On rootless operators and operators without logarithms

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1. Introduction

Let $\mathfrak{H}$ be an infinite dimensional, separable, complex Hilbert space, and denote by $\mathfrak{L}(\mathfrak{H})$ the algebra of all bounded linear operators on $\mathfrak{H}$. Topological properties of subsets of $\mathfrak{L}(\mathfrak{H})$ under discussion always refer to the uniform operator topology. An operator $A \in \mathfrak{L}(\mathfrak{H})$ is said to be *rootless* if for every positive integer $n \geqq 2$, A fails to have any n-th roots. Throughout this note, the set of *invertible, rootless* operators on $\mathfrak{H}$ is denoted by $\mathfrak{R}$. That $\mathfrak{R}$ is non-empty was first proved by HALMOS, LUMER, and SCHÄFFER in [2], and that $\mathfrak{R}$ has non-void interior was established in [3], and again proved in [5] and [6]. In fact, all previously known examples of operators in $\mathfrak{R}$ are interior points of $\mathfrak{R}$, and this caused LUMER [5] to ask if $\mathfrak{R}$ is open. In this note we generalize the methods of [1], which led to the construction of a certain class of operators without square roots, and thereby prove the following

Theorem 1. *The set $\mathfrak{R}$ is not open and is not closed relative to the invertible operators.*

Closely related to the question of whether an invertible operator A has roots is the question of whether A has a logarithm; i.e., whether there is some $B \in \mathfrak{L}(\mathfrak{H})$ satisfying $\exp(B) = A$. We denote the set of all invertible operators on $\mathfrak{H}$ that *fail* to have a logarithm by $\mathfrak{L}$. (A necessary and sufficient condition that an operator belong to $\mathfrak{L}$ is known [4, page 285], but it has not yielded specific examples of operators in $\mathfrak{L}$.) It is clear that $\mathfrak{R} \subset \mathfrak{L}$, and the existence of invertible operators with square roots but no fourth roots [1, 6] implies that the above inclusion is proper. That $\mathfrak{L}$ has non-void interior follows from the fact that $\mathfrak{R}$ does. However, the questions as to whether $\mathfrak{L}$ is open, or closed relative to the invertible operators, seem to have gone unanswered, and we furnish answers as follows:

Theorem 2. *The set $\mathfrak{L}$ is not open and is not closed relative to the invertible operators.*

2. Preliminaries

Before discussing the idea used in the proofs of Theorems 1 and 2, we introduce the following terminology.

Let N denote the set of integers greater than 2, and let $N_2, N_3, \ldots$ be infinite disjoint subsets of N whose union is N. The sets N and N_p, $p = 2, 3, \ldots,$ will remain fixed throughout the paper, and will frequently be regarded as increasing sequences without further apology. Notation such as $\lim_{n\in N_p}\{a_n\}$ will be used for simplicity, and should be given the obvious interpretation.

For each $n\in N$, let $\mathfrak{H}_n$ be n-dimensional complex Hilbert space, and let $\mathfrak{L}(\mathfrak{H}_n)$ be the algebra of all (linear) operators on $\mathfrak{H}_n$. Denote by $\mathfrak{H}$ the Hilbert space $\sum_{n\in N}\oplus\mathfrak{H}_n$, and let the algebra $\mathfrak{I}\subset\mathfrak{L}(\mathfrak{H})$ be the C^*-sum $\mathfrak{I}=\sum_{n\in N}\oplus\mathfrak{L}(\mathfrak{H}_n)$. Then, of course, $\mathfrak{I}$ consists of all operators $A=\sum_{n\in N}\oplus A_n$ where $A_n\in\mathfrak{L}(\mathfrak{H}_n)$ and the sequence $\{\|A_n\|\}_{n\in N}$ is bounded.

Note that to prove that neither $\mathfrak{R}$ nor $\mathfrak{L}$ is open it clearly suffices to prove the following proposition.

I) *There exists an operator $A\in\mathfrak{I}\cap\mathfrak{R}$ and a sequence $\{G^{(k)}\}$ of operators in $\mathfrak{I}$ such that each $G^{(k)}$ has a logarithm and such that $\|G^{(k)}-A\|\to 0$.*

On the other hand, to show that neither $\mathfrak{R}$ nor $\mathfrak{L}$ is closed relative to the class of invertible operators, it is enough to prove following proposition.

II) *There exists an operator $C\in\mathfrak{I}$ having a logarithm and a sequence $\{B^{(k)}\}$ of operators in $\mathfrak{I}\cap\mathfrak{R}$ such that $\|B^{(k)}-C\|\to 0$.*

Our task of proving Theorems 1 and 2 will be accomplished by proving I) and II). To do this, we show that it suffices to prove the following

Theorem 3. *There exists a sequence $\{A_n\}_{n\in N}$, $A_n\in\mathfrak{L}(\mathfrak{H}_n)$, satisfying:*

a) $\sum_{n\in N}\oplus A_n\in\mathfrak{R}\cap\mathfrak{I}$,

b) *every operator $B=\sum\oplus B_n$ in $\mathfrak{I}$ such that* spectrum $B_n=$ spectrum A_n *for $n\in N$ and $B_n=A_n$ for all sufficiently large n, satisfies $B\in\mathfrak{R}$.*

Furthermore, there exists a sequence $\{C_n\}_{n\in N}$, $C_n\in\mathfrak{L}(\mathfrak{H}_n)$, satisfying:

c) spectrum $C_n=$ spectrum A_n *for* $n\in N$,

d) $\|C_n-A_n\|\to 0$,

e) $\sum_{n\in N}\oplus C_n$ *has a logarithm in $\mathfrak{I}$ (and thus is itself an invertible operator in $\mathfrak{I}$).*

Proof of I) and II) using Theorem 3. With the notation as above, define:

$$A=\sum_{n\in N}\oplus A_n,\qquad C=\sum_{n\in N}\oplus C_n,$$

and for each $k\in N$,

$$B^{(k)}=C_3\oplus\ldots\oplus C_k\oplus A_{k+1}\oplus A_{k+2}\oplus\ldots,$$

$$G^{(k)}=A_3\oplus\ldots\oplus A_k\oplus C_{k+1}\oplus C_{k+2}\oplus\ldots.$$

Since C has a logarithm in $\mathfrak{I}$, and every invertible operator on a finite dimensional space has a logarithm, each $G^{(k)}$ has a logarithm. Since obviously $\|G^{(k)}-A\|\to 0$, I) is proved. Since each $B^{(k)}\in\mathfrak{R}\cap\mathfrak{I}$ and obviously $\|B^{(k)}-C\|\to 0$, II) is proved.

Thus to prove Theorems 1 and 2, it suffices to prove Theorem 3, and the remainder of the paper is devoted to this task.

3. A construction

To begin the proof of Theorem 3, we wish to produce for each $p>1$ an operator of the form $\sum_{n\in N_p} \oplus A_n$ that has no p-th root. Thus we must suitably modify and generalize the lemmas of [1, § 2] to make them applicable to the problem of p-th roots. Throughout the paper we denote by ω_n^r the n-th root of unity

$$\omega_n^r = e^{2\pi i r/n} \qquad (r = 0, \pm 1, \pm 2, \ldots).$$

Lemma 3.1. *Let $p\geqq 2$ and $n\geqq 3$ be integers, and let $\{\lambda_n^r\}_{r=1}^n$ be complex numbers such that $(\lambda_n^r)^p=\omega_n^r$ for $r=1, 2, \ldots, n$. Then*

$$\lambda_n^r = \omega_{pn}^r \omega_p^{k(r)} \quad \textit{for} \quad r = 1, 2, \ldots, n,$$

where each $k(r)$ is some integer satisfying $1\leqq k(r)\leqq p$, and either

(α) for some r satisfying $1\leqq r\leqq n-1$,

$$\left|\frac{(\lambda_n^r)^p-(\lambda_n^{r+1})^p}{\lambda_n^r-\lambda_n^{r+1}}\right| \leqq \frac{2\pi}{n|1-\omega_{3p}^2|}, \qquad \textit{or}$$

(β) $k(1)=k(2)=\ldots=k(n)$, in which case,

$$\left|\frac{(\lambda_n^1)^p-(\lambda_n^n)^p}{\lambda_n^1-\lambda_n^n}\right| \leqq \frac{2\pi}{n|1-\omega_{3p}^2|}.$$

Proof. Suppose there is an r satisfying $1\leqq r\leqq n-1$ such that $k(r)\neq k(r+1)$, and note that

$$\left|\frac{(\lambda_n^r)^p-(\lambda_n^{r+1})^p}{\lambda_n^r-\lambda_n^{r+1}}\right| = \left|\frac{(\omega_{pn}^r)^p-(\omega_{pn}^{r+1})^p}{\omega_{pn}^r\omega_p^{k(r)}-\omega_{pn}^{r+1}\omega_p^{k(r+1)}}\right| =$$

$$= \left|\frac{\omega_n^r-\omega_n^{r+1}}{\omega_{pn}^r[\omega_p^{k(r)}-\omega_{pn}^1\omega_p^{k(r+1)}]}\right| = \left|\frac{1-\omega_n^1}{1-\omega_{pn}^1\omega_p^{[k(r+1)-k(r)]}}\right|.$$

Now $k=k(r+1)-k(r)$ is a non-zero integer satisfying $-(p-1)\leqq k\leqq p-1$, and it is clear that the distance from 1 to $\omega_{pn}^1\omega_p^{-1}$ along the unit circle is less than or equal to the distance from 1 to the point $\omega_{pn}^1\omega_p^k$ along the unit circle. Thus

$$|1-\omega_{pn}^1\omega_p^k| \geqq |1-\omega_{pn}^1\omega_p^{-1}|.$$

Furthermore, $|1-\omega_{pn}^1\omega_p^{-1}|$ attains its minimum as a function of n at $n=3$, so

$$|1-\omega_{pn}^1\omega_p^k| \geqq |1-\omega_{3p}^1\omega_p^{-1}| = |1-\omega_{3p}^2|$$

and thus

$$\left|\frac{1-\omega_n^1}{1-\omega_{pn}^1\omega_p^k}\right| \leqq \left|\frac{1-\omega_n^1}{1-\omega_{3p}^2}\right| \leqq \frac{2\pi}{n|1-\omega_{3p}^2|}.$$

On the other hand, if $k(1)=k(2)=\ldots=k(n)$, then

$$\left|\frac{(\lambda_n^1)^p-(\lambda_n^n)^p}{\lambda_n^1-\lambda_n^n}\right| = \left|\frac{1-\omega_n^1}{1-\omega_{pn}^1\omega_p^{-1}}\right| \leqq \frac{2\pi}{n|1-\omega_{3p}^2|},$$

as before.

Lemma 3. 2. *Suppose that $K \in \mathfrak{L}(\mathfrak{H})$ has the distinct eigenvalues $\{\lambda_i\}_{i \in I}$, and suppose that for $i \in I$, the eigenspace corresponding to λ_i is spanned by the vector x_i. If J is any operator on $\mathfrak{H}$ that commutes with K, and $\mathfrak{K}$ is any subspace of $\mathfrak{H}$ spanned by some subset of the $\{x_i\}_{i \in I}$, then $\mathfrak{K}$ is an invariant subspace for J.*

Proof. It suffices to show that for $i \in I$, $Jx_i = \alpha_i x_i$ for some scalar α_i. If $Jx_i = y_i$, then $Ky_i = KJx_i = JKx_i = \lambda_i y_i$, so that by hypothesis $y_i = \alpha_i x_i$.

The following corollaries are immediate.

Corollary 3. 3. *If p and n are positive integers, T is an $n \times n$ complex matrix in upper triangular form having n distinct eigenvalues, and $R^p = T$, then R is also in upper triangular form.*

Corollary 3. 4. *Suppose $K \in \mathfrak{J}$ and satisfies the hypotheses of Lemma 3. 2 and the additional hypothesis that the vectors $\{x_i\}_{i \in I}$ span $\mathfrak{H}$. Suppose also that $J \in \mathfrak{L}(\mathfrak{H})$ and satisfies $J^p = K$ for some positive integer p. Then $J \in \mathfrak{J}$.*

The following easy computation is designated as a lemma for convenience in referring to it later.

Lemma 3. 5. *Let T be the $n \times n$ complex matrix*

$$\left(\begin{array}{c|c} B & z \\ \hline 0 & \lambda \end{array}\right)$$

where B is an $(n-1) \times (n-1)$ matrix, z is a $(n-1)$-vector, and λ is a scalar not in the spectrum of B. If p is any positive integer, then T^p is the matrix

$$\left(\begin{array}{c|c} B^p & x \\ \hline 0 & \lambda^p \end{array}\right),$$

where x is the $(n-1)$-vector $x = (B-\lambda)^{-1}(B^p - \lambda^p)z$.

(We call an $n \times n$ matrix *upper triangular* if the elements below its diagonal are all equal to 0.)

The following lemma is obtained from Lemma 3. 5 by induction on the size of the matrix.

Lemma 3. 6. *Let p and n be positive integers larger than 1. Let T be an upper triangular $n \times n$ matrix whose diagonal elements are $\mu_1, \ldots, \mu_n$, where the μ_i are distinct complex numbers. For $i = 1, 2, \ldots, n$, let λ_i be such that $(\lambda_i)^p = \mu_i$. Then there exists exactly one upper triangular $n \times n$ matrix R whose diagonal elements are $\lambda_1, \ldots, \lambda_n$, and for which $R^p = T$.*

With these preparatory lemmas out of the way, we proceed with some additional definitions needed to prove Theorem 3. The sequences of operators we shall consider can most easily be described matricially, so we assume given for each $n \in N$ an orthonormal basis X_n for $\mathfrak{H}_n$, and the matrices exhibited hereafter are to be regarded as the corresponding operators. For each pair (p, n) of integers with $p > 1$ and

$n \in N_p$, we define Q_n to be the unique operator on $\mathfrak{H}_n$ whose matrix is upper triangular and has the diagonal elements $(1+1/n)^{1/p}\,\omega_{pn}^k$ $(k=1, \ldots, n)$, satisfying

$$(Q_n)^p = (1+1/n)\begin{pmatrix} \omega_n^1 & n^{-\frac{1}{2}} & & & \\ & \omega_n^2 & n^{-\frac{1}{2}} & & \\ & & \ddots & \ddots & \\ & & & \omega_n^{n-1} & n^{-\frac{1}{2}} \\ & & & & \omega_n^n \end{pmatrix}$$

Next define (for each $p>1$) the sequence $\{c_n\}_{n\in N_p}$ of complex numbers by

$$c_n = \left(\frac{\omega_n^1 - 1}{\omega_{pn}^1 - \omega_p^1}\right) q_{1n}^{(n)}$$

Finally, (for each $p>1$) define the sequence $\{f_n\}_{n\in N_p}$ as follows:

γ) if the sequence $\{c_n\}_{n\in N_p}$ does not converge to zero, set $f_n=0$ for $n\in N_p$;

δ) if the sequence $\{c_n\}_{n\in N_p}$ converges to zero, set $f_n=0$ for all $n\in N_p$ such that $|c_n|>1$, and set $f_n=1-c_n$ for those $n\in N_p$ such that $|c_n|\leqq 1$.

Lemma 3. 7. *For each pair (p, n) with $p>1$ and $n\in N_p$, let A_n be the operator*

$$A_n = (1+1/n)\begin{pmatrix} \omega_n^1 & n^{-\frac{1}{2}} & & & f_n \\ & \omega_n^2 & n^{-\frac{1}{2}} & & \\ & & \ddots & \ddots & \\ & & & \omega_n^{n-1} & n^{-\frac{1}{2}} \\ & & & & \omega_n^n \end{pmatrix}$$

and let T_n be the unique operator of the form

$$T_n = (1+1/n)^{1/p}\begin{pmatrix} \omega_{pn}^1 & & & \\ & \omega_{pn}^2 & t_{ij}^{(n)} & \\ & & \ddots & \\ & & & \omega_{pn}^n \end{pmatrix}$$

satisfying $(T_n)^p = A_n$. Then $t_{1n}^{(n)} = (c_n+f_n)\left(\dfrac{\omega_{pn}^1 - \omega_p^1}{\omega_n^1 - 1}\right)$.

The proof of this lemma is an easy calculation using Lemmas 3. 5 and 3. 6 and is omitted.

4. The proof of Theorem 3

Note that if for each integer $p>1$ a sequence $\{t_n\}_{n\in N_p}$ has been defined, then these sequences give rise in an obvious way to a sequence $\{t_n\}_{n\in N}$. The following lemma proves the first half of Theorem 3.

Lemma 4. 1. *For each pair of integers (p, n) with $p>1$ and $n\in N_p$, let $A_n\in\mathfrak{L}(\mathfrak{H}_n)$ be as in Lemma 3. 7. Let $n_0\geqq 3$ be a fixed integer, and let $B=\sum_{n\in N}\oplus B_n$ belong to $\mathfrak{J}$ and satisfy:*

(ϱ) *for $3\leqq n\leqq n_0$, the eigenvalues of B_n are identical with those of A_n, and*

(τ) *for $n>n_0$, $B_n=A_n$.*

Then $B\in\mathfrak{R}$; i.e., B is an invertible rootless operator.

Proof. The inverse on $\mathfrak{H}_n$ of each A_n can be computed directly, and an easy calculation shows that

$$\|A_n^{-1}\| \leqq (1-1/\sqrt{n})^{-1}+|f_n|.$$

Since the sequence $\{f_n\}_{n\in N}$ is bounded by construction, $B^{-1}=\sum_{n\in N}\oplus B_n^{-1}\in\mathfrak{J}$. Now suppose that for some $p>1$ there is an operator $S\in\mathfrak{L}(\mathfrak{H})$ satisfying $S^p=B$. Since B satisfies the hypotheses of Corollary 3.4, $S\in\mathfrak{J}$, and we write $S=\sum_{n\in N}\oplus S_n$. By Corollary 3. 3, for $n>n_0$, S_n is in upper triangular form. Thus for each $n\in N_p$ satisfying $n>n_0$, let

$$S_n=(1+1/n)^{1/p}\begin{pmatrix} \lambda_n^1 & & & s_{ij}^{(n)} \\ & \lambda_n^2 & & \\ & & \ddots & \\ & & & \lambda_n^n \end{pmatrix}.$$

Direct computation shows that, for each n,

$$s_{i,\,i+1}=\left(\frac{\lambda_n^i-\lambda_n^{i+1}}{(\lambda_n^i)^p-(\lambda_n^{i+1})^p}\right)(1/\sqrt{n}), \qquad i=1, 2, \dots, n-1,$$

and we note that the $\{\lambda_n^r\}_{r=1}^n$ satisfy the hypotheses of Lemma 3. 1. Thus by Lemmas 3. 1 and 3. 6, for each $n\in N_p$ satisfying $n>n_0$

$$|s_{i,\,i+1}^{(n)}| \geqq \frac{\sqrt{n}\,|1-\omega_{3p}^2|}{2\pi}$$

for some i satisfying $1\leqq i\leqq n-1$ or $S_n=\omega_p^{k(n)}T_n$, where T_n is as defined in Lemma 3. 7 and $k(n)$ is some integer. Since $\|S_n\|\geqq|s_{i,\,i+1}^{(n)}|$ and S is assumed to be bounded, there must exist an integer $n_1\geqq n_0$ such that $S_n=\omega_p^{k(n)}T_n$ for all $n\in N_p$ satisfying $n>n_1$. But then by Lemma 3. 7,

$$s_{1n}^{(n)}=\omega_p^{k(n)}\frac{(c_n+f_n)(\omega_{pn}^1-\omega_p^1)}{(\omega_n^1-1)} \quad \text{for} \quad n\in N_p, \quad n>n_1,$$

and applying Lemma 3. 1 [case (β)],

$$|s_{1n}^{(n)}| \geqq \frac{|c_n+f_n|\cdot|1-\omega_{3p}^2|\,n}{2\pi}.$$

By construction, the sequence $\{c_n+f_n\}_{n\in N_p}$ does not converge to zero, so the sequence $\{s_{1n}^{(n)}\}_{n\in N_p}$ is unbounded, contradicting $S\in\mathfrak{L}(\mathfrak{H})$. Thus the lemma is proved.

The following lemma completes the proof of Theorem 3.

Lemma 4. 2. *For each $n\in N$ let $C_n\in\mathfrak{L}(\mathfrak{H}_n)$ be the operator*

$$C_n=(1+1/n)\begin{pmatrix} \omega_n^1 & & & & f_n \\ & \omega_n^2 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ & & & & \omega_n^n \end{pmatrix}$$

where f_n is as previously defined, and let $D_n\in\mathfrak{L}(\mathfrak{H}_n)$ be the operator

$$D_n=\log(1+1/n)I+\begin{pmatrix} 2\pi i/n & & & & & d_n \\ & 4\pi i/n & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & 2(n-1)\pi i/n & \\ & & & & & 0 \end{pmatrix},$$

where $d_n=\dfrac{2f_n\pi i}{n(\omega_n^1-1)}$. Then for $n\in N$, $\exp(D_n)=C_n$ and $\|D_n\|\leqq 11$, so that $D=\sum_{n\in N}\oplus D_n\in\mathfrak{I}$, and $\exp(D)=C$.

Proof. Compute, using the fact that $|f_n|\leqq 2$.

Question. Is it possible for an invertible operator to have roots of all orders and yet fail to have a logarithm?

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