

The algebraic structure of non self-adjoint operators

By NOBORU SUZUKI in Kanazawa (Japan) *

The purpose of this paper is to give an algebraic approach to the theory of non self-adjoint operators on (complex) Hilbert space by means of the theory of von Neumann algebras. In the spectral theory, the principal problem is to reduce a given operator to simpler operators. We shall consider this problem, from the algebraic view point, for a certain class of non self-adjoint operators.

Let A be an operator on Hilbert space. We shall denote by $R(A)$ the von Neumann algebra generated by A (i.e., the smallest von Neumann algebra containing A) and we say that A is *primary* if $R(A)$ is a factor. Then the spectral decomposition of a normal operator A essentially means the decomposition of A into primary normal operators (which are scalar operators). Moreover, we know that an isometry is decomposed into the direct sum of a unitary operator and a unilateral shift. As we have shown in [9; Lemma 2] (cf. [3; Theorem 1]), a unilateral shift is a primary operator. From this fact, we can easily see that a non-scalar isometry is a unilateral shift if and only if it is primary. Therefore, the decomposition of an isometry V mentioned above is essentially that of V into primary isometric operators with the aid of the spectral theorem for a unitary operator. From this point of view, the decomposition of an operator A into primary operators may be regarded as a kind of spectral decomposition of A .

We shall concern ourselves with the class of operators whose imaginary parts are completely continuous. M. S. BRODSKIĀ and M. S. LIVŠIĆ, cf. [1], [5], have developed a theory of the triangular form for operators whose imaginary parts belong to the trace class. Our purpose is to establish the decomposition of an operator with completely continuous imaginary part into primary operators belonging to the same class and to show that a primary operator of this class is the direct sum of copies of an irreducible operator of the same class by making use of the theory of von Neumann algebras. Consequently, we shall be able to see some algebraic aspects of operators with completely continuous imaginary part. This paper contains the details of the research announcement appeared in [8].

For the sake of simplicity, we shall assume that our Hilbert space is separable. By an operator we always understand a bounded linear transformation on a Hilbert space. By a von Neumann algebra we understand a self-adjoint operator algebra with the identity operator I which is closed in the weak topology. The set of operators each of which commutes with every operator in a von Neumann algebra M will

* This research was partially supported by the National Science Foundation (NSF Contract G 24295) at the University of Minnesota (1964—1965).

be called the commutant of M and be denoted by M' . The commutant M' is again a von Neumann algebra and $M = M''$. A factor means a von Neumann algebra whose center consists of scalar multiples of the identity operator. For terminology, notation and basic results, we shall refer to the book of J. DIXMIER [2].

1. The structure of operators with completely continuous imaginary part

In this section, we shall restrict our consideration to an operator A on a Hilbert space H whose imaginary part $\text{Im}(A) = \frac{1}{2i}(A - A^*)$ is completely continuous. Our object is to prove the following

Theorem 1. *An operator A with completely continuous imaginary part on a Hilbert space H is decomposed by a unique countable family of mutually orthogonal central projections P_0, P_i ($i \in I$) in $R(A)$ into the form*

$$A = A_{P_0} \oplus \sum_{i \in I} \oplus A_{P_i},$$

where the restriction A_{P_0} of A to P_0H is a self-adjoint operator, the restriction A_{P_i} of A to P_iH ($i \in I$) is a primary operator with completely continuous imaginary part and $P = \sum_{i \in I} P_i$ is the projection on the subspace generated by vectors of the form $A^n \varphi$ ($\varphi \in \text{Im}(A)H$; $n = 0, 1, 2, \dots$).

Certainly the essence of our result is in the reduction theory of VON NEUMANN [6], that is, in the direct integral decomposition of $R(A)$ into factors, but it should be noticed that the character of the operator A has induced a more simple and concrete decomposition of $R(A)$. Before beginning the proof, we shall provide some lemmas. We shall denote by K the range of $\text{Im}(A)$, i. e.,

$$K = \frac{1}{2i}(A - A^*)H$$

and the projection on the subspace \bar{K} will be denoted by E . In what follows, M always means the von Neumann algebra $R(A)$ generated by A . Since $\text{Im}(A)$ is a self-adjoint completely continuous operator, it is well known that there exists an orthonormal basis in H whose elements are proper vectors of $\text{Im}(A)$. Therefore, if we denote by $\{\mu_k\}$ ($k \in N$) the countable family of all distinct non-zero proper values of $\text{Im}(A)$ and by E_k the projection on the proper subspace corresponding to μ_k , each E_kH is finite dimensional and $E = \sum_{k \in N} E_k$. As the first step, we observe that each projection E_k belongs to M and hence the projection E is in M . This notable fact is the direct consequence of the following

Proposition 1. *Let B be an operator in the von Neumann algebra M . Then a projection on a proper subspace of B belongs to M .*

Proof. Let μ be a proper value of B and let $\mathcal{N}(\mu)$ the proper subspace corresponding to μ . We denote by F the projection on $\mathcal{N}(\mu)$. In order to prove that F

belongs to M , it is sufficient to show that F commutes with all operators belonging to the commutant M' of M . Let A' be an arbitrary operator in M' . Then, for every vector φ in the proper subspace $\mathcal{N}(\mu)$, the equality $B(A'\varphi) = A'B\varphi = A'\mu\varphi = \mu A'\varphi$ yields $A'\varphi \in \mathcal{N}(\mu)$. Similarly we have $A'^*\varphi \in \mathcal{N}(\mu)$ for every vector $\varphi \in \mathcal{N}(\mu)$. Thus the subspace $\mathcal{N}(\mu)$ reduces A' . This means that F commutes with A' .

We consider the subspace H_1 generated by vectors of the form $A^n\varphi$ ($\varphi \in K$; $n=0, 1, 2, \dots$) and denote by P the projection on H_1 . As is well known, the projection P plays a very important role in the study of our operator A , and so we need to find the exact relation between A and P .

Lemma 1. *The subspace H_1 coincides with the subspace $[MK]$ generated by vectors of the form $B\varphi$ ($B \in M, \varphi \in K$). That is, the projection P belongs to the center of M .*

Proof. It is clear that H_1 is invariant by A , and so its orthogonal complement $H_2 = H \ominus H_1$ is invariant by A^* . For each vector $\varphi \in H_2$, we have $\left\langle \frac{1}{i} (A - A^*)\varphi, \psi \right\rangle = \left\langle \varphi, \frac{1}{i} (A - A^*)\psi \right\rangle = 0$ for every vector $\psi \in H$. Thus $A\varphi = A^*\varphi$ for every vector $\varphi \in H_2$. This means that the subspace H_2 is invariant by A . Therefore, the subspace H_2 reduces A , that is to say, the projection Q on H_2 belongs to M' . Thus the projection $P = I - Q$ belongs to M' . For each operator $B \in M$ and for each vector $\varphi \in K$, the equality $B\varphi = BP\varphi = PB\varphi$ implies $[MK] \subset H_1$. On the other hand, obviously H_1 is contained in $[MK]$. Consequently, we obtain that the subspace H_1 coincides with the subspace $[MK]$.

Next we observe that the subspace $[MK]$ reduces every operator $B' \in M'$. In fact, since the projection E is in M by Proposition 1, $B'MK = MB'K = MB'EK = MEB'K \subset MK$ for each operator $B' \in M'$. In the same way, we can get $B'^*MK \subset MK$. It follows from this fact that P commutes with every operator belonging to M' . Thus the projection P belongs to M . Consequently, the projection P belongs to the center $M \cap M'$ of M .

Remark. In the theory of von Neumann algebras, the projection on $[MK]$ is called the central support of E . Indeed, it is the minimal central projection containing E . We have shown that the projection P is the central support of the projection E .

The following lemma on von Neumann algebras is essentially known, but, for the sake of completeness, we shall give the proof.

Lemma 2. *Let F be a minimal projection in M ¹⁾ with the central support R . Then there exists a countable family of orthogonal, equivalent projections $\{F_j\}$ ($j \in J$) such that $R = \sum_{j \in J} F_j$ and $F_{j_0} = F$ for a fixed $j_0 \in J$.*

¹⁾ A minimal projection in M means a non-zero projection F belonging to M such that $G \leq F$ and $O \neq G \in M$ implies $G = F$.

Proof. Let $F_j (j \in J)$ be a maximal family of orthogonal, equivalent projections such that $F_{j_0} = F$. Then $F_j \leq R$ for all $j \in J$. Put $G = R - \sum_{j \in J} F_j$. By using the theorem of comparison (cf. [1: Ch. III, Theorem 1]), we can find a central projection Q such that

$$GQ \prec FQ \quad \text{and} \quad F(I-Q) \prec G(I-Q).$$

If $F(I-Q) \neq 0$, $F(I-Q) = F$ since F is minimal in M . It follows that $F \leq I-Q$ and $F \prec G(I-Q) \leq G$. This contradicts to the maximality of $\{F_j\} (j \in J)$. Thus $F(I-Q)$ must be zero, and so $F \leq Q$. Then we have $GQ \prec F$. Since F is minimal in M , $GQ = 0$ or $GQ \sim F$. Therefore, $GQ = 0$ since $GQ \sim F$ obviously yields the contradiction. It follows that

$$0 = GQ = RQ - \sum_{j \in J} F_j Q = RQ - \sum_{j \in J} F_j.$$

Keeping in mind that R is the central support of F , we get

$$R = RQ = \sum_{j \in J} F_j.$$

Proof of Theorem 1. From Lemma 1 it follows that the operator A is decomposed by the central projection P into the form

$$A = A_{I-P} \oplus A_P$$

where A_{I-P} is a self-adjoint operator on $(I-P)H$ and P is the central support of E . As we have already seen, the projection E is expressed as the direct sum of finite dimensional projections $E_k (k \in N)$ in M . Since each projection E_k is finite dimensional and P is the central support of E , we can choose a family of minimal projections $F_i (i \in I)$ in M contained in some of E_k such that the central supports P_i of F_i are mutually orthogonal and $P = \sum_{i \in I} P_i$. By making use of Lemma 2, we can get a family of orthogonal, equivalent projections $F_{ij} (j \in J)$ such that $F_i \simeq F_{ij_0}$ and $P_i = \sum_{j \in J} F_{ij}$. Then the restriction M_{P_i} of M to $P_i H$ is spatially isomorphic to $M_{F_i} \otimes \mathcal{L}(l_2(J))^2$ where M_{F_i} is the restriction of M to $F_i H$ and $\mathcal{L}(l_2(J))$ means the algebra of all operators on $l_2(J)$. Since F_i is minimal in M , M_{F_i} is the scalar multiples of the identity operator on $F_i H$ and hence M_{P_i} is a factor. Note that P_i is a central projection. Then we obtain that the factor M_{P_i} is generated by A_{P_i} , that is to say, each operator M_{P_i} is a primary operator. In addition, it is obvious that $\text{Im}(A_{P_i})$ is completely continuous. Putting $P_0 = I - P$, we obtain the desired result since the uniqueness of a family $\{P_0, P_i\} (i \in I)$ directly follows from the fact that each operator $A_{P_i} (i \in I)$ is primary.

What our theorem means is quite well illustrated by taking a normal operator of this class. Indeed, Theorem 1 yields the spectral decomposition of the non-self-adjoint part of this operator.

²⁾ The notation \otimes always means the tensor product of Hilbert spaces, operators, or von Neumann algebras.

Corollary 1. *Let A be a normal operator with completely continuous imaginary part. Then A is uniquely expressed by a countable family of mutually orthogonal projections $P_0, P_i (i \in I)$ in $R(A)$ as follows:*

$$A = AP_0 + \sum_{i \in I} \lambda_i P_i,$$

where each $P_i (i \in I)$ is finite dimensional and $I = P_0 + \sum_{i \in I} P_i$, moreover $\{\lambda_i\} (i \in I)$ is a family of non-real proper values of A and AP_0 is a self-adjoint operator.

In fact, since A is normal, each operator A_{P_i} in Theorem 1 must be a scalar operator $\lambda_i I_i$ (where I_i is the identity operator on $P_i H$). Furthermore, since A_{P_i} is a non self-adjoint operator (in this case $P_i = F_i \cong E_k \cong E$) and has a completely continuous imaginary part, each λ_i is a non-real number and $P_i H$ must be finite dimensional. Then, by Theorem 1, $A = AP_0 + \sum_{i \in I} \lambda_i P_i$ and clearly λ_i is a proper value of A . Thus our result is the decomposition of the non-real spectrum of this operator.

Next we shall mention a very important special class of our operators. That is, we shall consider the class of operators whose imaginary parts are finite dimensional operators. Let A be an operator with finite dimensional imaginary part. Then the dimension r of the range of $\text{Im}(A)$ is called the *non-hermitian rank* of A . In this case, Theorem 1 may be stated as follows.

Corollary 2. *An operator A with non-hermitian rank r is decomposed by a unique family of mutually orthogonal central projections $P_0, P_1, \dots, P_n (n \leq r)$ in $R(A)$ into the form*

$$A = A_{P_0} \oplus A_{P_1} \oplus \dots \oplus A_{P_n},$$

where A_{P_0} is a self-adjoint operator and $\{A_{P_1}, \dots, A_{P_n}\}$ is a family of primary operators with non-hermitian rank k_i such that

$$\sum_{i=1}^n k_i = r.$$

In fact, from the proof of Theorem 1 we can easily see that a family $P_i (i \in I)$ is finite and the non-hermitian rank k_i of $A_{P_i} (i = 1, 2, \dots, n)$ is equal to $\dim(EP_i)$. Accordingly we have

$$\sum_{i=1}^n k_i = \sum_{i=1}^n \dim(EP_i) = \dim \left(E \sum_{i=1}^n P_i \right) = \dim(EP) = \dim(E) = r.$$

2. The algebraic type of operators with completely continuous imaginary part

The structure of an operator A is closely related to the type of the von Neumann algebra $R(A)$ generated by A . An operator A is said to be of type I if $R(A)$ is of type I and moreover a primary operator A is said to be of type I_n (resp. type I_∞) if the factor $R(A)$ is of type I_n (resp. type I_∞). Then the question coming to our mind is this: which non-normal operators are of type I? Partial answers to this

question are known. We know that an isometry is of type I ([8]). Moreover, we have shown that a completely continuous operator is of type I ([8]). This result will be generalized in what follows. Indeed, from the proof of Theorem 1 it is easy to determine the type of operators with completely continuous imaginary part.

Theorem 2. *An operator A with completely continuous imaginary part is of type I.*

Proof. As we have seen in the proof of Theorem 1, each operator A_{P_i} generates a von Neumann algebra M_{P_i} of type I_α ($\alpha = n$ or ∞) (recall that M_{P_i} is spatially isomorphic to $(\lambda I_i) \otimes \mathcal{L}(l_2(J))$). Moreover, since $\{P_0, P_i\}$ ($i \in I$) is a family of mutually orthogonal central projections, the von Neumann algebra $M = R(A)$ is decomposed as the direct sum

$$M = M_{P_0} \oplus \sum_{i \in I} M_{P_i}.$$

Thus we can conclude the theorem since the abelian von Neumann algebra M_{P_0} is of type I (cf. [1; Ch. I, § 8, Prop. 1]).

Here is a very remarkable fact which illustrates the algebraic aspect of primary operators of our class. In our decomposition, it is possible that A_{P_i} has the type I_∞ (actually we may restrict our attention to this case), but the commutant $R(A_{P_i})'$ of the von Neumann algebra $R(A_{P_i})$ has necessarily the type I_n ($n = 1, 2, \dots$). To show this it is sufficient to consider only a non-scalar primary operator A . Let A be a non-scalar primary operator with completely continuous imaginary part. Then $R(A)$ contains obviously a finite dimensional minimal projection (recall that each projection on a proper subspace of $\text{Im}(A)$ corresponding to a non-zero proper value is finite dimensional). Since all minimal projections in the factor $R(A)$ are equivalent to each other, the dimension d of a minimal projection in $R(A)$ is uniquely determined by the operator A . In what follows, the dimension d will be called the *multiplicity* of the operator A .

Proposition 2. *Let A be a non-scalar primary operator with completely continuous imaginary part. Then the commutant $R(A)'$ of $R(A)$ is of type I_n where n is the multiplicity of A .*

Proof. Let F be a minimal projection in $M = R(A)$. Then $\dim(F) = n$. By Lemma 2, we can choose a family of mutually orthogonal, equivalent projections $\{F_j\}$ ($j \in J$) in M such that $F_{j_0} = F$ and $\sum_{j \in J} F_j = I$. Then M is spatially isomorphic to $M_F \otimes \mathcal{L}(l_2(J))$. The minimality of F implies that M_F is the scalar multiples of the identity operator on FH . Thus M' is spatially isomorphic to $(M_F)' \otimes \mathcal{L}(l_2(J))' = \mathcal{L}(FH) \otimes \mathcal{C}$ where \mathcal{C} is the von Neumann algebra of scalar multiples of the identity operator on $l_2(J)$. Since $\mathcal{L}(FH)$ is of type I_n , $\mathcal{L}(FH) \otimes \mathcal{C}$ is of type I_n . Therefore, M' is of type I_n .

Corollary. *A primary operator A with non-hermitian rank 1 is irreducible (i.e., A has no non-trivial reducing subspace).*

3. The decomposition of a primary operator into irreducible operators

In this section, we shall show that a primary operator A with completely continuous imaginary part is expressed as the direct sum of copies of an irreducible operator of the same class. Indeed, Proposition 2 makes it possible to decompose A into irreducible operators in a simple way. Consequently, the study of our operators may be reduced to the case of irreducible operators of our class. We shall mention here some examples of irreducible operators with completely continuous imaginary part.

Example 1. The simplest irreducible operator with non-hermitian rank 1 on an infinite dimensional Hilbert space is the integral operator on $L_2(0, 1)$ defined by

$$(Af)(x) = i \int_0^x f(t) dt.$$

Indeed, $L_2(0, 1)$ is generated by the vectors of the form $A^n \varphi$ ($n=0, 1, 2, \dots$) where $\varphi(x) \equiv 1$, and the range of $\text{Im}(A)$ consists of scalar multiples of the vector φ . Thus it follows from Theorem 1 that A is primary. By what was already seen in the preceding section the operator A is irreducible. Moreover, we know that this operator is quasi-nilpotent. Here we should mention that the integral operator A is characterized by these algebraic properties. That is, the notable result obtained in [2] and [4] may be stated as follows: a quasi-nilpotent primary operator with non-hermitian rank 1 is unitarily equivalent (up to a non-zero real scalar multiple) to the integral operator A on $L_2(0, 1)$.

Example 2. Let V be a unilateral shift on a Hilbert space H . That is, for an orthonormal basis $\{\varphi_n\}$ ($n=1, 2, \dots$) in H , $V\varphi_n = \varphi_{n+1}$ for all n . Now we consider an operator A of the form VB , where B is a positive completely continuous operator whose range spans H . We shall show that the operator A is irreducible. Put $M = R(A)$. Then the equality $B^2 = BV^*VB = (VB)^*(VB) = A^*A$ implies $B^2 \in M$. Thus $B = (B^2)^{1/2}$ belongs to M . Here, B is expressed in the form: $B = \sum_n \lambda_n E_n$, where $\{\lambda_n\}$ is the countable family of all distinct proper values of B , E_n is the projection on the proper subspace corresponding to λ_n and $\sum_n E_n = I$. Since the range of B spans H , $\lambda_n > 0$ for all n . By Proposition 1, each projection E_n belongs to M . Hence, for each k we have

$$\lambda_k VE_k = V(\sum_n \lambda_n E_n)E_k = VBE_k \in M,$$

and so $VE_k \in M$ for each k . Consequently, $V = V(\sum_n E_n) = \sum_n VE_n \in M$. Since V is irreducible, $R(V) = \mathcal{L}(H)$ is contained in M . Thus $M = \mathcal{L}(H)$, that is to say, A is irreducible.

Theorem 3. Let A be a non-scalar primary operator with completely continuous imaginary part and let m be the multiplicity of A . Then A is unitarily equivalent to an operator $V \otimes I_m$, where V is an irreducible operator with completely continuous imaginary part and I_m is the identity operator on an m -dimensional Hilbert space. In particular, if A has the non-hermitian rank r , A is unitarily equivalent to $V \otimes I_m$, where V is an irreducible operator with non-hermitian rank n and $r = mn$.

Proof. We shall take here the projection E, P in $M = R(A)$ as in the section 1. From Proposition 2 it follows that there exists a family of mutually orthogonal, equivalent (minimal) projections P_1, P_2, \dots, P_m in M' such that $I = \sum_{i=1}^m P_i$. Then each operator A_{P_i} ($i=1, 2, \dots, m$) is an irreducible operator with completely continuous imaginary part. In fact, since P_i belongs to M' , $A_{P_i} - A_{P_i}^* = (A - A^*)_{P_i}$. To see that A_{P_i} is irreducible we consider the von Neumann algebra M_{P_i} which is clearly generated by A_{P_i} . As is well known, $(M_{P_i})' = M_{P_i}$. Here the right-hand side consists of scalar multiples of the identity operator on $P_i H$ since P_i is a minimal projection in M' . This means that A_{P_i} is irreducible.

Let W_i be a partially isometric operator in M' such that $W_i^* W_i = P_i$ and $W_i W_i^* = P_i$. Then, for every vector $\varphi \in P_i H$,

$$W_i A_{P_i} W_i^* \varphi = W_i A_{P_i} W_i^* \varphi = W_i A W_i^* \varphi = A W_i W_i^* \varphi = A P_i \varphi = A_{P_i} \varphi.$$

Thus each operator A_{P_i} is unitarily equivalent to A_{P_1} by W_i . Now the assertion will be completed by the standard argument. Put $\mathfrak{H} = P_1 H$ and $V = A_{P_1}$, then, as is well known, $H = \mathfrak{H} \otimes l_2(N)$ where $N = \{1, 2, \dots, m\}$, and each vector $\varphi \in \mathfrak{H} \otimes l_2(N)$ is expressed in the form:

$$\varphi = \sum_{i=1}^m \varphi_i \otimes \varepsilon_i,$$

where $\{\varepsilon_i\}$ ($i=1, 2, \dots, m$) is an orthonormal basis of $l_2(N)$ and $\{\varphi_i\}$ ($i=1, 2, \dots, m$) is a family of vectors in \mathfrak{H} . Define a linear transformation W of H onto $\mathfrak{H} \otimes l_2(N)$ as follows:

$$W\varphi = \sum_{i=1}^m W_i P_i \varphi_i \otimes \varepsilon_i \quad \text{for each vector } \varphi \in H.$$

Then it is verified by straightforward computation that W is an isometry of H onto $\mathfrak{H} \otimes l_2(N)$ and $W^{-1} \left(\sum_{i=1}^m \varphi_i \otimes \varepsilon_i \right) = \sum_{i=1}^m W_i^* \varphi_i$. Therefore, we have

$$\begin{aligned} W A W^{-1} \left(\sum_{i=1}^m \varphi_i \otimes \varepsilon_i \right) &= W \left(\sum_{i=1}^m A W_i^* \varphi_i \right) = \\ &= \sum_{i=1}^m W_i P_i \left(\sum_{j=1}^m A W_j^* \varphi_j \right) \otimes \varepsilon_i = \sum_{i=1}^m W_i A_{P_i} W_i^* \varphi_i \otimes \varepsilon_i = \\ &= \sum_{i=1}^m A_{P_i} \varphi_i \otimes \varepsilon_i = (V \otimes I_m) \left(\sum_{i=1}^m \varphi_i \otimes \varepsilon_i \right). \end{aligned}$$

That is, we have shown that A is unitarily equivalent to $V \otimes I_m$.

If A has the non-hermitian rank r , $r = \dim(E) = \sum_{i=1}^m \dim(P_i E)$. Since $P_i E$ are equivalent to each other in $\mathcal{L}(H)$, $\dim(P_1 E) = \dots = \dim(P_m E)$. Thus $\dim(P_i E) = r/m = n$. Note that $(V - V^*)_{P_1} H = (A_{P_1} - A_{P_1}^*)_{P_1} H = P_1 (A - A^*) H = P_1 E H$. Then we obtain that V has the non-hermitian rank n .

Remark. We cannot express an arbitrary primary operator (not necessarily of type I) as the direct sum of copies of an irreducible operator. That is, from the fact that an operator with completely continuous imaginary part is of type I, our theorem has been effected.

Corollary. Let A be a primary operator with non-hermitian rank r . Then the multiplicity of A is equal to the rank r if and only if A is unitarily equivalent to $V \otimes I_r$, where V is an irreducible operator with non-hermitian rank 1 and I_r is the identity operator on an r -dimensional Hilbert space.

Here we shall concentrate our attention on the case when the non-hermitian rank r of a primary operator A is a prime number. Then the multiplicity m of A must be either 1 or r . In the case of $m=1$, A is irreducible. The case of $m=r$ is that of the above corollary. Thus we have the following

Theorem 4. Let A be a primary operator with non-hermitian rank r . If r is a prime number, then A is either irreducible or unitarily equivalent to $V \otimes I_r$, where V is an irreducible operator with non-hermitian rank 1 and I_r is the identity operator on an r -dimensional Hilbert space.

In closing this section, to see how our results illustrate the algebraic structure of an operator with finite non-hermitian rank, we shall mention here the special cases of our operators. Actually the structure of an operator A with non-hermitian rank r depends on the dimensions of minimal projections with respect to $R(A)$ contained in the projection E (E : the projection on the range of $\text{Im}(A)$). We shall state the possible forms of operators in the cases of non-hermitian rank $r=1, 2, 3$.

An operator A with non-hermitian rank 1 has the following structure:

$$A = A_s \oplus A_i$$

where A_s is a self-adjoint operator and A_i is an irreducible operator with non-hermitian rank 1.

An operator A with non-hermitian rank 2 has one of the following structures:

$$(1) \quad A = A_s \oplus A_i, \quad (2) \quad A = A_s \oplus A_{i_1} \oplus A_{i_2}, \quad (3) \quad A = A_s \oplus A_j,$$

where A_s is a self-adjoint operator; A_i is an irreducible operator with non-hermitian rank 2; A_{i_1} and A_{i_2} are irreducible operators with non-hermitian rank 1; A_j is unitarily equivalent to $V \otimes I_2$ (V, I_2 : operators in Theorem 4). (1) and (2) arise in the case when the projection E contains a minimal projection of dimension 1, and (3) arises in the case when the projection E is a minimal projection.

An operator A with non-hermitian rank 3 has one of the following structures:

$$(1) \quad A = A_s \oplus A_i, \quad (2) \quad A = A_s \oplus A_{i_1} \oplus A_{i_2}, \\ (3) \quad A = A_s \oplus A_{j_1} \oplus A_{j_2} \oplus A_{j_3}, \quad (4) \quad A = A_s \oplus A_k,$$

where A_s is a self-adjoint operator; A_i is an irreducible operator with non-hermitian rank 3; $A_{i_1}, A_{j_1}, A_{j_2}$, and A_{j_3} are irreducible operators with non-hermitian rank 1; A_{i_2} is a primary operator with non-hermitian rank 2 (cf. Theorem 4); A_k is unitarily equivalent to $V \otimes I_3$ in Theorem 4. (1), (2) and (3) arise in the case when the projection E contains a minimal projection of dimension 1. (4) arises in the case when the projection E is a minimal projection.

4. Spectral properties

The basic properties of the spectrum of an operator A with completely continuous imaginary part are known. From M. S. BRODSKIĬ—M. S. LIVŠIĆ [2], we know that every non-real point of the spectrum of the operator A is a proper value and its proper subspace is finite dimensional. Moreover, we know that the set of non-real points of the spectrum of the operator A is at most countable and a limit point of this set is on the real line. If we denote by $\sigma(A)$ (resp. $\sigma_p(A)$) the spectrum (resp. the point spectrum) of A , Theorem 1 yields that $\sigma(A)$ and $\sigma_p(A)$ are divided as follows:

$$\sigma(A) = \sigma(A_0) \cup \left(\bigcup_{i \in I} \sigma(A_i) \right) \quad \text{in case } I \text{ is finite and} \quad \sigma_p(A_0) = \sigma_p(A) \cup \left(\bigcup_{i \in I} \sigma_p(A_i) \right).$$

Hence, in studying the spectrum of our operator, we may concentrate our attention on that of our primary operator. In this case we should point out from Theorem 3 that the (resp. point) spectrum of a primary operator of this class coincides with the (resp. point) spectrum of an irreducible operator of the same class. Here is a significant and interesting problem: how does the algebraic simplicity of a primary operator effect its spectrum? Although many questions about it are left to be settled in the future, we shall have some comments on this subject. The following lemma may be viewed as a step toward our desire.

Lemma 3. *Let A be a non-scalar primary operator. Then every proper value of A lies in the open disc $D = \{\lambda: |\lambda| < \|A\|\}$.*

Proof. Suppose that A has a proper value λ with $|\lambda| = \|A\|$. Then there exists a non-zero vector φ such that $A\varphi = \lambda\varphi$. Put $B = \frac{1}{\lambda}A$. Then $\|B\| = 1$ and $B\varphi = \varphi$.

From this fact it follows that $B^*\varphi = \varphi$ (cf. [7: Chap. X, No 143]). Consequently, $A^*\varphi = \lambda\varphi$. Thus the proper subspace \mathcal{M} corresponding to λ reduces A . That is, the projection P on \mathcal{M} commutes with A . This means that P belongs to the commutant $R(A)'$ of $R(A)$. As we have already seen in Proposition 1, P is the projection in $R(A)$. Hence P belongs to the center $R(A) \cap R(A)'$. Since the non-zero projection P is not the identity operator by the assumption, this contradicts the fact that A is a primary operator.

Combining the known result mentioned above and Lemma 3, we can conclude the following

Proposition 3. *Every non-real point of the spectrum of a non-scalar primary operator A with completely continuous imaginary part lies in the open disc $D = \{\lambda: |\lambda| < \|A\|\}$.*

Now let us consider an operator A with non-hermitian rank 1 whose spectrum is real. We shall show that our decomposition (Theorem 1) induces the spectral decomposition of A in the sense that A is decomposed by a central projection in $R(A)$ into the form $A = B \oplus C$ where B (resp. C) has a pure point (resp. continuous) spectrum.³⁾

³⁾ If $\sigma(A)$ coincides with the point (resp. continuous) spectrum of A , we say that A has a pure point (resp. continuous) spectrum.

Lemma 4. *Let A be a primary operator with non-hermitian rank 1 and let λ be a real scalar. Then the range of $A - \lambda I$ is dense in H .*

Proof. Let \mathcal{M} be the range of $A - \lambda I$ and let \mathcal{N} be the orthogonal complement of \mathcal{M} . Then, for each pair of vectors φ, ψ in \mathcal{N} , we have

$$\begin{aligned} \langle (A - A^*)\varphi, \psi \rangle &= \langle [(A - \lambda I) - (A^* - \lambda I)]\varphi, \psi \rangle = \\ &= \langle (A - \lambda I)\varphi, \psi \rangle - \langle \varphi, (A - \lambda I)\psi \rangle = 0. \end{aligned}$$

Since $K = \text{Im}(A)H$ is one dimensional, the subspace $[(A - A^*)\mathcal{N}]$ must be K or $\{0\}$. Therefore, \mathcal{N} is contained in $H \ominus K$. Keeping in mind that $A\psi = A^*\psi$ for every vector $\psi \in H \ominus K$, we have $A\varphi = A^*\varphi$ for every vector $\varphi \in \mathcal{N}$. This implies that \mathcal{N} reduces A since \mathcal{N} is invariant by $A^* - \lambda I$, i.e., A^* . In other words, the projection P on \mathcal{N} belongs to $R(A)'$. As we have seen in the section 2, A is irreducible. Thus P must be I or O . But obviously $P \neq I$, and so $P = O$, that is, $\mathcal{N} = \{0\}$. This states that \mathcal{M} is dense in H .

Lemma 5. *A primary operator A with non-hermitian rank 1 does not have a real proper value.*

Proof. Suppose that A has a real proper value λ . Let \mathcal{M} be a proper subspace of A corresponding to λ . Then it is immediately seen that \mathcal{M} is orthogonal to the range of $A^* - \lambda I$. Since A^* has also the non-hermitian rank 1, the range of $A^* - \lambda I$ is dense in H by Lemma 4. Thus \mathcal{M} contains only the zero vector, which is contradiction.

Proposition 4. *Let A be an operator with non-hermitian rank 1 whose spectrum is real. Then A is decomposed by a central projection R in $R(A)$ into the form:*

$$A = A_R \oplus A_{I-R},$$

where A_R has a pure point spectrum and A_{I-R} has a pure continuous spectrum.

Proof. By Theorem 1, Corollary 2, A is decomposed by a central projection P in $R(A)$ into the form:

$$A = A_P \oplus A_{I-P},$$

where A_P is a self-adjoint operator and A_{I-P} is a primary operator with non-hermitian rank 1. Furthermore, as is well known, the self-adjoint operator A_P is decomposed by a projection R in $R(A_P)$ (which is a central projection in $R(A)$) as follows:

$$A_P = A_R \oplus A_{P-R},$$

where A_P has a pure point spectrum and A_{P-R} has a pure continuous spectrum. Since the spectrum of A_{I-P} is real, the preceding lemmas mean that A_{I-P} has a pure continuous spectrum. Consequently, we can easily see that $A_{I-R} = A_{P-R} \oplus A_{I-P}$ has a pure continuous spectrum.

References

- [1] M. S. BRODSKIĬ and M. S. LIVŠIČ, Spectral analysis of non self-adjoint operators and intermediate systems, *Uspehi Mat. Nauk*, **13** (1958), 3—85 (Russian); *Amer. Math. Soc. Transl.*, (2) **13** (1960), 265—346.
- [2] J. DIXMIER, *Les algèbres d'opérateurs dans l'espace hilbertien* (Paris, 1957).
- [3] P. R. HALMOS, Shifts on Hilbert spaces, *J. reine angew. Math.*, **208** (1961), 102—112.
- [4] G. K. KALISCH, Direct proofs of spectral representation theorems, *J. Math. Anal. and Appl.*, **8** (1964), 356—363.
- [5] M. S. LIVŠIČ, On the spectral decomposition of linear non self-adjoint operators, *Mat. Sbornik*, **34** (1956), 145—198 (Russian); *Amer. Math. Soc. Transl.*, (2) **5** (1957), 67—114.
- [6] J. VON NEUMANN, On rings of operators. Reduction theory, *Ann. of Math.*, (2) **50** (1949), 401—485.
- [7] F. RIESZ and B. SZ.-NAGY, *Leçons d'analyse fonctionnelle* (Budapest, 1952).
- [8] N. SUZUKI, Algebraic aspects of non self-adjoint operators, *Proc. Jap. Acad.*; **41** (1965), 706—710.
- [9] N. SUZUKI, Isometries on Hilbert spaces, *Proc. Jap. Acad.*, **39** (1963), 435—438.
- [10] N. SUZUKI, On the type of completely continuous operators, *Proc. Jap. Acad.*, **40** (1964), 683—685.

UNIVERSITY OF MINNESOTA, MINNEAPOLIS (U. S. A.)
AND KANAZAWA UNIVERSITY, KANAZAWA (JAPAN)

(Received February 9, 1966)