Proximity structures in Boolean Algebras

By K. N. MEENAKSHI in Madras (India)

This paper is an attempt to study the notion of proximity structure (cf. I. S. GÁL) in the class of classical topological Boolean Algebras (cf. G. NÖBELING). A topological proximity Boolean algebra is defined in a manner similar to that of a proximity space. In the first section we prove that a classical topological Boolean algebra is completely regular if and only if it is a topological proximity algebra. Then we proceed to show that there exists a coarsest uniform structure compatible with a proximity structure of a classical topological Boolean algebra. Using this we prove that a classical topological Boolean algebra is completely regular if and only if it is homeomorphic to an invariant subalgebra of a compact regular space.

In section 2 we study quotient algebras of the form $S(X)/I$ where X is a completely regular space of topological weight m and I is an m-additive ideal of $S(X)$ (cf. SIKORSKI). We introduce the notion of quotient uniformity and quotient proximity in $S(X)/I$ and discuss the permutability of the two operations of taking quotient proximity and quotient uniformity.

A similar concept is studied by A. S. ŠVARC (cf. *Math. Reviews*, **19** (1958), p. 436) in his paper on "Proximity spaces and lattices". In Section 3 the connection between the concept of δ-lattices of ŠVARC and our notion of proximity Boolean algebras is explained.

The author is grateful to DR. V. S. KRISHNAN for the provision of both advice and encouragement in the work presented here and to the Council of Scientific and Industrial Research, Government of India, for a Senior Research Fellowship.

1. Definition 1. A *proximity relation* $\bar\delta$ for a Boolean algebra $\mathfrak{B}$ is a binary relation which satisfies the following axioms:

P. 1: $A\bar\delta 0$ for every element $A\in\mathfrak{B}$ where 0 is the zero element of $\mathfrak{B}$.

P. 2: $A_1\bar\delta A_2 \Leftrightarrow A_2\bar\delta A_1$ for any two elements A_1, A_2 in $\mathfrak{B}$.

P. 3: $A_1\wedge A_2>0 \Rightarrow A_1\delta A_2$ (i.e. not $A_1\bar\delta A_2$), where $A_1\wedge A_2$ denotes the Boolean product of A_1, A_2 in $\mathfrak{B}$.

P. 4: $A\bar\delta(B+C)\Leftrightarrow A\bar\delta B$ and $A\bar\delta C$ where $B+C$ denotes the Boolean sum of B and C in $\mathfrak{B}$.

P. 5: $A_1\bar\delta A_2\Rightarrow$ there exist elements B_1, B_2 in $\mathfrak{B}$ such that $A_i\bar\delta cB_i$ for $i=1,2$ and $B_1\bar\delta B_2$ where cB denotes the complement of B in $\mathfrak{B}$.

Definition 2. Let $(\mathfrak{B}, \tau)$ be a classical topological Boolean algebra. Then a proximity relation $\bar\delta$ defined in $\mathfrak{B}$ is said to be *compatible with the topology* τ on $\mathfrak{B}$ if for each element A in $\mathfrak{B}$, $\Sigma(U|U\bar\delta cA)$ exists in $\mathfrak{B}$ and $\mathrm{int}A=\Sigma(U|U\bar\delta cA)$.

Note. The proximity relation $\bar{\delta}$ is compatible with the topology τ of $(\mathfrak{B}, \tau)$ if and only if the open elements of $(\mathfrak{B}, \tau)$ are precisely the elements (A) of the form $A = \Sigma(U | U\bar{\delta}cA)$.

$(\mathfrak{B}, \tau)$ is called a topological proximity algebra if there exists a proximity relation on $\mathfrak{B}$ compatible with its topology τ.

Definition 3. (See NÖBELING, p. 91.) Let $(\mathfrak{B}, \tau)$ be a classical topological Boolean algebra. If for each dyadic rational $t = m/2^n$ ($m = 0, 1, 2, \ldots, 2^n$; $n = 1, 2, \ldots$) H_t is an open element of $\mathfrak{B}$ such that $\bar{H}_{t'} < H_{t''}$ for $t' < t''$, then we call the set $\{H_t\}$ of open elements H_t from $\mathfrak{B}$ a *binary scale.*

Definition 4. A classical topological Boolean algebra $\mathfrak{B}$ is *completely regular* if for any two non-zero elements A_0 and F_1 of $\mathfrak{B}$ such that (1) $A_0 \wedge F_1 = 0$ and (2) F_1 is closed, there exists a binary scale (H_t) such that $A_0 \wedge H_0 > 0$ and $F_1 \wedge H_1 = 0$.

Proposition 1. *Let $(\mathfrak{B}, \tau, \bar{\delta})$ be a classical topological proximity Boolean algebra. Then*

(1) $A_i \leqq B_i$ $(i = 1, 2)$ *and* $B_1\bar{\delta}B_2 \Rightarrow A_1\bar{\delta}A_2$;

(2) $A_1\bar{\delta}A_2 \Rightarrow \bar{A}_1\bar{\delta}\bar{A}_2$ *where $\bar{A}$ denotes the closure of A in* $(\mathfrak{B}, \tau)$;

(3) $A_1\bar{\delta}cA_3 \Rightarrow$ *there exists an element A_2 in $\mathfrak{B}$ such that* $A_1\bar{\delta}cA_2$ *and* $A_2\bar{\delta}cA_3$;

(4) $A_1\bar{\delta}A_2 \Rightarrow$ *there exist open elements G_1, G_2 in $\mathfrak{B}$ with* $A_i\bar{\delta}cG_i$, $i = 1, 2$ *and* $G_1\bar{\delta}G_2$; *and*

(5) $(\mathfrak{B}, \tau)$ *is completely regular.*

Proof. (1), (2), (3) and (4) follow by simple arguments using P_1, P_2, P_3, P_4 and P_5. For example we shall prove (2). To prove (2) it suffices to show that $A_1\bar{\delta}A_2 \Rightarrow \bar{A}_1\bar{\delta}A_2$. $A_1\bar{\delta}A_2 \Rightarrow$ there exist elements B_1, B_2 with $A_i\bar{\delta}cB_i$, $i = 1, 2$, and $B_1\bar{\delta}B_2 \Rightarrow$ there exist open elements H_1, H_2 such that $A_i < H_i$ and $H_1 \wedge H_2 = 0 \Rightarrow$ $\Rightarrow \bar{A}_1 \wedge A_2 = \bar{A}_2 \wedge A_1 = 0$. Hence $A_1\bar{\delta}A_2 \Rightarrow$ there exist B_1, B_2 with $A_i\bar{\delta}cB_i$ and $B_1\bar{\delta}B_2 \Rightarrow$ there exist B_1, B_2 with $A_i\bar{\delta}cB_i$ and $\bar{B}_1 \wedge B_2 = B_1 \wedge \bar{B}_2 = 0 \Rightarrow$ there exist B_1, B_2 with $A_i\bar{\delta}cB_i$, $\bar{A}_1 < \bar{B}_1 < cB_2\bar{\delta}A_2$, $\bar{A}_2 < \bar{B}_2 < cB_1\bar{\delta}A_1 \Rightarrow \bar{A}_1\bar{\delta}A_2$ and $\bar{A}_2\bar{\delta}A_1$.

Proof of (5). Let $A_0 > 0$ and $F_1 > 0$ be any two elements such that (i) F_1 is closed and (ii) $A_0 \wedge F_1 = 0$. Set $cF_1 = H_1$. Then $A_0 < H_1 = \Sigma(U : U\bar{\delta}F_1)$. Therefore there exists an open element H_0 such that

$$A_0 \wedge H_0 > 0 \quad \text{and} \quad H_0\bar{\delta}F_1.$$

Now $H_0\bar{\delta}F_1 \Rightarrow \bar{H}_0\bar{\delta}F_1 \Rightarrow$ there exist open elements G_1, G_2 with $\bar{H}_0 < G_1$, $F_1 < G_2$, $\bar{H}_0\bar{\delta}cG_1$, and $G_1\bar{\delta}G_2 \Rightarrow$ there exists an open element G_1 such that $\bar{H}_0 < G_1 < \bar{G}_1 < H_1$, $\bar{G}_1\bar{\delta}F_1$, and $\bar{H}_0\bar{\delta}cG_1$.

Setting $G_1 = H_{\frac{1}{2}}$ we have

(1) $$\bar{H}_0 < H_{\frac{1}{2}} < \bar{H}_{\frac{1}{2}} < H_1,$$

(2) $$\bar{H}_0\bar{\delta}cH_{\frac{1}{2}}, \quad \text{and} \quad (3) \quad \bar{H}_{\frac{1}{2}}\bar{\delta}F_1.$$

Replacing A_0 and F_1 by H_0 and $cH_{\frac{1}{2}}$ we can construct an open element $H_{\frac{1}{4}}$ such that (1) $\bar{H}_0 < H_{\frac{1}{4}} < \bar{H}_{\frac{1}{4}} < H_{\frac{1}{2}}$ and (2) $\bar{H}_0\bar{\delta}cH_{\frac{1}{4}}$.

We can construct H_t for every dyadic fraction $t=\frac{m}{2^n}$ by induction on n as follows.

Having defined H_t for $t=\frac{m}{2^n}$ $(m=0, 1, 2, \ldots, 2^n)$, we define $H_{t'}$ for $t'=\frac{m}{2^{n+1}}$. If m is even, $\frac{m}{2^{n+1}}=\frac{m'}{2^n}$ and $H_{\frac{m'}{2^n}}$ is defined. Let m be odd. Then $H_{\frac{m-1}{2^{n+1}}}$ and $H_{\frac{m+1}{2^{n+1}}}$ are already defined satisfying

(i) $\overline{H_{\frac{m-1}{2^{n+1}}}} < H_{\frac{m+1}{2^{n+1}}}$ and (ii) $\overline{H_{\frac{m-1}{2^{n+1}}}}\,\bar{\delta}c\left(H_{\frac{m+1}{2^{n+1}}}\right)$.

Now as before we can construct an open element G such that

(i) $$\overline{H_{\frac{m-1}{2^{n+1}}}} < G < \bar{G} < H_{\frac{m+1}{2^{n+1}}},$$

(ii) $$\overline{H_{\frac{m-1}{2^{n+1}}}}\,\bar{\delta}cG,$$

(iii) $$\bar{G}\bar{\delta}\, c\left(H_{\frac{m+1}{2^{n+1}}}\right).$$

Set $G=H_{\frac{m}{2^{n+1}}}$. Thus for each dyadic rational t we can define an open element H_t in $\mathfrak{B}$ such that

(i) $\bar{H}_0 < H_t < \bar{H}_t < H_{t'} < \bar{H}_{t'} < H_1$ for every $t<t'$;

(ii) $A_0 \wedge H_0 > 0$ and (iii) $F_1 \wedge H_1 = 0$.

Hence $\mathfrak{B}$ is completely regular.

Proposition 2. *Every classical topological completely regular Boolean algebra* $(\mathfrak{B}, \tau)$ *is a topological proximity Boolean algebra.*

Proof. $(\mathfrak{B}, \tau)$ is uniformisable (cf. NÖBELING p. 195). Let Ω be the index set corresponding to a uniformity $\mathfrak{A}$ of $\mathfrak{B}$. Then for each element $A\in\mathfrak{B}$ and to each index $\alpha\in\Omega$ an element $V_\alpha(A)$ in $\mathfrak{B}$ is uniquely associated satisfying the uniformity axioms U_1 to U_6 (cf. NÖBELING, p. 169).

Define for any two elements A, B in $\mathfrak{B}$, $A\bar{\delta}B \Leftrightarrow$ there exists an $\alpha\in\Omega$ such that $V_\alpha(A)\wedge V_\alpha(B)=0$.

We shall show that $\bar{\delta}$ is a proximity relation on $\mathfrak{B}$ compatible with the topology τ. Axioms P_1, P_2, P_3 and P_4 follow easily from axioms U_1-U_6. Suppose $A_1\bar{\delta}A_2$. Then there exists an $\alpha\in\Omega$ such that $V_\alpha(A_1)\wedge V_\alpha(A_2)=0$. Given $\alpha\in\Omega$ there exists a $\beta\in\Omega$ such that $V_\beta(V_\beta(V_\beta(A)))\leqq V_\alpha(A)$ for each $A\in\mathfrak{B}$. Let $B_i=V_\beta(V_\beta(A_i))$ for $i=1, 2$. Then $V_\beta(B_1)\wedge V_\beta(B_2)\leqq(V_\alpha(A_1))\wedge V_\alpha(A_2)=0$. This implies $B_1\bar{\delta}B_2$. Again $B_i\wedge cB_i=0$ $\Rightarrow V_\beta(V_\beta(A_i))\wedge cB_i=0\Rightarrow V_\beta(A_i)\wedge V_\beta(cB_i)=0\Rightarrow A_i\bar{\delta}cB_i$. Thus axiom P_5 is also satisfied.

Now we shall show that an element $A\in(\mathfrak{B}, \tau)$ is open if and only if $A=\Sigma(U|U\bar{\delta}cA)$.

Suppose A is open. Then cA is closed and therefore $cA=\bigwedge_{\alpha\in\Omega} V_\alpha(cA)$. This implies $A=\sum_{d\in\Omega} c(V_d(cA))$.

$V_\alpha(cA)\wedge c(V_\alpha(cA))=0$ implies $cA\bar{\delta}c(V_\alpha(cA))$. This in turn implies

$$A=\sum_{\alpha\in\Omega} c(V_\alpha(cA))\leqq \Sigma(U|U\bar{\delta}cA).$$

Conversely suppose $A=\Sigma(U|U\bar{\delta}cA)$. Then $A=\Sigma(U|U\wedge V_\alpha(cA)=0$ for some $\alpha\in\Omega)$. This implies $A\wedge(\bigwedge_{\alpha\in\Omega} V_\alpha(cA))=0$ and therefore $A\wedge(\overline{cA})=0$. Therefore $\overline{cA}=cA$. Hence cA is closed and A is open.

Propositions 1 and 2 lead to the following result:

Proposition 3. *A topological Boolean algebra* $(\mathfrak{B},\tau)$ *is completely regular if and only if there exists a proximity relation* $\bar{\delta}$ *compatible with the topology of* $\mathfrak{B}$.

Let $(\mathfrak{B},\tau,\bar{\delta})$ be a topological proximity algebra. We call a finite covering $(A_1, A_2, \ldots A_n)$ of $\mathfrak{B}$ a $\bar{\delta}$ covering or a proximity covering if there exists another covering $(B_1, B_2, \ldots, B_n)$ of $\mathfrak{B}$ such that $B_i\bar{\delta}cA_i$ or $B_i\ll A_i$ for $i=1,2,\ldots,n$. Here by a covering we mean a set of elements whose Boolean sum is the unit element of $\mathfrak{B}$.

The following properties of $\bar{\delta}$-coverings are easily proved. (i) If $\mathfrak{A}_\alpha$ and $\mathfrak{A}_\beta$ are two $\bar{\delta}$-coverings of a proximity topological algebra $(\mathfrak{B},\tau,\bar{\delta})$ then the covering $\mathfrak{A}_\alpha\wedge\mathfrak{A}_\beta=(A_1\wedge A_2|A_1\in\mathfrak{A}_\alpha, A_2\in\mathfrak{A}_\beta)$ is also a $\bar{\delta}$-covering, and (ii) if $\mathfrak{A}=(A_1, A_2, \ldots, A_n)$ is a $\bar{\delta}$-covering of $(\mathfrak{B},\tau,\bar{\delta})$ then $c(\sum_{i\in I} A_i)\ll\sum_{i\notin I} A_i$.

The concept of a $\bar{\delta}$-covering of a topological proximity algebra $(\mathfrak{B},\tau,\bar{\delta})$ is the extension of the notion of proximity coverings defined in ([1]). We use this concept in the proof of the following theorem:

Proposition 4. *Let* $(\mathfrak{B},\tau)$ *be a classical topological Boolean algebra and let* $\bar{\delta}$ *be a proximity relation compatible with the topology of* $\mathfrak{B}$. *Then there exists a coarsest uniform structure* $\mathfrak{A}$ *on* $\mathfrak{B}$ *compatible with* $\bar{\delta}$.

The proof of this result runs almost parallel to the proof of the corresponding theorem on proximity spaces (cf. [1]). So we shall give only the important steps in the proof.

Let $\mathfrak{A}=(\mathbf{U}_\alpha:\alpha\in\Omega)$ be the family of all finite $\bar{\delta}$-coverings of $(\mathfrak{B},\tau,\bar{\delta})$. For each element $A\in\mathfrak{B}$ and for each $\beta\in\Omega$ define $U_\beta(A)=\Sigma(A^\beta:A^\beta\in\mathbf{U}_\beta$ with $A^\beta\wedge A\neq 0)$. This defines the coarsest uniform structure on $\mathfrak{B}$ compatible with $\bar{\delta}$ i.e. for any two elements A_1, A_2 of $\mathfrak{B}$, $A_1\bar{\delta}A_2$ if and only if $U_\alpha(A_1)\wedge U_\alpha(A_2)=0$ for some $\alpha\in\Omega$. Axioms U_1, U_2, U_4 and U_5 are easily seen to hold good.

To prove U_3 suppose $\mathbf{U}_\alpha=(A_i;\ i=1,2,\ldots n)\in\mathfrak{A}$. Let I be a subset of $(1,2,\ldots n)$ and $\mathbf{U}_{\alpha_I}$ be the covering $\mathbf{U}_{\alpha_I}=(A_I, A_{I'})$ where I' is the set complement of I in $(1,2,\ldots n)$ and $A_I=\bigvee_{i\in I} A_i$. Then we have (1) U_{α_I} is a $\bar{\delta}$-covering for each I and $U_\alpha(A)=$ $=\bigwedge_I U_{\alpha_I}(A)$ and (2) for each $\mathbf{U}_{\alpha_I}$ there exists another $\bar{\delta}$-covering $\mathbf{U}_{\beta_I}$ such that

$U_{\beta_I}(U_{\beta_I}(A)) \leqq U_{\alpha_I}(A)$. To prove (1) suppose $U_\alpha(A) \ngeqq \wedge U_{\alpha_I}(A)$. Then there exists an element $B \in \mathfrak{B}$ with $B \neq 0$, $B \wedge U_\alpha(A) = 0$ and $B < \bigwedge_I U_{\alpha_I}(A)$. Let I be the set of all indices in $i = 1, 2, \ldots, n$ such that $A \wedge A_i \neq 0$. Then $U_\alpha(A) = A_I = U_{\alpha_I}(A)$ and $B \wedge U_{\alpha_I}(A) = B \wedge U_\alpha(A) = 0$ and this contradicts $B < \bigwedge_I U_{\alpha_I}(A)$. Hence $U_\alpha(A) = \bigwedge_I U_{\alpha_I}(A)$.

Using property (ii) of $\bar{\delta}$-coverings and result (3) of Proposition 1, we can construct for each subset I of $(1, 2, \ldots, n)$ elements K_{Ii} $(i = 1, 2, 3, 4)$ such that $cA_{I'} \ll K_{I1} \ll \ll K_{I2} \ll K_{I3} \ll K_{I4} \ll A_I$. Define $B_{I_1} = K_{I2}$, $B_{I_2} = K_{I4} \wedge cK_{I1}$, and $B_{I_3} = cK_{I3}$. Then $B_{I_1} \wedge B_{I_3} = 0$, $B_{I_1} + B_{I_2} \ll A_I$ and $B_{I_2} + B_{I_3} \leqq A_{I'}$. Now $K_{I1} \ll K_{I2} \ll K_{I3} \ll K_{I4} \Rightarrow$ there exist elements L_{I_1}, L_{I_2} such that $K_{I1} \ll L_{I_1} \ll K_{I2} \ll K_{I3} \ll L_{I_2} \ll K_{I4}$. Set $M_{I_1} = L_{I_1}$, $M_{I_2} = L_{I_2} \wedge cL_{I_1}$ and $M_{I_3} = cL_{I_2}$. Then $(M_{I_1}, M_{I_2}, M_{I_3})$ is a covering of $\mathfrak{B}$ with $M_{I_i} \ll B_{I_i}$ for $i = 1, 2, 3$. This shows that $\mathbf{U}_{\beta_I} = (B_{I_i} : i = 1, 2, 3)$ is a $\bar{\delta}$-covering of $\mathfrak{B}$. Clearly for any element $A \in \mathfrak{B}$, $U_{\beta_I}(U_{\beta_I}(A)) \leqq A_I$, or $A_{I'}$ or $A_I + A_{I'}$ and in all these cases $U_{\beta_I}(U_{\beta_I}(A)) \leqq U_{\alpha_I}(A)$. This completes the proof of (2).

Let U_β be the intersection of all the coverings U_{β_I}. Then U_β is again a $\bar{\delta}$-covering and $U_\beta(A) \leqq \bigwedge_I U_{\beta_I}(A)$. Now

$$U_\beta(U_\beta(A)) \leqq \bigwedge_I (U_{\beta_I}(\wedge U_{\beta_I}(A))) \leqq \bigwedge_I U_{\beta_I}(U_{\beta_I}(A)) \leqq \bigwedge_I U_{\alpha_I}(A) \leqq U_\alpha(A).$$

Thus we have shown that axiom U_3 is satisfied.

Before proving U_6 we shall show that the uniform structure is compatible with $\bar{\delta}$. Suppose $A_1 \bar{\delta} A_2$. Then (cA_1, cA_2) is a $\bar{\delta}$-covering $= \mathbf{U}_\alpha \in \mathfrak{A}$ and $U_\alpha(A_1) = cA_2$ and therefore $U_\alpha(A_1) \wedge A_2 = 0$. Given $U_\alpha \in \mathfrak{A}$ there exists a $U_\beta \in \mathfrak{A}$ such that $U_\beta(U_\beta(A_1)) < U_\alpha(A_1)$ and for this β, $U_\alpha(A_1) \wedge A_2 = 0 \Rightarrow U_\beta(A_1) \wedge U_\beta(A_2) = 0$. Thus $A_1 \bar{\delta} A_2 \Rightarrow U_\alpha(A_1) \wedge U_\alpha(A_2) = 0$ for some $\alpha \in \Omega$. Conversely suppose $U_\alpha(A_1) \wedge U_\alpha(A_2) = 0$. Let U_α be the covering $(B_i : i = 1, 2, \ldots, n)$. Then there exists a subset I of $(1, 2, \ldots n)$ such that $A_1 \leqq cB_I$ and $A_2 \leqq cB_{I'}$. By property (ii) of $\bar{\delta}$-coverings $cB_I \bar{\delta} cB_{I'}$ and therefore $A_1 \bar{\delta} A_2$. Hence $A_1 \bar{\delta} A_2 \Leftrightarrow$ there exists an $\alpha \in \Omega$ such that $U_\alpha(A_1) \wedge U_\alpha(A_2) = 0$.

To prove U_6 suppose $A \in \mathfrak{B}$. Since $\bar{\delta}$ is compatible with τ,

$$c\bar{A} = \Sigma(U | U \bar{\delta} \bar{A})$$

and therefore

$$\bar{A} = \wedge(cU | U \bar{\delta} A) = \wedge(cU | U \wedge U_\alpha(A) = 0 \quad \text{for some } \alpha \in \Sigma) \geqq \wedge U_\alpha(A).$$

This proves $\bar{A} = \wedge U_\alpha(A)$.

To complete the proof of Proposition 4 we have only to show that given any uniform structure $\mathfrak{V}$ on $\mathfrak{B}$ compatible with $\bar{\delta}$ and any $\alpha \in \Omega$ there exists a $V \in \mathfrak{V}$ such that $V(A) \leqq U_\alpha(A)$ for all $A \in \mathfrak{B}$. Suppose $\mathbf{U}_\alpha = (A_i : i = 1, 2, \ldots n)$. Since U_α is a $\bar{\delta}$-covering there exists another covering $(B_i$: $i = 1, 2, \ldots n)$ such that $B_i \ll A_i$. Now $B_i \ll A_i \Rightarrow B_i \bar{\delta} cA_i \Rightarrow V_i(B_i) \wedge cA_i = 0$ for some $V_i \in \mathfrak{V}$. Given $(V_i$: $i = 1, 2, \ldots n) \in \mathfrak{V}$, there exists $V \in \mathfrak{V}$ such that $V(A) \leqq V_i(A)$ for $i = 1, 2, \ldots n$ and for all $A \in \mathfrak{B}$. We shall show that $V(A) \leqq U_\alpha(A)$ for all $A \in \mathfrak{B}$.

$$V(A) = \Sigma(V(A \wedge B_i) | A \wedge B_i \neq 0) \leqq \Sigma(V(B_i) | A \wedge B_i \neq 0) \leqq \Sigma(A_i | A \wedge B_i \neq 0) \leqq U_\alpha(A).$$

Now we proceed to study the problem of imbedding a topological proximity algebra in a compact regular space. We call a subalgebra $\mathfrak{B}$ of a compact regular space $S(X)$ an invariant subalgebra provided for each element $A \in \mathfrak{B}$ $U_\alpha(A) \in \mathfrak{B}$ for each $\mathbf{U}_\alpha \in \mathfrak{A}$ and $\wedge U_\alpha(A) \in \mathfrak{B}$, where $\mathfrak{A}$ is the unique uniform structure on $S(X)$.

Proposition 5. *Let $S(X)$ be a compact regular space. Then every invariant subalgebra of $S(X)$ is completely regular and therefore a proximity Boolean algebra.*

The proof is evident.

Proposition 6. *Every topological proximity algebra $(\mathfrak{B}, \tau, \bar{\delta})$ is $\bar{\delta}$-isomorphic to an invariant subalgebra of a compact regular space.*

Proof. Let $(\mathfrak{B}, \Omega)$ be the coarsest uniform structure on $\mathfrak{B}$ compatible with $\bar{\delta}$ constructed in the proof of Proposition 4.

Let M be the set of all ultrafilters $(\mathbf{F})$ in $\mathfrak{B}$. For each $A \in \mathfrak{B}$ let φA be the set of all ultrafilters in M to which A belongs. Then $\varphi A \in S(M)$. For each $\mathbf{F} \in M$ define $U_\alpha(\mathbf{F}) = \bigcap [\varphi(\mathbf{U}_\alpha(A)) | A \in \mathbf{F}]$ where $\bigcap \varphi U_\alpha(A)$ is the set intersection of the subsets $(\varphi(U_\alpha(A)) : A \in F)$ of M. With this uniformity, M is a complete uniform space (cf. Nöbeling, p. 202) and $(\mathfrak{B}, \tau, \bar{\delta})$ is $\bar{\delta}$ isomorphic to the subalgebra $(\varphi A : A \in \mathfrak{B})$ of $S(M)$.

To show that M is compact it is enough to prove that the uniform structure $\mathfrak{B}$ defined above, is totally bounded. Let $U_\alpha \in \mathfrak{B}$ correspond to the $\bar{\delta}$-covering $(A_i : i = 1, 2, \dots n)$ of $\mathfrak{B}$. Let $(\mathbf{F}_i : i = 1, 2, \dots n)$ be ultrafilters in M such that $A_i \in \mathbf{F}_i$. Let N be the finite subset $N = (\mathbf{F}_i : i = 1, 2, \dots n)$ of M. Then clearly $U_\alpha(N) = M$. This completes the proof that $\mathfrak{B}$ is totally bounded and therefore M is compact.

Theorem 1. *A topological Boolean algebra $(\mathfrak{B}, \tau)$ is completely regular if and only if $(\mathfrak{B}, \tau)$ is homeomorphic to an invariant subalgebra of a compact regular space.*

The proof follows from Propositions 5 and 6.

2. Definition. Let X be a topological space of topological weight m and let I be an m-addive ideal of $S(X)$. Then we can define a topology in $S(X)/I$ as follows: an element $[A]$ in $S(X)/I$ is closed if and only if $A \equiv F \pmod I$ where F is a closed element of $S(X)$. (cf. Sikorski). We can call this the *quotient topology* on $S(X)/I$. Now we proceed to define and study quotient uniformity and quotient proximity in $S(X)/I$ where X is a completely regular space.

Proposition 2.1. *Let $(X, \bar{\delta})$ be a proximity space and let I be an ideal of $S(X)$. Then we can define a proximity structure in the quotient algebra $S(X)/I$ as follows: For any two elements $[A_1]$, $[A_2]$ in $S(X)/I$, $[A_1]\bar{\delta}[A_2] \Leftrightarrow$ there exist elements B_1, B_2 in $S(X)$ such that $A_i \equiv B_i \pmod I$ and $B_1 \bar{\delta} B_2$.*

Proof. *P. 1.* For any subset A of X, $A\bar{\delta}\varphi$ where φ is the null set and this implies $[A]\bar{\delta}[0]$ in $S(X)/I$.

P. 2. Clearly $[A_1]\bar{\delta}[A_2] \Leftrightarrow [A_2]\bar{\delta}[A_1]$.

P. 3. $[A_1] \wedge [A_2] > [0] \Rightarrow B_1 \cap B_2 \notin I$ for $B_i \equiv A_i$, $i = 1, 2$, $\Rightarrow B_1 \delta B_2 \Rightarrow [A_1]\delta[A_2]$.

P. 4. $[A]\bar{\delta}([B]+[C]) \Leftrightarrow [A]\bar{\delta}[B+C] \Leftrightarrow$ there exist A_1, B_1, C_1 such that $A \equiv A_1$ $B \equiv B_1$, $C \equiv C_1$ and $A_1\bar{\delta}(B_1+C_1) \Leftrightarrow [A]\bar{\delta}[B]$ and $[A]\bar{\delta}[C]$.

P. 5. Suppose $[A_1]\bar{\delta}[A_2]$. Then there exist B_1, B_2 in $[A_1]$, $[A_2]$ such that $B_1\bar{\delta}B_2$. This implies that there exist elements C_1, C_2 in $S(X)$ such that $B_i\bar{\delta}cC_i$ $(i=1,2)$ and $C_1\bar{\delta}C_2$. Therefore $[A_1]\bar{\delta}[A_2] \Rightarrow$ there exist $[C_1]$, $[C_2]$ such that $[A_i]\bar{\delta}c[C_i]$ and $[C_1]\bar{\delta}[C_2]$.

Proposition 2.2. *Let $(X, \mathfrak{B})$ be a uniform space of topological weight m and let I be an m-additive ideal of $S(X)$. Then we can define a uniformity in the quotient $S(X)/I$ as follows: for each element $[A]$ in $S(X)/I$ $U_\alpha(A)=[U_\alpha(A^*)]$ where*

$$A^* = c\big(\Sigma(G|G \wedge A \in I,\ G \text{ open in } S(X))\big).$$

Proof. $A_1 \equiv A_2 \pmod{I} \Rightarrow A_1^* = A_2^*$ (cf. SIKORSKI).

U. 1. $[A] \leqq [A^*] \leqq [U_\alpha(A^*)] = U_\alpha[A]$.

U. 2. Given α, β there exists a γ such that $U_\alpha(A^*) \cap U_\beta(A^*) \supset U_\gamma(A^*)$ and for this γ, $U_\alpha[A] \wedge U_\beta[A] \geqq U_\gamma[A]$.

U. 3. Given α, there exists a γ such that $(U_\gamma \cdot U_\gamma)(A^*) \subset U_\alpha(A^*)$ in $S(X)$ and given γ there exists a β such that $\overline{U_\beta(A^*)} \subset U_\gamma(A^*)$. Now

$$U_\beta(U_\beta[A]) = U_\beta[U_\beta(A^*)] = [U_\beta(U_\beta(A^*))^*] \leqq [U_\beta(\overline{U_\beta(A^*)})] \leqq [U_\gamma(U_\gamma(A^*))] \leqq [U_\alpha(A^*)] = \\ = U_\alpha[A].$$

U. 4. $$U_\alpha[A] \wedge [B] = [0] \Rightarrow U_\alpha(A^*) \cap B \in I \Rightarrow U_\alpha(A^*) \cap B^* = 0$$
$$\Rightarrow A^* \cap U_\alpha(B^*) = 0 \Rightarrow [A] \wedge U_\alpha[B] = [0].$$

U. 5. $[A_1] \leqq [A_2] \Rightarrow A_1^* \leqq A_2^* \Rightarrow U_\alpha(A_1^*) \leqq U_\alpha(A_2^*) \Rightarrow U_\alpha[A_1] \leqq U_\alpha[A_2]$.

U. 6. $\wedge\, U_\alpha[A] = \wedge\, [U_\alpha(A^*)] = \wedge\, \overline{[U_\alpha(A^*)]} = \overline{[\cap\, U_\alpha(A^*)]} = [\cap\, U_\alpha(A^*)] = [A^*] = \overline{[A]}$.

Theorem 2. *Let $(X, \mathfrak{B})$ be a uniform space of topological weight m and let I be an m-additive ideal of $S(X)$. Also let $\bar{\delta}$ be the proximity defined by $\mathfrak{B}$ on X. Then the quotient proximity on $S(X)/I$ defined by $\bar{\delta}$ (denoted by $\bar{\delta}_I$) is the same as the proximity defined in $S(X)/I$ by the quotient uniformity $\mathfrak{B}_I$ (denoted by $\bar{\delta}_{\mathfrak{B}}$).*

Proof. $[A_1]\bar{\bar{\delta}}_I[A_2] \Leftrightarrow B_1\bar{\bar{\delta}}B_2$ for some $B_i \equiv A_i \pmod{I}$, $(i=1,2,) \Rightarrow B_1^*\bar{\bar{\delta}}B_2^*$ in $S(X) \Leftrightarrow U_\alpha(B_1^*) \cap U_\alpha(B_2^*) = \varphi$ for some $U_\alpha \in \mathfrak{B} \Rightarrow U_\alpha[A_1] \cap U_\alpha[A_2] = [0]$ in $S(X)/I$. $\Rightarrow [A_1]\bar{\bar{\delta}}_{\mathfrak{B}}[A_2] \to$ (i). Conversely $[A_1]\bar{\bar{\delta}}_{\mathfrak{B}}[A_2] \Rightarrow U_\alpha[A_1] \wedge U_\alpha[A_2] = [0]$ for some $\alpha \in \Omega \Rightarrow$ $\Rightarrow U_\alpha(A_1^*) \cap U_\alpha(A_2^*) \in I \Rightarrow A_1^* \cap U_\alpha(A_2^*) \in I \to$ (ii). Again $A_1^* \cap U_\alpha(A_2^*) \in I \Rightarrow A_1 \cap U_\alpha(A_2^*) \in I \to$ (iii). Now given $\alpha \in \Omega$, there exists a $\beta \in \Omega$ such that $U_\beta(A_2^*) \leqq \text{int } U_\alpha(A_2^*)$. Therefore for this β, $A_1 \cap U_\alpha(A_2^*) \in I \Rightarrow A_1 \cap \text{int } U_\alpha(A_2^*) \in I \Rightarrow A_1^* \cap \text{int } U_\alpha(A_2^*) \in I \Rightarrow$ $\Rightarrow A_1^* \cap U_\beta(A_2^*) \in I \to$ (iv).

From (ii), (iii) and (iv) we have $[A_1]\bar{\bar{\delta}}_{\mathfrak{B}}[A_2] \Rightarrow A_1^* \cap \underline{U_\beta(A_2^*) = \varphi}$ for some $\beta \in \Omega \Rightarrow$ $\Rightarrow U_\gamma(A_1^*) \cap U_\gamma(A_2^*) = \varphi$ for some $\gamma \in \Sigma \Rightarrow A_1^*\bar{\bar{\delta}}A_2^*$ in $S(X) \Rightarrow \overline{[A_1]}\bar{\bar{\delta}}_I\overline{[A_2]}$ in $S(X)/I \Rightarrow$ $\Rightarrow [A_1]\bar{\bar{\delta}}_I[A_2] \to$ (v). Hence from (i) and (v) we have $[A_1]\bar{\bar{\delta}}_I[A_2] \Leftrightarrow [A_1]\bar{\bar{\delta}}_{\mathfrak{B}}[A_2]$.

Theorem 3. *Let $(X, \bar{\delta})$ be a proximity space of topological weight m and let I be an m-additive ideal of $S(X)$. Also let $\mathfrak{U}$ denote the coarsest uniformity on $S(X)$ compatible with $\bar{\delta}$. Then the coarsest uniformity on $S(X)/I$ computible with $\bar{\delta}_I$ is the same as the quotient uniformity $\mathfrak{U}_I$.*

Proof. $\mathfrak{U}$ has a base consisting of surroundings of the form $U_\alpha = \bigcup_{i=1}^{n} (A_i \times A_i)$ where $(A_i: i = 1, 2, \ldots, n)$ is an open covering of $S(X)$. If $(A_i: i = 1, 2, \ldots, n)$ is a $\bar{\delta}$-covering of $(X, \bar{\delta})$ then $([A_i]: i = 1, 2, \ldots, n)$ is a $\bar{\delta}_I$-covering of $S(X)/I$. This implies $\mathfrak{A}_I$ is coarser than the coarsest uniformity compatible with $\bar{\delta}_I$ and therefore $\mathfrak{U}_I$ is the coarsest uniformity compatible with $\bar{\delta}_I$ in $S(X)/I$.

3. In this section we shall prove a proposition connecting the concept of proximity lattices of Švarc and our notion of proximity Boolean algebras.

Definition 5. A subset $\mathfrak{U}$ of a proximity Boolean algebra $(\mathfrak{B}, \bar{\delta})$ is called *Švarc open* if (i) for each $U \in \mathfrak{U}$, there exists a $V \in \mathfrak{U}$ such that $U \bar{\delta} c V$, and (ii) if U, V are in $\mathfrak{U}$ then the set $(W: U \bar{\delta} c W$ and $W \bar{\delta} c V) \subset \mathfrak{U}$.

Proposition 3.1. *Let $(\mathfrak{B}, \tau, \bar{\delta})$ be a topological proximity algebra. Then any element A of $\mathfrak{B}$ of the form $A = \Sigma(U: U \in \mathfrak{U})$ where $\mathfrak{U}$ is a Švarc open set of $\mathfrak{B}$ is open. Conversely if A is an open element of $\mathfrak{B}$ then the set $(U: U \bar{\delta} c A)$ is a Švarc open set of $\mathfrak{B}$.*

Proof. Suppose $A = \Sigma(U: U \in \mathfrak{U})$ where $\mathfrak{U}$ is a Švarc open set of $\mathfrak{B}$.

Now $U \in \mathfrak{U} \Rightarrow$ there exists a $V \in \mathfrak{U}$ such that $U \bar{\delta} c V \Rightarrow$ there exists a $V \leqq A$ such that $U \bar{\delta} c V \Rightarrow U \bar{\delta} c A \Rightarrow U \leqq \operatorname{int} A$. These imply $A \leqq \operatorname{int} A$ and hence A is open. Conversely suppose A is an open element of $(\mathfrak{B}, \tau, \bar{\delta})$. Let $\mathfrak{U} = (U: U \bar{\delta} c A)$. $U \bar{\delta} c A \Rightarrow$ there exists a V such that $U \bar{\delta} c V$ and $V \bar{\delta} c A \Rightarrow$ there exists a $V \in \mathfrak{U}$ such that $U \bar{\delta} c V$. This proves $\mathfrak{U}$ satisfies condition (i) of Definition 5. Clearly $\mathfrak{U}$ satisfies condition (ii) also. Hence $\mathfrak{U}$ is a Švarc open set of $\mathfrak{B}$.

References

[1] E. M. Alfsen and J. E. Fenstad, On the equivalence between proximity structures and totally bounded uniform structures, *Math. Scandinavica*, 7 (1959), 353–360.
[2] I. S. Gál, Proximity relations and precompact structures, *Indagationes Math.*, 21 (1959), 304—314, 315—326.
[3] R. Sikorski, Closure Algebras, *Fund. Math.*, 36 (1949), 165—206.
[4] G. Nöbeling, *Grundlagen der Analytischen Topologie* (Berlin—Heidelberg—Göttingen, 1954).
[5] A. S. Švarc, Prostranstva blizosti i struktury, *Ivanov. Gos. Ped. Inst. Uč. Zap. Fiz.-Mat. Nauki*, 10 (1956), 55–60

(Received December 1, 1964)