Homomorphisms of a semigroup onto normal bands

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1. Introduction and summary

Homomorphisms of an arbitrary semigroup S onto semigroups belonging to a special class $\mathscr{C}$ of semigroups furnish some information about the structure of S. Of particular interest is the case when $\mathscr{C}$ is a class of bands, for in such a case all the classes of the decomposition of S induced by such homomorphisms are subsemigroups of S. The problem of finding a sufficiently explicit characterization of homomorphisms onto arbitrary bands appears difficult. In [3] and [4], we have found such a characterization for the case when $\mathscr{C}$ is the class of all semilattices and the class of all rectangular bands, respectively. In the present work we solve the problem when $\mathscr{C}$ is the class of all [left] normal bands (for definitions see section 2). Normal bands have been studied by YAMADA and KIMURA [9], and right normal bands by VAGNER [8], and SHAIN [5], [6] (the latter two call right normal bands "restrictive semigroups").

In section 2 we give the definitions and notation used in the paper. In it we introduce a number of concepts which are used in succeeding sections; section 3 discusses some of their properties. Section 4 contains main results of the paper, viz., characterization of congruences induced by homomorphisms onto [left] normal bands. In section 5 we discuss some general properties of normal bands and, in section 6, subdirect products of such bands. Section 7 contains a representation of normal bands as subsets of a set under certain multiplication.

Some of the results in this paper parallel those in [3] and [4] (similar methods are used); we will not expressly mention similarity with these papers.

2. Definitions and notation

Throughout S will denote an arbitrary semigroup unless stated otherwise. A one element set $X=\{x\}$ will be simply denoted by x. For properties of most of the concepts that are introduced below see [3] and [4]. Let H be a non-empty subset of S and let $x, y, z \in S$.

H is said to be *left (right) dense* if $xy \in H$ implies $x \in H$ ($y \in H$), *quasi dense* if (i) $x^2 \in H$ implies $x \in H$ and (ii) $xz \in H$ if and only if $xyz \in H$. A left dense and right dense subsemigroup of S is called a *face* of S. The smallest face of S containing H is denoted by $N(H)$. H is said to be a *left (right) normal complex* (abbreviated l. n.

complex [r. n. complex]) if H is a left dense right ideal (l.d.r.i.) [right dense left ideal (r.d.l.i.)] of $N(H)$. H ist said to be a *normal complex* (n complex) if H is a quasi dense subsemigroup of $N(H)$. By $A(x)$ [$B(x)$] denote the smallest l.n. [r.n.] complex of S containing x. (It is easy to see that the non-empty intersection of l.n. [r.n.] complexes is again a l. n. [r. n.] complex.)

By σ_H we denote the equivalence relation on S whose classes are the non-empty sets in the family of sets: H, $N(H)\setminus SN$, $\setminus N(H)$. If $\mathcal{F}$ is a non-empty family of non-empty subsets of S, we set $\sigma_{\mathcal{F}} = \bigcap_{H\in\mathcal{F}} \sigma_H$; if $\mathcal{F}$ is empty, $\sigma_{\mathcal{F}}$ denotes the universal relation on S. We let $\lambda_x=\sigma_{A(x)}$, $\varrho_x=\sigma_{B(x)}$, $\tau_x=\lambda_x\cap\varrho_x$. For $a\in S$, we write $H^{\cdot}.a= =\{x\in S|xa\in H\}$, $H.^{\cdot}a=\{x\in S|ax\in H\}$.

Following [9], we say that a band S is *left (right) normal* if it satisfies the identity $xyz=xzy$ [$xyz=yxz$], *normal* if it satisfies the identity $xyzx=xzyx$. If ξ is a congruence on an arbitrary semigroup S such that S/ξ is a left normal, right normal, or normal band, respectively, ξ is called a *left normal, right normal, normal congruence* (l.n., r.n., n. congruence, respectively).

By S^0 we denote the semigroup S with zero adjoined (irrespective of whether S has a zero or not). U, L, R will, respectively, stand for a one element semigroup, two element left zero semigroup, two element right zero semigroup. If A is any set, $|A|$ denotes its cardinality.

If S_α, $\alpha\in A$, is a non-empty family of semigroups, $\prod_{\alpha\in A} S_\alpha$ denotes their *Cartesian (or direct) product*, that is, the semigroup defined on the Cartesian product of sets S_α with coordinatewise multiplication; if $A=\{1,2\}$ we write $S_1\times S_2$ instead of $\prod_{i=1}^{2} A_i$. S is a *subdirect product* of semigroups S_α if S isomorphic to a subsemigroup S' of $\prod_{\alpha\in A} S_\alpha$ such that for all $\alpha\in A$, $\pi_\alpha(S')=S_\alpha$ (π_α is the α-th projection). If $\{B_i\}_{i=1}^n$ is a partition of A and for every i, $1\leqq i\leqq n$, all semigroups S_α with $\alpha\in B_i$ are isomorphic to a semigroup T_i, we say that S is a subdirect product of $|B_1|$ copies of T_1, $|B_2|$ copies of T_2, ..., $|B_n|$ copies of T_n. Subdirect irreducibility is taken in the usual sense (a one element semigroup is excluded).

For all concepts and notation not mentioned above the reader is referred to [2]. We will omit all statements that can be obtained from our results by the left-right duality.

3. Basic properties of concepts used

We will repeatedly use the next proposition without express mention.

Proposition 1 (*cf.* [9], Theorem 10). *Any normal band S satisfies the identity*

$$ax_1x_2\ldots x_nb=ax_{i_1}x_{i_2}\ldots x_{i_n}b, \tag{1}$$

where $\{i_1, i_2, \ldots, i_n\}$ is a permutation of the set $\{1, 2, \ldots, n\}$.

Proof. We prove the case $n=2$; the general case is treated by induction. It is clear that normality implies (1) for $a=b$. Thus

$$axyb=(axyb)(axyb)=(axyba)(xyb)=(ayxba)(xyb)=$$
$$=(ayx)(baxyb)=(ayx)(bayxb)=(ayxb)(ayxb)=ayxb.$$

Theorem 1. *The intersection of a* l.n. *congruence and a* r.n. *congruence is a* n. *congruence. Conversely, every* n. *congruence* $\sim$ *is the intersection of the finest* l.n. *congruence on* S *containing* $\sim$ *and the finest* r.n. *congruence on* S *containing* $\sim$.

Proof. The first statement of the theorem is immediate. Hence let $\sim$ be a n. congruence and for any $x, y \in S$, define

$$x \overset{l}{\sim} y \quad \text{if and only if} \quad x \sim yx \quad \text{and} \quad y \sim xy,$$

$$x \overset{r}{\sim} y \quad \text{if and only if} \quad x \sim xy \quad \text{and} \quad y \sim yx.$$

We show that $\overset{l}{\sim}$ is the finest l.n. congruence on S containing $\sim$; the case of $\overset{r}{\sim}$ is treated analogously. If $x \overset{l}{\sim} y$ and $y \overset{l}{\sim} z$, then

$$x \sim yx \sim zyx \sim (zy)(zx) \sim y(zx) \sim (yz)x \sim zx$$

and analogously $z \sim xz$. Thus $x \overset{l}{\sim} z$, and $\overset{l}{\sim}$ is an equivalence relation (symmetry and transitivity are obvious). If $x \overset{l}{\sim} y$, then for any $z \in S$,

$$xz \sim yxz \sim (yz)(xz),$$

similarly $yz \sim (xz)(yz)$ so that $xz \overset{l}{\sim} yz$; analogously $zx \overset{l}{\sim} zy$ and hence $\overset{l}{\sim}$ is a congruence and is clearly a l.n. congruence. Let $\overset{l}{\approx}$ be any l.n. congruence containing $\sim$. Then for $x \overset{l}{\sim} y$, we have $x \sim yx$, $y \sim xy$ and thus $x \overset{l}{\approx} yx$, $y \overset{l}{\approx} xy$. Consequently

$$x \overset{l}{\approx} yx \overset{l}{\approx} y(xy) \overset{l}{\approx} yy \overset{l}{\approx} y,$$

that is, $\overset{l}{\sim}$ is contained in $\overset{l}{\approx}$. It follows easily that $\sim = \overset{l}{\sim} \cap \overset{r}{\sim}$.

The next theorem establishes a connection between l.n. complexes and l.n. congruences.

Theorem 2. *The following conditions on a complex* H *of* S *are equivalent:*

a) H *is a* l.n. *complex of* S;

b) σ_H *is a* l.n. *congruence on* S;

c) *for all* $a \in N(H)$, $H^{\cdot}.a = H$.

Proof. a) implies b). If $N(H) = S$, then H is a l.d.r.i. of S and $S/\sigma_H \cong L$, and if $N(H) = H$, then H is a face of S and σ_H is a semilattice congruence. Hence suppose that $H \neq N(H) \neq S$, and let $A = N(H) \setminus H$, $B = S \setminus N(H)$. Then by the definition of a l.n. complex, the following inclusions hold:

	H	A	B
H	H	H	B
A	A	A	B
B	B	B	B

where, e.g., $HA \subseteq H$, etc. Defining multiplication according to this table, we see that $\{H, A, B\} \cong L^0$ and thus σ_H is a l.n. congruence.

b) implies c). Since σ_H is a l.n. congruence and it has at most 3 classes, S/σ_H is isomorphic to one of the semigroups U, U^0, L, L^0. If $S/\sigma_H \cong U$, $H = S$; if $S/\sigma_H \cong U^0$, H is a face of S; if $S/\sigma_H \cong L$, H is a l.d.r.i. of S; in any of these cases,

c) is established without difficulty. Finally, if $S/\sigma_H \cong L^0$ and $a \in N(H)$, then c) follows easily from the above table since this case then reduces to considering the semigroup $\{H, A\}$ which is isomorphic to L.

c) implies a). This follows easily from the definition of a l.n. complex.

Proposition 2. *A complex H is a* l.n. *complex of S if and only if H is left dense in S and has the property: if $x \in S$ is such that every prime ideal of S which contains x also intersects H, then $Hx \subset H$.*

Proof. Necessity. Since H is left dense in $N(H)$, it is also left dense in S. If $x \in S$ has the property stated above, then $x \in N(H)$ since otherwise $S \setminus N(H)$ would be a prime ideal of S containing x and not intersecting H. The inclusion $Hx \subseteqq H$ now follows by the definition of a l.n. complex.

Sufficiency. As before, we conclude that $x \in S$ with the above property must be contained in $N(H)$. Thus H is a l.d.r.i. of $N(H)$ as desired.

Recall that $A(x)$ is the smallest l.n. complex of S containing x.

Theorem 3. *Let x be an element of S, let*

$$A_1(x) = x \cup xN(x),$$

$$A_{2n}(x) = \{y \in S \mid A_{2n-1}(x) \cap R(y) \neq \square\},$$

$$A_{2n+1}(x) = A_{2n}(x) \cup A_{2n}(x)N(A_{2n}(x)),$$

for $n = 1, 2, \ldots$. Then $A(x) = \bigcup_{n=1}^{\infty} A_n(x)$.

Proof. We write A_n and A instead of $A_n(x)$ and $A(x)$, respectively, and let $T = \bigcup_{n=1}^{\infty} A_n$. Since $a \in A$, we have $N(x) \subseteqq N(A)$ and thus $xN(x) \subseteqq AN(A) \subseteqq A$ whence $A_1 \subseteqq A$. Suppose that $A_n \subseteqq A$, $n \geqq 1$. If n is even, then $A_{n+1} = A_n \cup A_nN(A_n)$ and $A_n \subseteqq A$ implies $A_nN(A_n) \subseteqq AN(A) \subseteqq A$, that is, $A_{n+1} \subseteqq A$. If n is odd, then for $y \in A_{n+1}$ we have $A_n \cap R(y) \neq \square$. Thus $yz \in A_n$ for some $z \in S^1$; hence $yz \in A$ so that $y \in A$. Consequently $A_{n+1} \subseteqq A$ and by induction we conclude that $T \subseteqq A$.

For the opposite inclusion, it suffices to show that T is a l.n. complex. Let $yz \in T$; then $yz \in A_n$ for some n. We may suppose that n is even since $A_1 \subseteqq A_2 \subseteqq \ldots$. Then $A_{n-1} \cap R(yz) \neq \square$ whence $A_{n-1} \cap R(y) \neq \square$ and thus $y \in A_n \subseteqq T$. Next let $y \in T$ and $z \in N(T)$. Since $A_1 \subseteqq A_2 \subseteqq \ldots$, we have $N(A_1) \subseteqq N(A_2) \subseteqq \ldots$ and hence

$$z \in N(T) = N\left(\bigcup_{n=1}^{\infty} A_n\right) = \bigcup_{n=1}^{\infty} N(A_n).$$

We thus have $y \in A_m$ and $z \in N(A_n)$ and we may suppose that $m = n$ and n is even. Hence $yz \in A_nN(A_n) \subseteqq A_{2n+1} \subseteqq T$. Consequently T is a l.n. complex of S and thus $A = T$.

Proposition 3. *Let H be a* n. *complex of S and let a be an element of S. Then*

a) $A(H) = H^{\cdot}.a$;

b) $H = (H^{\cdot}.a) \cap (H.^{\cdot}a)$;

c) $N(H) = N(H^{\cdot}.a)$.

Proof. Items a) and b) follow easily from the definitions and Theorem 2, [4]. Clearly $H^{\cdot}.a \subseteqq N(H)$, which implies $N(H^{\cdot}.a) \subseteqq N(H)$. Conversely, $H \subseteqq H^{\cdot}.a$ implies $N(H) \subseteqq N(H^{\cdot}.a)$ and c) is established.

Theorem 4. *The non-empty intersection H of a* l.n. *complex C and a* r. n. *complex D is a* n. *complex and $N(H)=N(C)\cap N(D)$. Conversely, if H is a* n. *complex, then $H=A(H)\cap B(H)$ and $N(H)=N(A(H))=N(B(H))$.*

Proof. Let C and D be as above and $H=C\cap D\neq\square$. Let $E=N(C)\cap N(D)$; $H=C\cap D\subseteqq E$ so that $N(H)\subseteqq E$ since E is then a face. Let $C'=C\cap E$ and $D'=D\cap E$; then

$$H=C\cap D=(C\cap N(C))\cap(D\cap N(D))=(C\cap E)\cap(D\cap E)=C'\cap D',$$

where $C'\subseteqq E$, $D'\subseteqq E$. Further, $C'E\subseteqq CE\subseteqq CN(C)\subseteqq C$ and $C'E\subseteqq E$ which implies $C'E\subseteqq C'$. Also $xy\in C'\subseteqq C$ implies $x\in C$ which together with $x\in E$ (since $xy\in C'\subseteqq E$ and E is a face) implies $x\in C'$. Consequently C' is a l.d.r.i. of E; similarly D' is a r.d.l.i. of E, and thus $H=C'\cap D'$ is a quasi dense subsemigroup of E. If $x\in E$, then for any $c\in C'$, $d\in D'$, $cxd\in C'\cap D'=H\subseteqq N(H)$ whence $x\in N(H)$. Thus $E\subseteqq N(H)$ the opposite inclusion being obvious, we have $E=N(H)$. Therefore $N(H)=N(C)\cap\cap N(D)$.

For the converse it suffices to apply Proposition 3.

Remark. It follows from the definitions that a complex H of S is a n. complex if and only if for all $x, y, z\in S$:

a) $x, y\in H$ implies $xy\in H$;

b) $x^2\in H$ implies $x\in H$;

c) $xyz\in H$ implies $xz\in H$;

d) $xz\in H$, $y\in N(H)$ implies $xyz\in H$.

4. Homomorphisms onto normal bands

Let S be a fixed semigroup, $\mathscr{A}$ the family of all proper l.n. complexes of S together with the empty set, and $\mathfrak{A}$ the set of all l.n. congruences on S. On the set $\mathfrak{P}(\mathscr{A})$ of all non-empty subsets of $\mathscr{A}$ define the function α by: $\alpha(\mathscr{A}')=\sigma_{\mathscr{A}'}$ (for notation see section 2). Then we have the following result which is fundamental for most of this paper.

Theorem 5. *The function α maps $\mathfrak{P}(\mathscr{A})$ onto $\mathfrak{A}$ and is inclusion inverting.*

Proof. If $\mathscr{A}'=\square$, $\sigma_{\mathscr{A}'}$ is the universal relation and hence $\sigma_{\mathscr{A}'}\in\mathfrak{A}$. Otherwise $\mathscr{A}'\neq\square$ which implies that for every $A\in\mathscr{A}'$, σ_A is a l.n. congruence by Theorem 2; consequently $\sigma_{\mathscr{A}'}=\bigcap_{A\in\mathscr{A}'}\sigma_A\in\mathfrak{A}$. This shows that α maps $\mathfrak{P}(\mathscr{A})$ into $\mathfrak{A}$.

We show next that α maps $\mathfrak{P}(\mathscr{A})$ onto $\mathfrak{A}$. Hence let $\sim$ be any proper l.n. congruence on S. For every $x\in S$, let

$$A_x=\{y\in S\mid x\sim yx\},$$

and let $\mathscr{A}'$ be the family of all distinct sets A_x such that $A_x\neq S$, as x ranges over all elements of S. We will show that $\mathscr{A}'\in\mathfrak{P}(\mathscr{A})$ and that $\sigma_{\mathscr{A}'}=\sim$.

$\mathscr{A}'$ is not empty for in such a case we would have $A_x = S$ for all $x \in S$ which would imply for all $x, y \in S$, $x \sim yx$, $y \sim xy$. But then

$$x \sim yx \sim xyx \sim xy \sim y$$

contradicting the hypothesis that $\sim$ is proper. Thus to show that $\mathscr{A}' \in \mathfrak{P}(\mathscr{A})$, it suffices to prove that for all $x \in S$, $A_x \in \mathscr{A}$ if $A_x \neq S$. We fix $x \in S$ and let

$$T_x = \{y \in S \mid x \sim xy\}.$$

If $y, z \in T_x$, then $x \sim xy \sim xz$ and thus $x \sim (xy)(xz) \sim xyz$, that is, $yz \in T_x$. Conversely, if $yz \in T_x$, then $x \sim xyz$ and hence

$$x \sim xyz \sim xxyz \sim (xy)(xyz) \sim xyx \sim xxy \sim xy,$$

that is, $y \in T_x$; similarly $z \in T_x$. Consequently T_x is a face of S. If $y \in A_x$, then $x \sim yx$ whence $x \sim xyx \sim xy$ and thus $y \in T_x$. Hence $A_x \subseteq T_x$ and thus $N(A_x) \subseteq T_x$ since T_x is a face. Further, if $y \in T_x$, then $x \sim xy$ whence $x \sim (xy)x$ so that $xy \in A_x \subseteq N(A_x)$. But $xy \in N(A_x)$ implies $y \in N(A_x)$ which proves $T_x \subseteq N(A_x)$. Consequently $T_x = N(A_x)$.

If $yz \in A_x$, then $x \sim yzx$ and we have $yx \sim y(yzx) \sim yzx \sim x$ so that $y \in A_x$. If $y \in A_x$, $z \in T_x$, then $x \sim yx \sim xz$ and thus $x \sim yx \sim yxz \sim (yz)x$, that is, $yz \in A_x$. Consequently A_x is a l.d.r.i. of $T_x = N(A_x)$ and hence $A_x \in \mathscr{A}$ if $A_x \neq S$.

We show next that $\sigma_{\mathscr{A}'} = \sim$. Suppose that $x \sigma_{\mathscr{A}'} y$. Then $x \in A_x$ implies that $y \in A_x$, that is, $x \sim yx$; dually $y \sim xy$ and thus $x \sim yx \sim xyx \sim xy \sim y$.

Conversely, suppose that $x \sim y$ and let $z \in S$. If $x \in A_z$, then $z \sim xz$ and thus $z \sim yz$, that is, $y \in A_z$. If $x \in N(A_z) \setminus A_z$, then $z \sim zx$ since $x \in N(A_z) = T_z$, which implies $z \sim zy$ and $y \in N(A_z)$. If y were an element of A_z, then $z \sim yx$ which would imply $z \sim yz \sim xz$, that is, $x \in A_z$ contradicting the hypothesis. Consequently $y \in N(A_z) \setminus T_z$. The implications established also prove that $x \notin N(A_z)$ implies $y \notin N(A_z)$. By symmetry we conclude that $x \sigma_{A_z} y$ and since z is arbitrary, also $x \sigma_{\mathscr{A}'} y$. Therefore $\sigma_{\mathscr{A}'} = \sim$.

The last statement of the theorem is now clear.

Corollary. *$\sigma_{\mathscr{A}}$ is the finest* l.n. *congruence on S.*

Let $\mathscr{C}$ be the family of all proper l.n. complexes and proper r.n. complexes of S together with the empty set, and $\mathfrak{C}$ be the set of all l.n. congruences and r.n. congruences on S. On the set $\mathfrak{P}(\mathscr{C})$ of all non-empty subsets of $\mathscr{C}$ define the function γ by: $\gamma(\mathscr{C}') = \sigma_{\mathscr{C}'}$.

Theorem 6. *The function γ maps $\mathfrak{P}(\mathscr{C})$ onto $\mathfrak{C}$ and is inclusion inverting.*

Proof. This follows easily from Theorem 5 and its dual, and Theorem 1.

Corollary 1. *$\sigma_{\mathscr{C}}$ is the finest* n. *congruence on S.*

Letting $\mathscr{D}$ be the family of all n. complexes of S, we have the following result by the preceding corollary and Theorem 1.

Corollary 2. *$\sigma_{\mathscr{D}}$ is the finest* n. *congruence on S.*

Note that $\mathscr{C}$ can not be replaced by $\mathscr{D}$ in Theorem 6.

Recall that for any $x \in S$, $\lambda_x = \sigma_{A(x)}$, $\varrho_x = \sigma_{B(x)}$, $\tau_x = \lambda_x \cap \varrho_x$. Hence by Theorem 2 (its dual) $\lambda_x[\varrho_x]$ is a l.n. [r.n.] congruence and hence by Theorem 1, τ_x is a n. congruence. It is not hard to show that $\sigma_{\mathscr{A}} = \bigcap_{x \in S} \lambda_x$ and $\sigma_{\mathscr{Q}} = \sigma_{\mathscr{C}} = \bigcap_{x \in S} \tau_x$. The next proposition follows easily from the definitions.

Proposition 4. *For any $x \in S$, we have*

a) $S/\lambda_x \cong U$ *if and only if* $A(x) = S$;

b) $S/\lambda_x \cong L$ *if and only if* $A(x) \neq N(x) = S$;

c) $S/\lambda_x \cong U^0$ *if and only if* $A(x) = N(x) \neq S$;

d) $S/\lambda_x \cong L^0$ *if and only if* $A(x) \neq N(x) \neq S$.

The next theorem characterizes the n. congruences τ_x.

Theorem 7. *For any $x \in S$, we have:*

a) $S/\tau_x \cong U$ *if and only if* $A(x) = S = B(x)$;

b) $S/\tau_x \cong L$ *if and only if* $A(x) \neq S = B(x)$;

c) $S/\tau_x \cong R$ *if and only if* $A(x) = S \neq B(x)$;

d) $S/\tau_x \cong L \times R$ *if and only if* $A(x) \neq N(x) = S \neq B(x)$;

e) $S/\tau_x \cong U^0$ *if and only if* $A(x) = B(x) \neq S$;

f) $S/\tau_x \cong L^0$ *if and only if* $A(x) \neq N(x) = B(x) \neq S$;

g) $S/\tau_x \cong R^0$ *if and only if* $B(x) \neq N(x) = A(x) \neq S$;

h) $S/\tau_x \cong (L \times R)^0$ *if and only if* $A(x) \neq N(x) \neq B(x)$, $N(x) \neq S$,

and these are all homomorphic images S/τ_x.

Proof. As a sample we outline the proof of h); the other cases are treated analogously (most of them are simpler to prove than h)). We note first that by Theorem 4, $N(x) = N(A(x)) = N(B(x))$.

Necessity of h). Since S/τ_x has a zero, we must have $N(x) \neq S$. Then $N(x)/\xi_x \cong L \times R$, where ξ_x is the restriction of τ_x to $N(x)$, which by d) (d) in turn follows easily from Proposition 4, part b) and its dual) implies that $A(x) \neq N(x) \neq B(x)$. Sufficiency of h) is proved by essentially reversing the steps in the proof of necessity.

The last statement of the theorem follows by enumerating all possible cases of relationships among $A(x)$, $B(x)$, $N(x)$, and S.

5. Some properties of normal bands

We now investigate the properties of l.n. [r.n.] complexes of normal bands. The next theorem will be useful, it is also of independent interest.

Theorem 8. *Let S be any semigroup such that for all $a, x, y, b \in S$, $axyb = ayxb$. Then for any $x \in S$, we have:*

a) $N(x) = \{y \in S \mid SyS \cap \langle x \rangle \neq \Box\}$ *(the smallest face containing x)*;

b) $P(x)=\{y\in S|\ yS\cap xS\neq\Box\}$ *(the smallest* l.d.r.i. *containing x)*;

c) $A(x)=\{y\in S|\ yS\cap\langle x\rangle\neq\Box\}$ *(the smallest* l.n. *complex containing x)*.

Proof. a) Let $T(x)=\{y\in S|\ SyS\cap\langle x\rangle\neq\Box\}$. If $a, b\in T(x)$, then $uav=x^m$, $zbw=x^n$ for some $u, v, z, w\in S$ and some m, n. Hence $(uz)ab(vw)=(uav)(zbw)==x^{m+n}$ and thus $ab\in T(x)$. Conversely, if $ab\in T(x)$, then $uabv=x^m$ for some $u, v\in S$ and some m, which implies $a, b\in T(x)$. Thus $T(x)$ is a face of S containing x and hence $N(x)\subseteq T(x)$. The opposite inclusion being evident, we have $N(x)=T(x)$.

b) Let $T(x)=\{y\in S|\ yS\cap xS\neq\Box\}$. If $a\in T(x)$, $b\in S$, then $au=xv$ for some $u, v\in S$. Thus $aub^2=xvb^2$ whence $ab(ub)=x(vb^2)$ so that $ab\in T(x)$. Conversely, if $ab\in T(x)$, then obviously $a\in T(x)$. Hence $T(x)$ is a l.d.r.i. of S containing x and thus $P(x)\subseteq T(x)$. Again the opposite inclusion is obvious, and we have $P(x)=T(x)$.

c) Let $T(x)=\{y\in S|\ yS\cap\langle x\rangle\neq\Box\}$. By the definition of $A(x)$ and parts a), b) of the present theorem, $A(x)=\{y\in N(x)|\ yN(x)\cap xN(x)\neq\Box\}$. If $a\in T(x)$, then $au=x^n$ for some $u\in S$ and some n. Hence $a(ux)=xx^n$, where clearly $ux, x^n\in N(x)$ and $a\in A(x)$. Conversely, suppose that $a\in A(x)$. Then $a\in N(x)$ and $au=xv$ for some $u, v\in N(x)$. By part a), $zvw=x^m$ for some $z, w\in S$ and some m. Consequently $a(uzw)=xvzw=x(zvw)=x^{m+1}$, and $a\in T(x)$. Therefore $A(x)=T(x)$.

Corollary 1. *In a normal band S, for any $x\in S$, we have:*

a) $N(x)=\{y\in S|\ x=xyx\}$;

b) $P(x)=\{y\in S|\ ya=xa$ *for some* $a\in S\}$;

c) $A(x)=\{y\in S|\ x=yx\}$.

Proof. a) This is valid in any band S (6.2, [3]).

b) If $yu=xv$, then $y(vu)=(yu)(vu)=(xv)(vu)=x(vu)$.

c) If $yu=x$, then $yx=yu=x$.

Corollary 2. *In a normal band S, the following statements are equivalent for any $x\in S$:*

a) N_x *is a left zero semigroup*;

b) x *is a left zero of* $N(x)$;

c) $B(x)=N(x)$.

Proof. We prove only that c) implies a). If $a\in N_x$, then $x=xax$ by part a) of Corollary 1 which together with the hypothesis and the dual of part c) of Corollary 1 implies $x=xa$.

Note that a) and b) in Corollary 2 are equivalent in any band. Let S be a normal band, and let

$$D_1=\{x\in S|\ |N_x|=1\},$$
$$D_2=\{x\in S|\ |N_x|>1 \text{ and } N_x \text{ is a left zero semigroup}\},$$
$$D_3=\{x\in S|\ |N_x|>1 \text{ and } N_x \text{ is a right zero semigroup}\},$$
$$D_4=S\setminus(D_1\cup D_2\cup D_3).$$

Further, let K be the intersection of all ideals of S and let

$$F_i = D_i \cap K,$$
$$G_i = D_i \cap (S \setminus K),$$

for $i=1,2,3,4$. Finally let $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ denote, respectively, the semigroups U, L, R, $L \times R$. We are now ready to prove the main theorem of this section.

Theorem 9. *Let x be any element of a normal band S. Then for $i=1,2,3,4$,*

a) $S/\tau_x \cong \alpha_i$ *if and only if* $x \in F_i$,

b) $S/\tau_x \cong \alpha_i^0$ *if and only if* $x \in G_i$.

Proof. We note that $K=\{y \in S \mid N(y)=S\}$, that by Theorem 4, $N(x)=N(A(x))=N(B(x))$, and that $A(x)=B(x)$ implies $A(x)=N(x)$. The theorem now follows from Theorem 7 and Corollary 2 to Theorem 8 and its dual.

We say that a family $\mathscr{F}$ of equivalence relations on a set T distinguishes elements of T if $\bigcap_{\varphi \in \mathscr{F}} \varphi = \iota_T$, where ι_T is the identity relation on T.

Theorem 10. *S is a normal band if and only if the family $\{\tau_x\}_{x \in S}$ distinguishes elements of S.*

Proof. Necessity. Let $x, y \in S$ and suppose that for all $z \in S$, $x\tau_z y$. In particular $x\tau_x y$ which implies $y \in A(x) \cap B(x)$. By minimality of $A(x)$ and $B(x)$, we have $A(y) \subseteq A(x)$ and $B(y) \subseteq B(x)$. Similarly $x\tau_y y$ implies $A(x) \subseteq A(y)$, $B(x) \subseteq B(y)$. Consequently $A(x)=A(y)$, $B(x)=B(y)$. By Corollary 1 to Theorem 8 (part c) and its dual), we have for all $z, w, \in S$:

$$x = zx \quad \text{if and only if} \quad y = zy,$$
$$x = xw \quad \text{if and only if} \quad y = yw.$$

Since x and y are idempotent, $x = xy = y$.

Sufficiency. Since τ_x is a n. congruence, so is $\bigcap_{x \in S} \tau_x$ which by the hypothesis is equal to ι_S, the identity relation on S. But then S itself is a normal band.

One similarly establishes

Proposition 5. *S is a left normal band if and only if the family $\{\lambda_x\}_{x \in S}$ distinguishes elements of S.*

6. Normal bands and subdirect products

We next use some results of the previous section to obtain representations of normal bands as subdirect products of subdirectly irreducible normal bands.

Let S be a normal band. Then by Theorem 10, $\bigcap_{x \in S} \tau_x = \iota_S$ which by BIRKHOFF's theorem (Theorem 9, p. 92, [1]) implies that S is a subdirect product of semigroups S/τ_x, $x \in S$. Theorem 9 yields all the semigroups S/τ_x in terms of the sets F_i and G_i, $i=1,2,3,4$. In the notation of that theorem, $S/\tau_x \cong \alpha_1 = U$ if and only if $x \in F_1$.

Since $F_1 = D_1 \cap K$ it is clear that if $F_1 \neq \square$, F_1 consists of a single element which is the zero of S. Further, if $S/\tau_x \cong \alpha_4^0 = (L \times R)^0$, then by the definition of τ_x $(\tau_x = \lambda_x \cap \varrho_x)$, we must have $S/\lambda_x \cong L^0$, $S/\varrho_x \cong R^0$. It is easy to verify that U^0, L, R, L^0, R^0 are subdirectly irreducible. Using Theorem 9 for counting the number of copies of these semigroups among the semigroups S/τ_x and taking into account the above discussion, we obtain the next theorem which is the main result of this section.

Theorem 11. *Any normal band S having more than one element is the subdirect product of*

$$|F_2| + |F_4| \ \text{ copies of } \ L, \qquad |F_3| + |F_4| \ \text{ copies of } \ R,$$

$$|G_1| \ \text{ copies of } \ U^0, \quad |G_2| + |G_4| \ \text{ copies of } \ L^0, \quad |G_3| + |G_4| \ \text{ copies of } \ R^0.$$

The total number of copies is $|S|$ if S has no zero and $|S|-1$ if S has a zero. The semigroups L, R, U^0, L^0, R^0 are subdirectly irreducible.

Remark. We have already noted that if $F_1 \neq \square$, then it consists of a single element which is the zero of S. Similarly Corollary 2 to Theorem 8 implies that if $F_2 \neq \square$, then F_2 is the set of all left zeros of S and is the kernel of S; an analogous statement is valid for F_3. Hence at most one of the sets F_1, F_2, F_3 is non-empty.

Corollary 1. *All the semigroups considered contain more than one element.*

a) *A left normal band is the subdirect product of*

$$|F_2| \ \text{ copies of } \ L, \qquad |G_1| \ \text{ copies of } \ U^0, \qquad |G_2| \ \text{ copies of } \ L^0.$$

b) *A left zero semigroup is a subdirect product of*

$$|S| \ \text{ copies of } \ L.$$

c) *A semilattice is the subdirect product of*

$$|S|-1 \ \text{ copies of } \ U^0 \ \text{ if } \ S \ \text{ is finite,}$$

$$|S| \ \text{ copies of } \ U^0 \ \text{ if } \ S \ \text{ is infinite.}$$

Proof. Part a) follows from the theorem; it can also be derived directly, by the same method of proof as above, from Propositions 4 and 5. Part b) follows directly from the theorem. If S is a finite semilattice, then S has a zero. If S is an infinite semilattice with zero, then $|S| = |S \setminus 0| = |G_1|$. Hence c) holds.

Corollary 2 (*cf.* Theorem 1, [5] and Corollaire I, Théorème V, [7]). *These are the only subdirectly irreducible*

a) *normal bands:* L, R, U^0, L^0, R^0;

b) *left normal bands:* L, U^0, L^0;

c) *left zero semigroups:* L;

d) *semilattices:* U^0.

Corollary 3. (*cf.* Theorem 4, [9] which is a stronger statement). *Every normal band is a subdirect product of a left and a right normal band.*

Proof. This follows from the theorem since the product of, e.g., copies of L, U^0, L^0 is a left normal band; similarly for R, U^0, L^0. This corollary also follows from Theorem 1.

One might ask what kind of bands are subdirect products of, say, left zero semigroups and semilattices, or some other combination of classes of semigroups we have considered. The desired results can be obtained from Theorem 11 or certain of its corollaries.

7. A representation of normal bands

The following construction is an easy modification of the one given by SHAIN [5] for right normal bands. It gives a representation of a normal band by subsets of a set under certain multiplication.

Let B, C, and D be sets, let $E=\{1, 2\}$ and suppose that B, $C\times E$, $D\times E$ are pairwise disjoint. Let

$$A=B\cup(C\times E)\cup(D\times E),$$

and let $\mathcal{F}$ be the set of all subsets of A under the following multiplication: for $\mathfrak{A}, \mathfrak{B}\in\mathcal{F}$;

$$\begin{aligned} &\mathfrak{A}\cdot\mathfrak{B}\cap B=\mathfrak{A}\cap\mathfrak{B}\cap B,\\ &\mathfrak{A}\cdot\mathfrak{B}\cap(C\times E)=\mathfrak{A}\cap(C\times E),\\ &\mathfrak{A}\cdot\mathfrak{B}\cap(D\times E)=\mathfrak{B}\cap(D\times E).\end{aligned} \tag{1}$$

It is easy to see that $\mathcal{F}$ is a normal band; in analogy to [5] we call subsemigroups of $\mathcal{F}$ *special normal bands.*

Theorem 12. *Every normal band is isomorphic to a special normal band.*

Proof. Let S be a normal band; then using Theorem 11, we may suppose that $S\subseteq\prod_{i\in I} S_i$, where S_i is one of the semigroups: L, R, U^0, L^0, R^0, and projection $\pi_i(S)=S_i$ for all $i\in I$. Let

$$\begin{aligned} B&=\{i\in I\mid S_i=U^0\},\\ C&=\{i\in I\mid S_i=L \quad\text{or}\quad S_i=L^0\},\\ D&=\{i\in I\mid S_i=R \quad\text{or}\quad S_i=R^0\}.\end{aligned}$$

Let $U=\{1\}$, $L=\{l_1, l_2\}$, $R=\{r_1, r_2\}$. Let A, E, and $\mathcal{F}$ be as above and define the function φ: $S\to\mathcal{F}$ by letting $\varphi(x)\subseteq A$ be defined by:

$$\begin{aligned} &\varphi(x)\cap B = \{i\in B\mid \pi_i(x) = 1\},\\ &\varphi(x)\cap(C\times E) = \bigcup_{j=1}^{2}\{(i,j)\in C\times E\mid \pi_i(x) = l_j\},\\ &\varphi(x)\cap(D\times E) = \bigcup_{j=1}^{2}\{(i,j)\in D\times E\mid \pi_i(x) = r_j\}.\end{aligned} \tag{2}$$

A straightforward calculation shows that φ is an isomorphism; here $\varphi(S)$ is a special normal band.

Theorem 13 (*cf.* Theorem 2, [5] and Corollaire II, Théorème V, [7]). *With the notation as in the introduction to this section, we have:*

a) $\mathcal{F}$ *is a left normal band if and only if* $D=\square$;

b) $\mathcal{F}$ *is a left zero semigroup if and only if* $B=D=\square$;

c) $\mathcal{F}$ *is a semilattice if and only if* $C=D=\square$;

d) $\mathcal{F}$ *is a rectangular band if and only if* $B=\square$.

Defining a "special left normal band", a "special left zero semigroup", etc., analogously as a special normal band above, the statements corresponding to Theorem 12 *are valid for bands in* a)—d).

Proof. The proof is an adaptation of the proof of Theorem 12 and is omitted.

Remark. If we replace (1) by

$$\mathfrak{A}\cdot\mathfrak{B}\cap B=(\mathfrak{A}\cup\mathfrak{B})\cap B,$$

$\mathcal{F}$ is still a normal band and Theorem 12 remains valid where in the proof, (2) is replaced by

$$\varphi(x)\cap B=\{i\in B|\ \pi_i(x)=0\}$$

(0 is the zero of U^0).

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