sum |(A\varphi_i, \varphi_i)| < \infty$. The trace $t(A) = \sum (A\varphi_i, \varphi_i)$ of A is independent of the basis with respect to which it is computed. If $A, B \in (\tau c)$, X is any bounded operator, and c is a complex number, then

(1) $t(A^*) = \overline{t(A)}$,

(2) $t(cA) = ct(A)$,

(3) $(A+B) \in (\tau c)$ and $t(A+B) = t(A) + t(B)$,

(4) $AX, XA \in (\tau c)$ and $t(AX) = t(XA)$ (the traces of commutators are zero).

Definition. Let f be any continuous strictly increasing real valued function on the non-negative real numbers such that $f(0) = 0$. The class $(\tau c)_f$ is the set of all operators A such that $f([A]) \in (\tau c)$.

It is easy to see that an operator A is compact if and only if $f([A])$ is compact; thus $(\tau c)_f$ is a subset of the compacts. If A is compact, $A = \sum_k \mu_k W_k$ as in § III, then $[A] = \sum_k \mu_k E_k$, where E_k is the projection $W_k^* W_k$. We denote by f_A the convex support (see [3]) of the set of points $(\mu_k, 1/(k^2 \dim [\mathfrak{R}(E_k)]))$. If $\{\varphi_i\}$ is an orthonormal basis of $\mathfrak{H}$ consisting of eigenvectors of $[A]$, then

$$\sum_i (f([A])\varphi_i, \varphi_i) = \sum_k \dim [\mathfrak{R}(E_k)] f(\mu_k) \leqq \sum_k \dim [\mathfrak{R}(E_k)]/(k^2 \dim [\mathfrak{R}(E_k)]) < \infty,$$

so $A \in (\tau c)_{f_A}$. If A and B are compact and unitarily equivalent, then so are $[A]$ and $[B]$, so $f_A = f_B$. Thus, if I_f is a complete set of unitary invariants for $(\tau c)_f$, f_A and $I_{f_A}(A)$ form a complete set of unitary invariants for all compact operators.

Although we shall not need to make use of this fact, we note that an easy application of lemma 4. 1 shows that $\{t[(f(A))^n]\}_{n=1}^{\infty}$ is a complete set of isometric invariants for the positive members of $(\tau c)_f$.

Theorem 4. *Let Ω denote the free multiplicative semigroup of all words $\omega(x, y)$ in the free variables x and y. A complete set of isometric invariants for operators A in $(\tau c)_f$ is*

$$\{t[f([A])\omega(A, A^*)]: \omega(x, y) \in \Omega\}.$$

The addition of $\dim [\mathfrak{N}(A) \cap \mathfrak{N}(A^*)]$ *to the above set of isometric invariants yields a complete set of unitary invariants for $(\tau c)_f$.*

Proof. Since traces are independent of the bases with respect to which they are computed and since $t[f([A])\omega(A, A^*)]$ is not affected by the dimension of $\mathfrak{N}(A) \cap \cap \mathfrak{N}(A^*)$, $t[f([A])\omega(A, A^*)]$ is preserved under isometric equivalence.

Now suppose that A and B are in $(\tau c)_f$ and that $t[f([A])\omega(A, A^*)] = t[f([B]) \cdot \cdot \omega(B, B^*)]$ for each $\omega(x, y) \in \Omega$. Let $A = \sum_k \mu_k W_k$, $A_n = \sum_{k=1}^{n} \mu_k W_k$, $B = \sum_k \nu_k V_k$, and $B_n = \sum_{k=1}^{n} \nu_k V_k$ as in § III. By theorem 3, it suffices to show that $t[\omega(A_n, A_n^*)] = = t[\omega(B_n, B_n^*)]$ for each $\omega(x, y) \in \Omega$ and each n.

We first show that $\mu_k = \nu_k$ for each k. Choose an orthonormal set of vectors $\{\varphi_i\}$ such that $\varphi_{i_k}, \dots, \varphi_{i_{k+1}-1}$ is a basis of the initial space of W_k. Since $f([A])(A^*A)^p = \sum_k f(\mu_k)\mu_k^{2p} W_k^* W_k$ is in (τc), we have, for each positive integer p,

$$t[f([A])(A^*A)^p] = \sum_i (f([A])(A^*A)^p \varphi_i, \varphi_i) =$$
$$= \sum_k \sum_{i=i_k}^{i_{k+1}-1} (f([A])(A^*A)^p \varphi_i, \varphi_i) = \sum_k f(\mu_k) t(W_k^* W_k)\mu_k^{2p}.$$

Similarly

$$t[f([B])(B^*B)^p] = \sum_k f(\nu_k) t(V_k^* V_k)\nu_k^{2p}.$$

Setting $a_k = f(\mu_k)t(W_k^* W_k) \neq 0$ and $b_k = f(\nu_k)t(V_k^* V_k) \neq 0$, we conclude from lemma 4. 1 that $\mu_k = \nu_k$ for each k.

For each $\omega(x, y) \in \Omega$ we write $\omega(x, y) = \prod_{j=1}^{r} z_j$, where $z_j = x$ or $z_j = y$. Since the traces of commutators are zero and the trace of the adjoint of an operator is the complex conjugate of the trace of the operator, it suffices to show that $t[\omega(A_n, A_n^*)] = t[\omega(B_n, B_n^*)]$ for each ω such that $z_1 = y$.

In an induction argument later in the proof we shall consider products involving not only A and A^* but also the partial isometries $W_1, W_1^*, W_2, W_2^*, \dots$, and the corresponding products involving B, B^*, $V_1, V_1^*, V_2, V_2^*, \dots$. For this purpose we introduce the free semigroup $\hat{\Omega}$ of words $\hat{\omega}(x, y, x_1, y_1, x_2, y_2, \ldots) = \prod_{j=1}^{m} \zeta_j$ where $\zeta_j \in \{x, y, x_1, y_1, x_2, y_2, \ldots\}$. Denote by $\lambda(\hat{\omega})$ the number of j's, $1 \leq j \leq m$, such that $\zeta_j \in \{x_1, y_1, x_2, y_2, \ldots\}$. (Thus if no ζ_j is equal to x or y, $\lambda(\hat{\omega})$ is the length m of $\hat{\omega}$.) For simplicity of notation we write

$$\hat{\omega}(A) = \hat{\omega}(A, A^*, W_1, W_1^*, W_2, W_2^*, \ldots\}$$

and

$$\hat{\omega}(B) = \hat{\omega}(B, B^*, V_1, V_1^*, V_2, V_2^*, \ldots\}.$$

With each $\omega(x, y) = \prod_{j=1}^{r} z_j \in \Omega$ and each r-tuple $k_1, \ldots, k_r$ of positive integers we associate the member $\hat{\omega}_{\omega, k_1, \ldots, k_r}(x, y, x_1, y_1, \ldots) = \prod_{j=1}^{r} \zeta_j$ of $\hat{\Omega}$ such that $\zeta_j = x_{k_j}$ if $z_j = x$ and $\zeta_j = y_{k_j}$ if $z_j = y$. Then

$$\omega(A_n, A_n^*) = \sum_{k_1, \ldots, k_r = 1}^{n} \mu_{k_1} \ldots \mu_{k_r} \hat{\omega}_{\omega, k_1, \ldots, k_r}(A)$$

and

$$\omega(B_n, B_n^*) = \sum_{k_1, \ldots, k_r = 1}^{n} \nu_{k_1} \ldots \nu_{k_r} \hat{\omega}_{\omega, k_1, \ldots, k_r}(B).$$

We now give an example to illustrate the notation introduced above. If $\omega(x, y) = y^2 x$, the word $\hat{\omega}_{y^2 x, k_1, k_2, k_3}(x, y, x_1, y_1, \ldots)$ is then $y_{k_1} y_{k_2} x_{k_3}$. We have

$$\omega(A_n, A_n^*) = \left(\sum_{k_1=1}^{n} \mu_{k_1} A_{k_1}^*\right)\left(\sum_{k_2=1}^{n} \mu_{k_2} A_{k_2}^*\right)\left(\sum_{k_3=1}^{n} \mu_{k_3} A_{k_3}\right) =$$

$$= \sum_{k_1, k_2, k_3 = 1}^{n} \mu_{k_1} \mu_{k_2} \mu_{k_3} A_{k_1}^* A_{k_2}^* A_{k_3} = \sum_{k_1, k_2, k_3 = 1}^{n} \mu_{k_1} \mu_{k_2} \mu_{k_3} \hat{\omega}_{y^2 x, k_1, k_2, k_3}(A).$$

Since we already know that $\mu_k = \nu_k$ for all k, it suffices to show that $t[\hat{\omega}_{\omega, k_1, \ldots, k_r}(A)] = t[\hat{\omega}_{\omega, k_1, \ldots, k_r}(B)]$ for all $\omega(x, y) \in \Omega$ such that $z_1 = y$ and for all $k_1, \ldots, k_r$; that is, that $t[\hat{\omega}(A)] = t[\hat{\omega}(B)]$ for all $\hat{\omega} = \prod_{j=1}^{r} \zeta_j$ such that $\zeta_j \in \{x_1, y_1, x_2, y_2, \ldots\}$ and $\zeta_1 = y_{k_1}$. We note that for such an $\hat{\omega}$, since $W_k^* W_k W_{k_1}^* = \delta_{k, k_1} W_{k_1}$,

$$f([A])\hat{\omega}(A) = \sum_k f(\mu_k) W_k^* W_k \hat{\omega}(A) = f(\mu_{k_1})\hat{\omega}(A);$$

similarly,

$$f([B])\hat{\omega}(B) = f(\nu_{k_1})\hat{\omega}(B) = f(\mu_{k_1})\hat{\omega}(B).$$

Thus, for such an $\hat{\omega}$, if $t[f([A])\hat{\omega}(A)] = t[f([B])\hat{\omega}(B)]$, then $t[\hat{\omega}(A)] = t[\hat{\omega}(B)]$. We conclude the proof by proving the following by induction on $\lambda(\hat{\omega})$:

($*$) If $\hat{\omega} \in \hat{\Omega}$, then $t[f([A])\hat{\omega}(A)] = t[f([B])\hat{\omega}(B)]$.

Note that, since then traces of commutators are zero, if ($*$) holds for all $\hat{\omega} \in \hat{\Omega}$ such that $\lambda(\hat{\omega}) = q$, then $t[\hat{\omega}(A) f([A])] = t[\hat{\omega}(B) f([B])]$ if $\lambda(\hat{\omega}) = q$, and $t[\hat{\omega}_1(A) \cdot f([A])\hat{\omega}_2(A)] = t[\hat{\omega}_1(B) f([B])\hat{\omega}_2(B)]$ if $\lambda(\hat{\omega}_1) + \lambda(\hat{\omega}_2) = q$.

If $\lambda(\hat{\omega})=0$, then there is an $\omega(x, y)\in\Omega$ such that $\hat{\omega}(A)=\omega(A, A^*)$ and $\hat{\omega}(B)=\omega(B, B^*)$, so $(\ast)$ is true by hypothesis.

We now suppose that $(\ast)$ holds for $\lambda=q$, that $\lambda(\hat{\omega})=q+1$, and prove that $t[f([A])\hat{\omega}(A)]=t[f([B])\hat{\omega}(B)]$. By taking adjoints if necessary and using the fact that the traces of commutators are zero, it suffices to show that $t[CW_k]=t[DV_k]$ in the three cases

(i) $C=f([A])\hat{\omega}_0(A), \quad D=f([B])\hat{\omega}_0(B),\ \lambda(\hat{\omega}_0)=q,$

(ii) $C=\hat{\omega}_0(A)f([A]), \quad D=\hat{\omega}_0(B)f([B]),\ \lambda(\hat{\omega}_0)=q,$

and

(iii) $C=\hat{\omega}_1(A)f([A])\hat{\omega}_2(A), \quad D=\hat{\omega}_1(B)f([B])\hat{\omega}_2(B), \quad \lambda(\hat{\omega}_1)+\lambda(\hat{\omega}_2)=q.$

In each of the three cases, the induction hypothesis guarantees that

$$t[CA(A^*A)^p]=t[DB(B^*B)^p]$$

for each positive integer p. As above, we choose an orthonormal set of vectors $\{\varphi_i\}$ such that $\varphi_{i_k}, \ldots, \varphi_{i_{k+1}-1}$ is a basis of the initial space of W_k. Then

$$\mu_k^{2p+1}\, t[CW_k] = t[\mu_k^{2p+1}\, CW_k] = \sum_{i=i_k}^{i_{k+1}-1} (CA(A^*A)^p\varphi_i, \varphi_i),$$

so

$$t[CA(A^*A)^p] = \sum_i (CA(A^*A)^p\varphi_i, \varphi_i) =$$

$$= \sum_k \sum_{i=i_k}^{i_{k+1}-1} (CA(A^*A)^p\varphi_i, \varphi_i) = \sum_k \mu_k^{2p+1}\, t[CW_k].$$

Now, since $CA(A^*A)^p$ is in the trace class,

$$\sum_k \mu_k^{2p+1}\, |t[CW_k]| = \sum_k \left| \sum_{i=i_k}^{i_{k+1}-1} (CA(A^*A)^p\varphi_i, \varphi_i)\right| \leqq \sum_i |(CA(A^*A)^p\varphi_i, \varphi_i)| < \infty.$$

Similarly

$$t[DB(B^*B)^p] = \sum_k v_k^{2p+1}\, t[DV_k] = \sum_k \mu_k^{2p+1}\, t[DV_k]$$

and

$$\sum_k v_k^{2p+1}\, |t[DV_k]| < \infty.$$

Setting $a_k=\mu_k t[CW_k]$ and $b_k=\mu_k t[DV_k]$, we can conclude from lemma 4.1 that $t[CW_k]=t[DV_k]$ for all k, which completes the proof of theorem.

Corollary 4.2. *Let Ω denote the free multiplicative semigroup in the free variables x and y. Complete sets of isometric invariants for operators A in the trace- and Schmidt-classes are $\{t[\omega(A, A^*)]:\omega(x, y)\in\Omega\}$ and $\{t[(A^*A)\omega(A, A^*)]:\omega(x, y)\in\Omega\}$, respectively. The addition of* $\dim[\mathfrak{N}(A)\cap\mathfrak{N}(A^*)]$ *to the above sets of isometric invariants yields complete sets of unitary invariants.*

Proof. The Schmidt-class is the class $(\tau c)_f$ where $f(x)=x^2$, so the result for the Schmidt-class is a special case of theorem 4. The result for the trace-class follows from the fact that the trace-class is a subset of the Schmidt-class.

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(Received November 27, 1965)