Homomorphisms of certain commutative lattice ordered semigroups

By P. J. McCARTHY in Lawrence (Kansas, U. S. A.)*

Let S be a semigroup. It is well-known that a homomorphic image of S is, to within isomorphism, of the form S/θ where θ is a congruence on S. In this note we shall show that if S is a commutative lattice ordered semigroup [2, Chapter XII], with certain additional properties, then only congruences of a certain type are required to describe all of the homomorphic images of S. Then we shall point out a particularly interesting example of a class of semigroups which have all of these properties. In this note, when we refer to a homomorphism from one partially ordered semigroup into another we shall always mean one that preserves ordering.

First, assume that S is a commutative partially ordered semigroup, so that there is a partial ordering on S with the property that if $a, b, c \in S$ and if $a \leqq b$ then $ac \leqq bc$. Also assume that S has an identity e such that $a \leqq e$ for all $a \in S$.

Let M be a subsemigroup of S. For each $a \in S$ we set $a' = \{x \in S | mx \leqq a$ for some $m \in M\}$. Since $m \leqq e$ for all $m \in M$ we have $ma \leqq a$ and so $a \in a'$ for all $a \in S$. We define a relation θ on S by $a \equiv b(\theta)$ if and only if $a' = b'$. This is an equivalence relation on S and we easily verify the following facts:

(1) if $a \in b'$ then $a' \subseteqq b'$,

(2) if $a \equiv b(\theta)$ then $ac \equiv bc(\theta)$ for all $c \in S$,

(3) if $a \leqq c \leqq b$ and $a \equiv b(\theta)$ then $a \equiv c(\theta)$.

Thus, θ is a congruence on S and we can consider the semigroup S/θ. We denote the equivalence class of $a \in S$ with respect to θ by $\theta(a)$, and we also denote the natural homomorphism from S onto S/θ by θ.

From now on we shall assume that S is a lattice with respect to its partial ordering and that $a(b \vee c) = ab \vee ac$ for all $a, b, c \in S$.

(4) if $a \equiv b(\theta)$ we have $a \vee c \equiv b \vee c(\theta)$ and $a \wedge c \equiv b \wedge c(\theta)$ for all $c \in S$.

For, since $a \in b'$ there is an $m \in M$ such that $ma \leqq b$. Then $m(a \vee c) = ma \vee mc \leqq \leqq b \vee c$ and $m(a \wedge c) \leqq ma \wedge mc \leqq b \wedge c$. Hence $a \vee c \in (b \vee c)'$ and $a \wedge c \in (b \wedge c)'$, and so $(a \vee c)' \subseteqq (b \vee c)'$ and $(a \wedge c)' \subseteqq (b \wedge c)'$. By symmetry, $(b \vee c)' \subseteqq (a \vee c)'$ and $(b \wedge c)' \subseteqq (a \wedge c)'$.

*) Supported in part by NSF GP-1738.

We can therefore define operations $\vee$ and $\wedge$ on S/θ by $\theta(a)\vee\theta(b) = \theta(a\vee b)$ and $\theta(a)\wedge\theta(b)=\theta(a\wedge b)$, and with respect to these operations, S/θ is a lattice ordered semigroup. It is quite trivial that the properties required of the operations $\vee$ and $\wedge$ hold. It remains to verify that the ordering induced on S/θ by the lattice structure is compatible with the multiplication on S/θ. We have $\theta(a)\leqq\theta(b)$ if and only if $\theta(b)=\theta(a)\vee\theta(b)=\theta(a\vee b)$, and so if $\theta(a)\leqq\theta(b)$ we have for all $\theta(c)\in S/\theta$, $\theta(bc)= =\theta(b)\theta(c)=\theta(a\vee b)\theta(c)=\theta((a\vee b)c)=\theta(ac\vee bc)$. Hence $\theta(a)\theta(c)\leqq\theta(b)\theta(c)$. Note that $\theta(e)$ is the identity of S/θ and that $\theta(a)\leqq\theta(e)$ for all $\theta(a)\in S/\theta$.

Now consider a homomorphism h from S onto a partially ordered semigroup T. If we set $M=\{m\in S|h(m)=h(e)\}$ then M is a subsemigroup of S. Let θ be the congruence on S associated with M in the manner we have described. If $a\in S$ we set $f\theta(a)=h(a)$. If we show that f is a well-defined mapping from S/θ into T, then it is clear that f is a homomorphism from S/θ onto T such that $f\theta=h$. Suppose that $\theta(a)=\theta(b)$, i.e., $a\equiv b(\theta)$. Then there are elements $m, n\in M$ such that $ma\leqq b$ and $nb\leqq a$. Hence $nma\leqq nb\leqq a$ and so $h(a)=h(mna)\leqq h(nb)=h(b)\leqq h(a)$. Thus $h(a)=h(b)$ and we conclude that f is well-defined.

We seek conditions under which f will be an isomorphism. A suitable condition, for our purposes, is that both S and T be residuated [2, p. 189] and that h preserve residuals. For, suppose that this is the case, and that $h(a)=h(b)$. Then $h(e)h(b)=h(a)$ and so $h(e)\leqq h(a):h(b)$. Hence $h(e)=h(a):h(b)=h(a:b)$, which means that $a:b\in M$. Since $(a:b)b\leqq a$ we have $b'\subseteqq a'$. Similarly $a'\subseteqq b'$ and therefore $a\equiv b(\theta)$ and $\theta(a)= =\theta(b)$. We can summarize all of this as the

Theorem. *Let S be a commutative residuated lattice ordered semigroup with an identity e such that $a\leqq e$ for all $a\in S$. Let T be a residuated partially ordered semigroup and suppose there is a homomorphism h from S onto T which preserves residuals. Then there is a subsemigroup M of S such that if θ is the congruence on S determined as above by M, then there is an isomorphism f from S/θ onto T such that $f\theta=h$.*

Remark 1. If S and T are as in the statement of the theorem, then T becomes a lattice ordered semigroup when we define meet and join on T by $h(a)\wedge h(b)= =h(a\wedge b)$ and $h(a)\vee h(b)=h(a\vee b)$.

Remark 2. Let S be as in the statement of the theorem, let M be a subsemigroup of S, and let θ be the congruence on S determined by M. Then the semigroup S/θ is residuated and the homomorphism θ preserves residuals. To show this we shall verify that $\theta(a:b)$ is the residual of $\theta(a)$ by $\theta(b)$. We have $\theta(a:b)\theta(b)=\theta((a:b)b)\leqq\theta(a)$. Furthermore, suppose that $\theta(c)\theta(b)\leqq\theta(a)$. Then $\theta(cb)\leqq\theta(a)$ and so $\theta(cb)= =\theta(a)\wedge\theta(cb)=\theta(a\wedge cb)$. Thus, for some $m\in M$, $mcb\leqq a\wedge cb\leqq a$. Hence $mc\leqq a:b$ and so $\theta(mc)\leqq\theta(a:b)$. Since $\theta(m)=\theta(e)$, as is easily seen, $\theta(mc)=\theta(m)\theta(c)=\theta(c)$. Hence $\theta(c)\leqq\theta(a:b)$. Therefore, $\theta(a:b)$ is the required residual [2, p. 189]. More generally, these conditions are satisfied by the residuated multiplicative lattices, which have been studied by WARD and DILWORTH (see [1] and the references at the end of that paper).

Let L be a residuated multiplicative lattice: then L is a commutative residuated lattice ordered semigroup with an identity I such that $A\leqq I$ for all $A\in L$. The formation of the multiplicative lattice L/θ, where θ is determined as above by a subsemigroup (i.e., multiplicatively closed set) of L, is an abstract construction of the lattice of ideals of a ring of quotients of a commutative ring with identity. A special case of this construction was discussed by DILWORTH [1, pp. 489—491]. Let L be a Noether lattice and let $D \in L$. If M is the set of all $A \in L$ such that D is not greater than or equal to any of the primes associated with a normal decomposition of A, then M is a subsemigroup of L, and if θ is the congruence on L determined as above by M, then L/θ is precisely the congruence lattice L_D of Dilworth. In particularl if D is a prime P of L then $A \in M$ if and only if $A \not\leqq P$, for if $A \leqq P$ then some minima, prime associated with a normal decomposition of A must be less than or equal to P.

References

[1] R. P. DILWORTH, Abstract commutative ideal theory, *Pacific J. Math.*, **12** (1962), 481—498.
[2] L. FUCHS, *Partially Ordered Algebraic Systems* (Reading, Mass., 1963).

(Received April 15, 1965)