The Riemann—Lebesgue theorem on groups

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In [2] HEWITT gives an interesting and elegant *constructive* proof of PLANCHEREL's theorem for L^2 functions on a locally compact abelian group (*LCAG*). His proof is modelled on the classical proof of F. RIESZ of the special case in which the group is the real line R^1. We notice that in HEWITT's proof the Riemann—Lebesgue theorem is taken as known. However, to our knowledge, no *constructive* proof of the Riemann—Lebesgue theorem for the general *LCAG* has ever been given. The proof of this theorem is easy in the case of R^1, but only because the explicit form of the group characters as functions is known. The theorem for the general *LCAG* is always deduced from the Gelfand theory (see, for example, [4]) via the Tychonoff—Alaoglu theorem and other far from trivial considerations. This approach completely obscures the relation of the group structure to the theorem. In this paper we give a constructive proof of the Riemann—Lebesgue theorem for the general *LCAG* (again modelled on a well-known proof of the case of R^1). In particular, some light will be thrown on the behavior of the group characters as functions. (See Definition B and Theorem H.)

We begin with the following well-known proof. (See [1], for example.)

Theorem A. (Riemann—Lebesgue theorem for R^1.) *Let $f\in L^1(R^1)$ and let $f^\wedge$ be the Fourier transform of f;*

$$f^\wedge(\gamma)=\int_{-\infty}^{\infty} e^{-i\gamma x}f(x)\,dx \qquad (\gamma\in R^1). \tag{1}$$

Then $\lim_{\gamma\to\pm\infty} f^\wedge(\gamma)=0$. That is, the Fourier transform of an L^1 function vanishes at infinity.

Proof. For $y\in R^1$ let $f_y(x)=f(x-y)$ $(y\in R^1)$. Given $\varepsilon>0$ choose $\delta>0$ such that $\|f-f_y\|_1<2\varepsilon$ if $|y|<\delta$. From (1) we have, for $\gamma\neq 0$,

$$-f^\wedge(\gamma)=\int_{-\infty}^{\infty} e^{-i\gamma\left(x+\frac{\pi}{\gamma}\right)}f(x)\,dx=\int_{-\infty}^{\infty} e^{-i\gamma x}f\left(x-\frac{\pi}{\gamma}\right)dx. \tag{2}$$

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[1]) This research was supported by the National Science Foundation Grants 2130 and 3930.

Subtracting (2) from (1) we have

$$2f^{\wedge}(\gamma) = \int_{-\infty}^{\infty} e^{-i\gamma x}\left[f(x)-f\left(x-\frac{\pi}{\gamma}\right)\right] dx, \quad 2|f^{\wedge}(\gamma)| \leqq \|f-f_{\frac{\pi}{\gamma}}\|_1,$$

and hence, if $|\pi/\gamma|<\delta$, then $2|f^{\wedge}(\gamma)|<2\varepsilon$. That is,

$$|f^{\wedge}(\gamma)|<\varepsilon \qquad \left(|\gamma|>\frac{\pi}{\delta}\right).$$

This proves the theorem.

As we shall demonstrate, the key to the proof of the theorem is the fact that at the point $x=\frac{\pi}{\gamma}$, the character $x\to e^{i\gamma x}$ takes the value -1. In particular, if U is the neighborhood $(-\delta, \delta)$ of 0, then there is a compact set $K=\left[-\frac{\pi}{\delta}, \frac{\pi}{\delta}\right]$ such that if $\gamma\in R^1-K$ then the character determined by γ (namely $x\to e^{i\gamma x}$) takes a value at some point of U (namely π/γ) whose real part is $\leqq 0$. It is this property that we shall demonstrate for the general *LCAG*.

Definition B. Let G be a *LCAG* with character group Γ. We say that G has the R$-$L *property*, if, for any neighborhood U of the identity 0 of G there exists a compact set K in Γ such that, if $\gamma\in\Gamma-K$ then there exists $x\in U$ with $\operatorname{Re}\gamma(x)\leqq 0$. (We call K a *compact set corresponding to* U.)

As we have seen, R^1 has the R$-$L property. It is now easy to show that if the *LCAG* G has the R$-$L property then the Riemann—Lebesgue theorem holds for G.

Theorem C. *Let G be a LCAG with the* R$-$L *property. If $f\in L^1(G)$ and $f^{\wedge}$ is the Fourier transform of f, i.e.*

$$f^{\wedge}(\gamma) = \int_G \overline{\gamma(x)} f(x)\, dx \qquad (\gamma\in\Gamma), \tag{3}$$

then $f^{\wedge}$ vanishes at infinity.

Proof. We simply imitate the proof in Theorem A. Given $\varepsilon>0$ choose a neighborhood U of 0 in G such that $\|f-f_y\|_1<\varepsilon$ if $y\in U$. (Here again, $f_y(x)=f(x-y)$.) According to the R$-$L property there exists a compact set K in Γ corresponding to U. Then if $\gamma\in\Gamma-K$ there exists x_0 in U with $\operatorname{Re}\gamma(x_0)\leqq 0$. So

$$\overline{\gamma(x_0)} f^{\wedge}(\gamma) = \int_G \overline{\gamma(x+x_0)} f(x)\, dx = \int \overline{\gamma(x)} f_{x_0}(x)\, dx, \tag{4}$$

and, subtracting (4) from (3),

$$|f^{\wedge}(\gamma)|\cdot|1-\overline{\gamma(x_0)}| \leqq \|f-f_{x_0}\|_1<\varepsilon.$$

Since $\operatorname{Re}\gamma(x_0)\leqq 0$ we must have $|1-\overline{\gamma(x_0)}|\geqq 1$. Thus $|f^{\wedge}(\gamma)|<\varepsilon$ for all $\gamma\in\Gamma$ outside of the compact set K. That is, $f^{\wedge}$ vanishes at infinity, which is what we wished to show.

In view of Theorem C, to show that the Riemann—Lebesgue theorem holds for an arbitrary *LCAG* G, it is sufficient to show that G has the $\mathrm{R-L}$ property. We do this in several steps ultimately making use of structure theory for the *LCAG*.

Lemma D. *If each of the locally compact abelian groups G_1 and G_2 has the* $\mathrm{R-L}$ *property, then so does $G_1 \times G_2$.*

Proof. Let $G=G_1\times G_2$. Then the character group Γ of G is $\Gamma_1\times\Gamma_2$ where Γ_i is the character group of G_i. Let U be any neighborhood of the identity in G. We may assume that $U=U_1\times U_2$ where U_i is a neighborhood of the identity 0_i in G_i. According to the $\mathrm{R-L}$ property for G_i $(i=1, 2)$, there exists a compact subset K_1 of Γ_i corresponding to U_i. Now let $K=K_1\times K_2$. If $\gamma=(\gamma_1, \gamma_2)\in\Gamma-K$ then either $\gamma_1\notin K_1$ or $\gamma_2\notin K_2$. We may assume $\gamma_1\notin K_1$. Then, by the $\mathrm{R-L}$ property for G_1, there exists $x_1\in U_1$ with $\operatorname{Re}\gamma_1(x_1)\leqq 0$. Let $x=(x_1, 0_2)$. Then $x\in U$ and $\gamma(x)=$ $=\gamma_1(x_1)\gamma_2(0_2)=\gamma_1(x_1)$ and hence $\operatorname{Re}\gamma(x)\leqq 0$. Thus K may be used as a compact set corresponding to U, and so G has the $\mathrm{R-L}$ property.

Next we shall show that for any *compact* abelian group G, the topology for G is generated by finite *independent* subsets of Γ. (The subset $\{\beta_1, \ldots, \beta_s\}$ of elements of a group is said to be independent if whenever $n_1, \ldots, n_s$ are integers with $\sum_{i=1}^{s} n_i\beta_i=0$ then $n_i\beta_i=0$ for all $i=1, \ldots, s$.)

If $C=\{\gamma_1, \ldots, \gamma_n\}$ is a finite set of characters of G and S is a symmetric open arc of the unit circle about 1 (that is, for some θ_0 with $0<\theta_0\leqq\pi$, $S=\{e^{i\theta}|-\theta_0<\theta<$ $<\theta_0\}$) then $U[C; S]$ denotes the set of x in G such that $\gamma_k(x)\in S$ for $k=1, \ldots, n$. It is well known that the collection of all such $U[C:S]$ forms a basic set of neighborhoods of 0 in G. Thus, to show that the finite *independent* sets in Γ generate the topology of G, it is enough to show

Lemma E. *Let G be a compact abelian group. Then any neighborhood $U[C; S]$ contains a neighborhood $U[B; S']$ where B is a finite independent subset of Γ.*

Proof. Let $[C]$ denote the subgroup of Γ generated by $C=\{\gamma_1, \ldots, \gamma_n\}$. Then $[C]$ is a finitely generated abelian group and is thus the direct sum of s cyclic subgroups. Let $\beta_1, \ldots, \beta_s$ be the generators of these cyclic subgroups. Then $B=\{\beta_1, \ldots \beta_s\}$ is an independent set. Now any γ_k in C may be expressed as $\gamma_k=n_{k1}\beta_1+\ldots+n_{ks}\beta_s$ where the n_{kj} are integers. (These representations may not be unique since some β's can have finite order. In any case for each $k=1, 2, \ldots, n$, fix one such representation.) Let $M_k=\sum_{j=1}^{s}|n_{kj}|$ and let $M=\max_{1\leqq k\leqq n} M_k$. If $S=\{e^{i\theta}|-\theta_0<\theta<\theta_0\}$ let $S'=\{e^{i\theta}|-\theta_0'<\theta<\theta_0'\}$ where $\theta_0'=\dfrac{\theta_0}{M}$. We shall show that $U[B; S']\subset U[C; S]$. Indeed, if $x\in U[B; S']$ then $\beta_j(x)=e^{i\theta_j}$ where $|\theta_j|<\theta_0'$. Hence $\gamma_k(x)=\exp(i[n_{k1}\theta_1+\ldots$ $\ldots+n_{ks}\theta_s])$. But $|n_{k1}\theta_1+\ldots+n_{ks}\theta_s|<M_k\theta_0'\leqq M\theta_0'=\theta_0$. Hence $\gamma_k(x)\in S$ for $k=1,\ldots,n$ and so $x\in U[C; S]$ which is what we wished to show.

We next prove

Lemma F. *Every compact abelian group G has the* $\mathrm{R-L}$ *property.*

Proof. Let U be any neighborhood of 0 in G. By the preceding lemma we may assume that $U=U[B;S]$ where $B=\{\alpha_1, \ldots, \alpha_r, \beta_1, \ldots, \beta_s\}$ is an independent set, the α_j having finite order and the β_j infinite order, and $S=\{e^{i\theta} \mid -\theta_0<\theta<\theta_0\}$ where $0<\theta_0 \leqq \pi$. Let q be the smallest positive integer such that $q\theta_0>\frac{\pi}{2}$. (Then $\operatorname{Re} e^{iq\theta_0} \leqq 0$.)

Let K be the (finite) set of γ in Γ which can be expressed $\gamma=\alpha+n_1\beta_1+\ldots+n_s\beta_s$ where α is an element of $[\alpha_1, \ldots, \alpha_r]$, the finite subgroup generated by $\alpha_1, \ldots, \alpha_r$, and $|n_1|+\ldots+|n_s| \leqq q$. (If there are no α_j — that is if every element in B has infinite order — use $\{0\}$ instead of $[\alpha_1, \ldots, \alpha_r]$. If there are no β_j, set $K=[\alpha_1, \ldots, \alpha_r]$ and use obvious modifications in the remainder of the proof.) We shall now show that K may be taken as a compact set corresponding to U. For suppose $\gamma \in \Gamma-K$. There are two possible cases.

I. Suppose $\gamma \in[B]$ where $[B]$ is the subgroup of Γ generated by B. Then $\gamma=\alpha+ +n_1\beta_1+\ldots+\ldots+n_s\beta_s$ for some $\alpha \in[\alpha_1, \ldots, \alpha_r]$ and for some integers $n_1, \ldots, n_s$. Since $\gamma \notin K$ we must have $M=|n_1|+\ldots+|n_s|>q$. For $j=1, \ldots, r$ let $x(\alpha_j)=1$. For $j=1, \ldots, s$ let $x(\beta_j)=e^{iq\theta_0/M}$ if $n_j>0$, let $x(\beta_j)=e^{-iq\theta_0/M}$ if $n_j<0$, and let $x(\beta_j)=1$ if $n_j=0$. Since $\alpha_1, \ldots, \alpha_r, \beta_1, \ldots, \beta_s$ are independent it is easy to verify that x may be extended multiplicatively to a character on $[B]$. We then have $x(\gamma)= =\exp\left((iq\theta_0/M)[|n_1|+\ldots+|n_s|]\right)=\exp(iq\theta_0)$. Now $[B]$ is a closed subgroup of Γ (since Γ is discrete). Hence we may extended x to a character on all of Γ, that is $x \in G$. But since $q\theta_0/M<\theta_0$ we have $\beta_j(x)=x(\beta_j) \in S$ for $j=1, \ldots, s$. Thus $x \in U$. Since $\operatorname{Re} \gamma(x)=\operatorname{Re} x(\gamma)=\operatorname{Re} e^{iq\theta_0} \leqq 0$, this shows that γ takes an appropriate value at x. (Note: if there are no β_j in B then case I cannot occur, since then $K=[B]$ and $\gamma \notin K$.)

II. Suppose $\gamma \notin[B]$. Then there is an element $y \in G$ such that y is in the annihilator of $[B]$ but $y(\gamma)=\gamma(y) \neq 1$. If $x=y^p$ for an appropriate positive integer p, we have $\operatorname{Re} \gamma(x)=\operatorname{Re}[\gamma(y)]^p \leqq 0$. But x is also in the annihilator of $[B]$ so that $\alpha_j(x)=x(\alpha_j)= =1=x(\beta_j)=\beta_j(x)$ for all $\alpha_j, \beta_j \in B$. Hence, $x \in U$ and the proof is complete.

Lemma G. *Let H be a LCAG which contains a compact open subgroup G. Then H has the* R$-$L *property.*

Proof. Let U be any neighborhood of the identity 0 of H. We may assume $U \subset G$. (Otherwise, since G is open, we could consider $U \cap G$ instead of U.) Lemma F shows that the compact group G has the R$-$L property. Thus there is a finite subset $K_0=\{\gamma_1, \ldots, \gamma_n\}$ of characters of G corresponding to U. Since G is compact, every γ_j may be extended to a character λ_j of H. If μ_j is any other character of H which is also an extension of γ_j then $\mu_j\lambda_j^{-1}$ is an element of the annihilator A of G.

(Here, of course, $A \subset H^{\wedge}$ where $H^{\wedge}$ is the character group of H.) That is, $\mu_j \in \lambda_j A$. Hence, if we set $K=\bigcup_{j=1}^{n} \lambda_j A$ then K is the set of all extensions of $\gamma_1, \ldots, \gamma_n$ to characters of H. Moreover, K is compact since A, being the annihilator of the open compact group G, is itself open and compact [3]. It is now easy to show that K may be used as a compact set corresponding to U. For if $\lambda \in H^{\wedge}-K$, then λ_G (the restriction of λ to G) is not one of the γ_j. That is, $\lambda_G \in G^{\wedge}-K_0$. Thus there exists $x \in U$ with $\operatorname{Re} \lambda_G(x) \leqq 0$. Obviously, then, $\operatorname{Re} \lambda(x) \leqq 0$ and we are done.

We now conclude with

Theorem H. *Every LCAG has the* R – L *property.*

Proof. Every *LCAG* may be factored as $R^n \times H$ for some $n = 0, 1, 2, \ldots$, where R^n is Euclidean n-space and H is a *LCAG* with a compact open subgroup [3]. After Definition B we observed that R^1 has the R – L property. Hence, by Lemma D, R^n also has the R – L property. This together with Lemma G and another application of Lemma D complete the proof.

Corollary I. *The Riemann—Lebesgue theorem holds for every LCAG.*

Proof. Theorem H and Theorem C.

References

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(Received February 6, 1965)