Characterization of some classes of measures

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I. Introduction

Let $M(G)$ be the set of (bounded regular Borel) measures μ on a locally compact abelian group G. The following theorem is a well-known characterization of those measures which are absolutely continuous (with respect to the Haar measure of G), given in terms of the translates μ_x (where $\mu_x(E)=\mu(E-x)$).

Theorem A. *Let $\mu\in M(G)$. Then μ is absolutely continuous if and only if* $\|\mu_x-\mu\|\to 0$ *as* $x\to 0$.

For a proof see ([5], p. 230). The norm in the statement of Theorem A is the usual measure norm (=total variation).

In this paper we introduce two other norms for $M(G)$. Using them we give 1) a characterization of the measures in $M(G)$ whose Fourier—Stieltjes transforms vanish at infinity, and 2) a characterization of the continuous (=non-atomic) measures in $M(G)$. In each case the necessary and sufficient condition is similar to that in Theorem A — namely, that as $x\to 0$, μ_x must approach μ in a suitable norm. In case 2) we must restrict ourselves to metrizable groups.

II. Characterization of measures whose Fourier—Stieltjes transforms vanish at infinity

Let Γ denote the character group of G. If $\mu\in M(G)$ let

$$\|\mu\|_\Gamma=\sup_{\gamma\in\Gamma}|\hat{\mu}(\gamma)|$$

where $\hat{\mu}$ is the Fourier—Stieltjes transform of μ — that is $\hat{\mu}(\gamma)=\int_G\overline{(\gamma,t)}\,d\mu(t)$. Since μ is determined by the values of $\hat{\mu}$ on Γ [5], we have $\|\mu\|_\Gamma=0$ if and only if μ is the 0 measure. The other conditions that $\|\cdot\|_\Gamma$ be a norm are readily verified. Here is our characterization.

Theorem B. *Let $\mu\in M(G)$. Then $\hat{\mu}$ vanishes at infinity if and only if* $\|\mu_x-\mu\|_\Gamma\to 0$ *as* $x\to 0$.

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Proof. Suppose first that $\hat{\mu}$ vanishes at infinity. Since

$$\hat{\mu}_x(\gamma) = \int_G \overline{(\gamma, t)}\, d\mu_x(t) = \int_G \overline{(\gamma, t)}\, d\mu(t-x) = \int_G \overline{(\gamma, t+x)}\, d\mu(t) = \overline{(\gamma, x)}\hat{\mu}(\gamma),$$

we have

(1) $$\|\mu_x - \mu\|_\Gamma = \sup_{\gamma \in \Gamma} |\hat{\mu}_x(\gamma) - \hat{u}(\gamma)| = \sup_{\gamma \in \Gamma} |(\gamma, x) - 1| \cdot |\hat{\mu}(\gamma)|.$$

Since, by assumption, $\hat{u}$ vanishes at infinity, given $\varepsilon > 0$ there exists a compact $K \subset \Gamma$ such that $|\hat{\mu}(\gamma)| < \frac{\varepsilon}{2}$ if $\gamma \in \Gamma - K$. Moreover, the set U of all x in G such that $|(\gamma, x) - 1| < \varepsilon / \|\mu\|_\Gamma$ for all $\gamma \in K$ is a neighborhood of 0 in G. If $\gamma \in K$ we thus have

(2) $$|(\gamma, x) - 1| \cdot |\hat{\mu}(\gamma)| < (\varepsilon / \|\mu\|_\Gamma) \cdot \|\mu\|_\Gamma = \varepsilon \quad (x \in U),$$

while if $\gamma \in \Gamma - K$ we have

(3) $$|(\gamma, x) - 1| \cdot |\hat{\mu}(\gamma)| < 2 \cdot \frac{\varepsilon}{2} = \varepsilon \qquad (x \in G).$$

From (1), (2), (3) it follows that $\|\mu_x - \mu\|_\Gamma < \varepsilon$ if $x \in U$. That is $\|\mu_x - \mu\| \to 0$ as $x \to 0$. This proves half the theorem.

To prove the other half we need a lemma (see [1]).

Lemma. *Let U be any neighborhood of 0 in the locally compact abelian group G. Then there exists a compact subset K of Γ (the character group of G) such that for any $\gamma \in \Gamma - K$ there exists $x \in U$ with* $\operatorname{Re}(\gamma, x) \leqq 0$.

Now suppose $\|\mu_x - \mu\|_\Gamma \to 0$ as $x \to 0$. We must show that $\hat{\mu}$ vanishes at infinity. Given $\varepsilon > 0$ choose a neighborhood U of 0 in G such that $\|\mu_x - \mu\|_\Gamma < \varepsilon$ $(x \in U)$. Then

(4) $$|(\gamma, x) - 1| \cdot |\hat{\mu}(\gamma)| < \varepsilon \quad (\gamma \in \Gamma;\ x \in U).$$

For this U choose $K \subset \Gamma$ according to the lemma. Then if $\gamma \in \Gamma - K$ there exists $x \in U$ with $\operatorname{Re}(\gamma, x) \leqq 0$ so that $|(\gamma, x) - 1| > 1$. Using this x in (4) we have $|\hat{\mu}(\gamma)| < \varepsilon$ if $\gamma \in \Gamma - K$. This completes the proof.

III. Characterization of continuous measures

If G is a non-discrete metrizable group then its character group Γ is σ-compact. In Γ there is a sequence of open subsets A_n with compact closure satisfying $A_1 \subset A_2 \subset \ldots$, $\lim_{n \to \infty} m(A_n) = \infty$, and such that

(5) $$M(f) = \lim_{n \to \infty} \frac{1}{m(A_n)} \int_{A_n} f(\gamma)\, d\gamma$$

exists for all almost periodic functions f on Γ and is equal to the mean value of f ([2]). Here $m(A_n)$ means the Haar measure of A_n. Hewitt and Stromberg ([3]) have shown that the limit in (5) will exist for many other functions as well, and, in fact they proved

Lemma. *Let $\mu \in M(G)$. Then μ is a continuous measure if and only if $M(|\hat{\mu}|)=0$.*

We now define our second norm. If $\mu \in M(G)$ where G is metrizable, let

$$\mathcal{N}(\mu) = \sup_n \frac{1}{m(A_n)} \int_{A_n} |\hat{\mu}(\gamma)|\, d\gamma.$$

Our second main result is

Theorem C. *Let G be a metrizable locally compact abelian group and let $\mu \in M(G)$. Then μ is a continuous measure if and only if $\mathcal{N}(\mu_x - \mu) \to 0$ as $x \to 0$.*

Proof. First suppose that μ is continuous. Then by the lemma we have $M(|\hat{\mu}|) = \lim_{n\to\infty} \frac{1}{m(A_n)} \int_{A_n} |\hat{\mu}(\gamma)| d\gamma = 0$. Hence, given $\varepsilon > 0$ there exists N such that

$$\frac{1}{m(A_n)} \int_{A_n} |\hat{\mu}(\gamma)|\, d\gamma < \varepsilon/2 \quad (n \geqq N). \tag{6}$$

Moreover, the set U of x in G such that $|(\gamma, x) - 1| < \varepsilon/\|\mu\|_\Gamma$ for all $\gamma \in \bar{A}_N$ is a neighborhood of 0 in G. Thus, if $n \geqq N$ we have from (6)

$$\frac{1}{m(A_n)} \int_{A_n} |(\gamma, x) - 1| \cdot |\hat{\mu}(\gamma)|\, d\gamma \leqq \frac{1}{m(A_n)} \int_{A_n} 2|\hat{\mu}(\gamma)| < \varepsilon \qquad (x \in G).$$

Also, if $n \leqq N$ then $A_n \subseteqq \bar{A}_N$ and so, if $x \in U$

$$\frac{1}{m(A_n)} \int_{A_n} |(\gamma, x) - 1| \cdot |\hat{\mu}(\gamma)|\, d\gamma \leqq \|\mu\|_\Gamma \cdot \frac{1}{m(A_n)} \int_{A_n} |(\gamma, x) - 1|\, d\gamma \leqq$$

$$\leqq \|\mu\|_\Gamma (\varepsilon/\|\mu\|_\Gamma) \cdot \frac{1}{m(A_n)} \int_{A_n} d\gamma = \varepsilon.$$

Thus, if $x \in U$,

$$\sup_n \frac{1}{m(A_n)} \int_{A_n} |(\gamma, x) - 1| \cdot |\hat{\mu}(\gamma)|\, d\gamma \leqq \varepsilon,$$

or

$$\sup_n \frac{1}{m(A_n)} \int_{A_n} |\hat{\mu}_x(\gamma) - \hat{\mu}(\gamma)|\, d\gamma \leqq \varepsilon,$$

or

$$\mathcal{N}(\mu_x - \mu) \leqq \varepsilon$$

Thus, $\mathcal{N}(\mu_x - \mu) \to 0$ as $x \to 0$. This proves half the theorem.

Now suppose that $\mathcal{N}(\mu_x - \mu) \to 0$ as $x \to 0$. We must prove that μ is continuous. Given $\varepsilon > 0$ choose a symmetric neighborhood U of 0 in G such that

$$\mathcal{N}(\mu_x - \mu) < \varepsilon \qquad (x \in U). \tag{7}$$

Let φ be the function on G defined by

$$\varphi(t)=1/m(U) \qquad (t\in U),$$
$$\varphi(t)=0 \qquad (t\in G-U).$$

(We are now denoting the Haar measure on G, as well as that on Γ, by m. We may clearly assume that $m(U)<\infty$.) Then $\hat{\varphi}$, the Fourier transform of φ, is real-valued (since U is symmetric) and $\hat{\varphi}$ vanishes at infinity by the Riemann—Lebesgue theorem. Thus, for some compact $K\subset\Gamma$, $\hat{\varphi}(\gamma)\leqq\frac{1}{2}$ if $\gamma\in\Gamma-K$. That is,

$$\hat{\varphi}(\gamma)=\int_G\overline{(\gamma,x)}\varphi(x)\,dx=\frac{1}{m(U)}\int_U(\gamma,x)\,dx\leqq\frac{1}{2} \qquad (\gamma\in\Gamma-K).$$

Hence

$$\frac{1}{m(U)}\int_U[1-(\gamma,x)]\,dx\geqq\frac{1}{2} \qquad (\gamma\in\Gamma-K),$$

and so

$$\frac{1}{m(U)}\int_U|(\gamma,x)-1|\,dx\geqq\frac{1}{2} \qquad (\gamma\in\Gamma-K). \tag{8}$$

Now from (7) we have for any n

$$\frac{1}{m(A_n)}\int_{A_n}|(\gamma,x)-1|\cdot|\hat{\mu}(\gamma)|\,d\gamma<\varepsilon \qquad (x\in U).$$

If we multiply by $\frac{1}{m(U)}$, integrate over U, and invert the order of integration we obtain

$$\frac{1}{m(A_n)}\int_{A_n}|\hat{\mu}(\gamma)|\,d\gamma\cdot\frac{1}{m(U)}\int_{A_n}|(\gamma,x)-1|\,dx<\varepsilon.$$

Then certainly

$$\frac{1}{m(A_n)}\int_{A_n-K}|\hat{\mu}(\gamma)|\,d\gamma\cdot\frac{1}{m(U)}\int_U|(\gamma,x)-1|\,dx<\varepsilon.$$

But if $\gamma\in A_n-K$ then, by (8), $\frac{1}{m(U)}\int_U|(\gamma,x)-1|dx\geqq\frac{1}{2}$. Hence

$$\frac{1}{m(A_n)}\int_{A_n-K}|\hat{\mu}(\gamma)|\,d\gamma<2\varepsilon.$$

Moreover, it is certainly true for large n that

$$\frac{1}{m(A_n)}\int_K|\hat{\mu}(\gamma)|\,d\gamma<\varepsilon$$

since $m(A_n)\to\infty$ as $n\to\infty$. Hence

$$\frac{1}{m(A_n)}\int\limits_{A_n}|\hat{\mu}(\gamma)|\,d\gamma<3\varepsilon$$

for large n, which proves that $M(|\hat{\mu}|)=\lim\limits_{n\to\infty}\frac{1}{m(A_n)}\int\limits_{A_n}|\hat{\mu}(\gamma)|d\gamma=0$. By the lemma, μ is continuous and the proof is complete.

IV. Remark on another class of measures

We have made an attempt at another result along the lines of Theorems A, B, C. Let Δ be the maximal ideal space of the measure algebra $M(G)$. That is, Δ is the space of continuous complex-valued homomorphisms h on $M(G)$. If for $\mu\in M(G)$ we define $\|\mu\|_\Delta=\sup\limits_{h\in\Delta}|h(\mu)|$ then, since $h(\mu)=\hat{\mu}(h)$ where $\hat{\mu}$ is the Gelfand transform of μ, $\|\mu\|_\Delta$ is the spectral norm of μ [4: p. 76]. It is now natural to ask: For what μ is it true that $\|\mu_x-\mu\|_\Delta\to 0$ as $x\to 0$?

For each $x\in G$ let σ_x be the unit mass concentrated at x. For $h\in\Delta$ the function χ_h defined by

$$\chi_h(x)=h(\sigma_x)\qquad(x\in G)$$

is easily seen to be a group character of G. However, χ_h need not be continuous. If we could answer positively a certain question about these χ_h we could give a characterization of the kernel of the hull of $L^1(G)$ in $M(G)$ — the set of all $\mu\in M(G)$ such that $\hat{\mu}(h)=0$ for all $h\in\Delta-\hat{G}$. The question whose answer we are unable to establish is this: Are the h for which χ_h is discontinuous dense in $\Delta-\hat{G}$? If the answer to this question is yes then we can easily establish the following:

Let $\mu\in M(G)$. Then μ is in the kernel of the hull of $L^1(G)$ if and only if $\|\mu_x-\mu\|_\Delta\to 0$ as $x\to 0$.

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