On a theorem of L. Rédei about complete oriented graphs

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The very first and perhaps most famous theorem about path problems of oriented graphs is that of L. RÉDEI, which reads as follows: "Every complete oriented finite graph has a hamiltonian path". (See [1], [2], [3], [4].)

However, this fails in the case of infinite oriented graphs, as it is pointed out in the Russian edition of the book of BERGE ([5] p. 123).

This paper gives a sufficient condition for an infinite complete graph to have a (one-sided infinite) hamiltonian path. The notations and definitions are those of C. BERGE. In this paper the term *graph* means always an *oriented graph*. Parallel edges (with coinciding or converse orientation) and selfloops are not permitted. Following BERGE's notation, we identify a graph with the ordered pair (X, Γ) where X is the vertex set of the graph and Γ is that multi-valued mapping of X into X which maps a vertex a into those vertices b for which an edge $\overrightarrow{ab}$ exists. If $A \subseteq X$ then let ΓA consist of those vertices b for which a vertex a exists satisfying $a \in A$ and $b \in \Gamma a$.

A sequence of vertices $(a_0, a_1, \dots, a_\nu)$ such that (a_i, a_{i+1}), $(i=0, 1, \dots, \nu-1)$ is an edge, is a path of length ν.

If such a path exists, we say that a_ν is a *consequent of order* ν of a_0 and we use the notation $a_\nu \in \Gamma^\nu a_0$. (The number ν is not unambigously determined by a_0 and a_ν but it depends on the path $(a_0, a_1, \dots, a_\nu)$. To the given vertices a, b there may be found several numbers ν such that $b \in \Gamma^\nu a$.)

The set $\hat{\Gamma} a$ is defined by

$$\hat{\Gamma} a = \{a\} \cup \Gamma a \cup \Gamma^2 a \cup \dots .$$

The path $(a_0, a_1, \dots, a_\nu)$ is *elementary* if its vertices are pairwise different. An elementary path $(a_0, a_1, \dots, a_k)$ is called *hamiltonian* if it contains all vertices of X.

A path $(a_0, a_1, \dots, a_k)$ where $a_0 = a_k$ and $a_0, a_1, \dots, a_{k-1}$ are pairwise different is a circuit. The definition of hamiltonian circuits is obvious.

A graph is *strongly connected* if for every pair of vertices a, b $(a \neq b)$, b is a consequent of a. A graph is *complete* if for every pair of vertices a, b, $(a \neq b)$ we have either $a \in \Gamma b$ or $b \in \Gamma a$.

If A is a subset of X the graph (A, Γ_A) is a subgraph of (X, Γ) where $\Gamma_A x = \Gamma x \cap A$. The definition of Γ_A^ν and $\hat{\Gamma}_A$ is obvious. For brevity, we shall often use the notation (A, Γ) instead of (A, Γ_A).

A hamiltonian path of the infinite graph (X, Γ) is a sequence of pairwise distinct vertices of (X, Γ), $(x_0, x_1, x_2, \ldots)$, $x_i \in X$ for which

a) $x_i \in \Gamma x_{i-1}$

b) $\{x_0, x_1, \ldots\} = X$.

Let (X, Γ) be arbitrary (finite or infinite) graph, and let M be a subset of X. We denote by $D(M)$ the set of all points x of M for which we have $\hat{\Gamma}_M x = M$.

It is easy to see, that if $D(M) \neq \emptyset$ then $(D(M), \Gamma_{D(M)})$ is strongly connected. Indeed, let (a, b) be an ordered pair of vertices of $D(M)$. Then there exists a path in (M, Γ_M) from a to b say $p = (a, x_1, x_2, \ldots, x_k, b)$. We have $b \in \hat{\Gamma}_M x_i$ $(i = 1, 2, \ldots, k)$. $\hat{\Gamma}_M b = M$, so $\hat{\Gamma}_M x_i = M$ i.e. $x_i \in D(M)$ $(i = 1, 2, \ldots k)$, and the path p passes in $(D(M), \Gamma_{D(M)})$.

Be (X, Γ) an infinite complete graph. We denote by $\mathfrak{M}$ the family of all subsets M of X for which $|X \setminus M| < \infty$.

Theorem. *The following conditions are together sufficient for* (X, Γ) *to have a hamiltonian path:*

C1. *X is countable;*

C2. *For an arbitrary subset $Q \in \mathfrak{M}$ there exist a set $N \subseteq Q$, $N \in \mathfrak{M}$ and a vertex $d \in D(N)$ such that one can find a hamiltonian path in the finite graph $((X \setminus N) \cup \{d\}, \Gamma)$ which terminates in d;*

C3. *There exists a subset $P \subset X$, $0 < |P| < \infty$ with the following properties:*

a) *The set of the points x of X for which $P \cap \hat{\Gamma} x \neq \emptyset$ is finite;*

b) *For any subset $M \in \mathfrak{M}$ we have $|D(M \setminus P)| < \infty$.*

Before proving the theorem we make some remarks about the conditions.

I. *The conditions* C1 *and* C2 *are also necessary.*

This is trivial for C1.

Now, assume that there exists a hamiltonian path in (X, Γ), say $(x_0, x_1, \ldots)$. Let $Q \in \mathfrak{M}$ i.e. $|X \setminus Q| < \infty$; let $X \setminus Q$ consist of the vertices $x_{i_1}, x_{i_2}, \ldots, x_{i_n}$. Denote by k the maximum of $i_1, i_2, \ldots, i_n$; the section $(x_0, x_1, \ldots, x_k)$ of the infinite hamiltonian path $(x_0, x_1, \ldots)$ contains all the points $x_{i_1}, x_{i_2}, \ldots, x_{i_n}$. Choose $N = X \setminus \{x_0, x_1, \ldots \ldots, x_k\}$; $d = x_{k+1} \in D(N)$. It is easy to see that condition C2 is fulfilled.

A counterexample will show that condition C3 is not necessary. Let us define a graph (X, Γ) as follows:

$$X = \{1, 2, 3, 4, \ldots\}$$

and for any two vertices i, j $(i, j \in X;\ i < j)$,

$$j \in \Gamma_i, \quad \text{if } j - i = 1,$$

$$i \in \Gamma_j, \quad \text{if } j - i > 1.$$

(X, Γ) has a hamiltonian path $(1, 2, \ldots)$ but, as it is easy to see, condition C3 does not hold.

II. *$Q \in \mathfrak{M}$, $Q' \supset Q$ imply $Q' \in \mathfrak{M}$.* (Obvious.)

Lemma. *Consider an infinite complete graph* (X, Γ) *which satisfies the conditions* C1 *and* C2. *Then* $\hat{\Gamma} a \in \mathfrak{M}$ *for an arbitrary vertex* $a \in X$.

Proof. Let $X \setminus \{a\} = Q$. Evidently $Q \in \mathfrak{M}$. Thus, making use of C2, there exist a subset N, $N \in \mathfrak{M}$, $d \in D(N)$ and a hamiltonian path of $((X \setminus N) \cup \{d\}, \Gamma)$ which passes through a and terminates in d. So $\hat{\Gamma} a \supseteqq \hat{\Gamma} d \supseteqq N$ and by Remark II we have $\hat{\Gamma} a \in \mathfrak{M}$.

Proof of the theorem

1) Let (X, Γ) be an infinite complete graph for which the conditions C1, C2, C3 hold. Denote by P^* the set of all points x of X for which $P \cap \hat{\Gamma} x \neq \emptyset$. Then $P \subseteqq P^*$, and, by C3 a), we have $|P^*| < \infty$.

2) Since $X \setminus P^* \in \mathfrak{M}$, we can use condition C2. There exist N, d such that $N \subseteqq X \setminus P^*$, $N \in \mathfrak{M}$, $d \in D(N)$ and one can find a hamiltonian path π in $((X \setminus N) \cup \{d\}, \Gamma)$ which terminates in d.

3) $p \in P^*$, $x \in X \setminus P^*$ *implies* $x \in \Gamma p$.

Proof. (X, Γ) is complete, hence for the pair of vertices p, x we have either $p \in \Gamma x$ or $x \in \Gamma p$. But $p \in \Gamma x$ implies $x \in P^*$, which is a contradiction.

4) Take $Q = X \setminus (N \cup P^*)$. We have $|Q| < \infty$ because $N \in \mathfrak{M}$. According to what has been said above, the hamiltonian path π of $((X \setminus N) \cup \{d\}, \Gamma)$ consists of two sections, the first of which goes through P^* and the second through $Q \cup \{d\}$ i.e.:

$$\pi = (p_1, p_2, \ldots, p_n, q_1, q_2, \ldots, q_m, d); \; p_i \in P^*, \; q_i \in Q, \; d \in D(N).$$

5) Let r be an arbitrary point of $N \setminus D(N)$. Then $r \in \Gamma n$ for any point $n \in D(N)$ because, on account of the definition of $D(N)$, $r \in N$, $n \in D(N)$, $n \in \Gamma r$ imply $\hat{\Gamma}_N r = N$, i.e. $r \in D(N)$.

Denote by R the set of all points r of N for which $\hat{\Gamma} r \cap Q \neq \emptyset$.

6) $|R| < \infty$.

Proof. $Q = \emptyset$ implies $R = \emptyset$ i.e. $|R| = 0 < \infty$ and the statement is true. So it may be supposed that $0 < |Q| = m$. We have $X \setminus P^* \in \mathfrak{M}$ and $(X \setminus P^*) \cap P = \emptyset$ thus making use of C3 b) we obtain $|D(X \setminus P^*)| < \infty$. On the other hand, $D(X \setminus P^*) \neq \emptyset$ because $q_1 \in D(X \setminus P^*)$ (q_1 is the first point of the hamiltonian path π in Q; see 4). Indeed, $q_i \in \hat{\Gamma} q_1$ for any point $q_i \in Q$ and for an arbitrary point $n \in N$, $n \in \hat{\Gamma} q_1$ because of $d \in \hat{\Gamma} q_1$, $d \in D(N)$.

Denote $D(X \setminus P^*)$ by D'. From the definition of D' it follows that $q_j \in D'$ and $i < j$ imply $q_i \in D'$, so D' contains a whole section $(q_1, q_2, \ldots, q_{k_1})$ of π. Suppose $k_1 < m$ i.e. $Q \setminus D' \neq \emptyset$. Let $D'' = D(X \setminus (P^* \cup D'))$. Then $|D''| < \infty$ because of C3 b), and $D'' \neq \emptyset$ because $q_{k_1+1} \in D''$. Assuming that $Q \setminus (D' \cup D'') \neq \emptyset$ we can continue this procedure and so we get the subsets $D', D'', \ldots, D^{(i)}, \ldots$ where

$$D^{(i+1)} = D(X \setminus (P^* \cup D' \cup D'' \cup \ldots \cup D^{(i)})).$$

Since $|Q|<\infty$, this procedure comes to an end, at the kth step, say:

$$Q \subseteqq \bigcup_{1}^{k} D^{(i)}; \ |D^{(i)}|<\infty, \ D^{(i)} \cap Q \neq \emptyset \quad (i=1, 2, \ldots, k).$$

In order to prove our statement it is sufficient to show, that $R \subseteqq \bigcup_{1}^{k} D^{(i)}$. Let $r \in R$ be an arbitrary point of R. By the definition of the subset R, we have $\hat{\Gamma} r \cap Q \neq \emptyset$. There exists an index i such that $D^{(i)} \cap \hat{\Gamma} r \neq \emptyset$; this implies $r \in D^{(i)}$.

7) Denote $Q \cup D(N) \cup R$ by D_1. We have $0<|D_1|<\infty$. ($D_1 \neq \emptyset$ because $d \in D_1$.)

$$d' \in D_1, p \in P^*, x \in X \setminus (P^* \cup D_1) \quad \textit{imply} \quad d' \in \Gamma p; \quad x \in \Gamma p; \quad x \in \Gamma d'.$$

Proof. In 3), we proved $d' \in \Gamma p, x \in \Gamma p$. Thus, we have only to show that $x \in \Gamma d'$.

We have $D_1 = Q \cup R \cup D(N)$. If $d' \in Q$ or $d' \in R$ then $d' \in \Gamma x$ would imply $x \in R$ which is impossible. The case $d' \in D(N)$ was examined in 5).

8) In the foregoing we have used only the following three properties of P^*:

a) $|P^*|<\infty$

b) $P \subseteqq P^*$

c) $p \in P^*, x \in X \setminus P^*$ imply $x \in \Gamma p$.

According to 6) and 7) the subset $P^* \cup D_1$ also possesses the properties a), b) and c). Taking $P^{**} = P^* \cup D_1$, the above construction (from 2) to 7)) can be applied to P^{**} instead of P^* and we get the subset D_2. The iteration of the procedure leads to the sequence of subsets $D_0 = P^*, D_1, D_2, \ldots, D_i, \ldots$, where D_i is derived from the construction applied to

$$P^{(i-1)} = P^* \cup D_1 \cup D_2 \cup \ldots \cup D_{i-1}.$$

Obviously the subsets $D_0, D_1, \ldots$ are pairwise disjoint. From the construction it follows that $a \in D_k, b \in D_l, k<l$ imply $b \in \Gamma a$ ($k=0, 1, 2, \ldots$).

9) All the graphs $(D_0, \Gamma), (D_1, \Gamma), (D_2, \Gamma), \ldots$ are non-empty *finite* complete ones, and so, by the theorem of RÉDEI, they have hamiltonian paths

$$p_0 = (x_0, x_1, \ldots, x_{n_0}),$$

$$p_1 = (x_{n_0+1}, \ldots, x_{n_1}),$$

$$p_2 = (x_{n_1+1}, \ldots, x_{n_2}),$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

By linking the paths $p_0, p_1, p_2, \ldots$, we get a hamiltonian path $(x_0, x_1, \ldots, x_{n_0}, x_{n_0+1}, \ldots)$ in the graph $\left(\bigcup_{0}^{\infty} D_i, \Gamma\right)$.

10) To prove our theorem we have only to show that $\bigcup_{0}^{\infty} D_i = X$. From the construction of the subsets $D_0, D_1, \ldots$ it follows that $z \in X, \hat{\Gamma} z \cap D_k \neq \emptyset$ imply

$z \in \bigcup_0^k D_i$. Now suppose $z \in X \setminus \bigcup_0^\infty D_i$. According to the last remark, this means that $\hat{\Gamma} z \cap \left(\bigcup_0^k D_i \right) = \emptyset$ $(k=0, 1, 2, \ldots)$ which contradicts our Lemma.

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