An embedding theorem for some countable groups

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Every countable soluble group can be embedded in a soluble 2-generator group, the solubility length increasing by no more than 2 in the process: this was shown in [5]. We here extend this result to some of the transfinite generalizations of soluble groups. The method of [5] has to be modified to do this, firstly as in [4] and secondly as in HALL's paper [1].

We use the following notation and terminology. An ascending series of subgroups of a group G is a family $\{L_\lambda\}_{0 \leqq \lambda \leqq \sigma}$ of subgroups of G indexed by the set of ordinals less than or equal to the ordinal σ, and such that $L_0 = \{1\}$ and, for $0 < \lambda \leqq \sigma$

$$L_\lambda = \bigcup_{\mu < \lambda} L_{\mu+1}. \tag{1}$$

[This condition ensures that $L_\mu \leqq L_\lambda$ whenever $\mu \leqq \lambda$, and simultaneously that L_λ is the union of its predecessors when λ is a limit ordinal.] If each L_λ is normal in its successor $L_{\lambda+1}$, or even in G, the series is called "normal" or "invariant", respectively. If for $0 \leqq \lambda < \sigma$

$$[L_{\lambda+1}, L_{\lambda+1}] \leqq L_\lambda, \quad \text{or even} \quad [G, L_{\lambda+1}] \leqq L_\lambda,$$

where $[A, B]$ stands for the mutual commutator group of A and B, then the series is called "soluble" or "central", respectively. A soluble series is necessarily normal, and a central series invariant.

If G has a soluble series with $L_\sigma = G$, then G is defined to be an SN^*-group; if the soluble series can be chosen invariant, then G is an SI^*-group; if G has a central series with $L_\sigma = G$, then G is a ZA-group. The ordinal σ is called a "length" of G — we do not assume it chosen minimal, and if G has SN^*-length or SI^*-length or ZA-length σ, then it has also every greater length.

We shall prove the following theorem.

Theorem. *Every countable SI^*-group G of length σ can be embedded in a 2-generator SI^*-group of length $\sigma+2$.*

The method of proof yields rather more than the theorem. To every countable group G, we contruct a 2-generator group H which embeds it. The new feature of H is that its second derived group is contained in a certain interdirect power N_σ of G. Let $\mathfrak{C}$ be a class of groups which is closed under the operations of taking subgroups and taking interdirect powers like N_σ. (The reader has to refer to the first paragraph of the proof: an interdirect power F is selected there, and N_σ is a restricted direct power of F.) It follows from our construction that *every countable group in* $\mathfrak{C}$ *can be embedded in a 2-generator group whose second derived group is in* $\mathfrak{C}$. Some examples of classes which satisfy the conditions on $\mathfrak{C}$ are those of SN^*-groups, ZA-groups, locally nilpotent groups, locally finite groups, periodic groups, etc. In particular, it follows that *every countable* SN^**-group of length* σ *is embeddable in a 2-generator* SN^**-group of length* $\sigma+2$.

We mention an easy consequence of the theorem itself:

Corollary. *There exist* SI^**-groups that are not locally soluble.*

This fact was pointed out by HALL in [2]; in the present context it follows by applying the theorem to a countable insoluble SI^*-group G, for instance to one of the characteristically simple groups of McLAIN [3].

Proof of the Theorem. In addition to the notation introduced above, we also use the definitions and notation of [5]. In the complete wreath product $P = G \operatorname{Wr} C$ of the given SI^*-group G and an infinite cyclic group C generated by an element c, we single out a subgroup that contains the restricted wreath product $G \operatorname{wr} C$, but is not much larger. In the base group of P, that is the cartesian power G^C consisting of all functions on C to G, we single out those functions f that are constant for all sufficiently large positive powers of c, and also for all sufficiently large negative powers of c, the constant in this latter case being 1; thus we consider those f to which there is an integer $p \geqq 0$, depending on f, such that

$$f(c^n) = 1 \text{ when } n < -p; \quad f(c^n) = f(c^{n+1}) \text{ when } n > p.$$

These functions form a subgroup F of G^C, and F is normalized by C. We put $FC = P^0$.

The cartesian powers L_λ^C are normal subgroups of G^C, but they will not in general form an ascending series in G^C, as the analogue of (1) may fail for limit ordinals λ. However, if we put, for $0 \leqq \lambda \leqq \sigma$,

$$M_\lambda = F \cap L_\lambda^C,$$

so that M_λ consists of those functions $f \in F$ that take values in L_λ, then each M_λ is a normal subgroup of $M_\sigma = F$ and indeed of P^0, and in fact $\{M_\lambda\}_{0 \leqq \lambda \leqq \sigma}$ is an ascending soluble invariant series of P^0. We omit the easy verification. If we put $M_{\sigma+1} = P^0$, then the thus augmented series shows that P^0 is an SI^*-group of length $\sigma+1$.

Next we take an infinite cyclic group B generated by an element b and form the complete wreath product

$$Q = P^0 \operatorname{Wr} B.$$

This contains in its base group P^{0B} the direct powers N_λ of the M_λ, that is the functions on B to M_λ with finite support. These are easily seen to form an ascending soluble invariant series $\{N_\lambda\}_{0 \leqq \lambda \leqq \sigma+1}$ in Q.

We now use the assumption that G is countable, and generate it by a family $\{g_i\}_{i \in I}$ of elements indexed by the set I of positive integers. To these we define elements $k_i \in F$ by

$$k_i(c^n) = 1 \text{ when } n < 0, \quad k_i(c^n) = g_i^{-1} \text{ when } n \geqq 0.$$

Put $g_{i1}=[k_i, c]$; then

$$g_{i1}(1)=g_i, \quad g_i(c^n)=1 \text{ when } n\neq 0.$$

Thus the family $\{g_{i1}\}_{i\in I}$ generates the coordinate subgroup G_1 of G^C; clearly $G_1\cong G$. Next we define an element $a\in P^{0B}$ by

$$a(b)=c, \quad a(b^{2^i})=k_i \text{ when } i\in I,$$

$$a(b^n)=1 \text{ when } n \text{ is not a power of } 2.$$

Let H be the subgroup of Q generated by a and b, and let A be the normal closure of a in H. Then A is generated by the conjugates

$$a^{b^n}=a_n,$$

say, of a, where n ranges over all integers.

We now show that the derived group A' of A is contained in N_σ. First we remark that A' is generated by all commutators $[a_m, a_0]$ and their conjugates under powers of b; and as b normalizes N_σ, it suffices to show that every $[a_m, a_0]$ lies in N_σ. Now $[a_m, a_0]$ is a function on B to P^0, and we compute its value at b^n:

$$[a_m, a_0](b^n)=[a_m(b^n), a_0(b^n)]=[a(b^{n-m}), a(b^n)];$$

this is 1 unless $n-m$ and n are distinct powers of 2, say $n-m=2^i$, $n=2^j$, with i,j non-negative integers. In this case $m=2^j-2^i$, and to any one m there is at most one such pair i,j. Thus the support of $[a_m, a_0]$ consists of at most one element of B; it only remains to show that the one non-trivial value of $[a_m, a_0]$, if it has one at all, lies in $M_\sigma=F$. Now if $m=2^j-2^i\neq 0$, then

$$\begin{aligned}[a_m, a_0](b^{2^j}) = [a(b^{2^i}), a(b^{2^j})] &= g_{j1}^{-1} \text{ if } i=0,\\ &= g_{i1} \text{ if } j=0,\\ &= [k_i, k_j] \text{ if } i\neq 0, j\neq 0.\end{aligned}$$

These values all lie in F, and it follows that $A'\leqq N_\sigma$ as claimed.

Incidentally the above argument also shows how G can be embedded in H; for if we put, for $i\in I$,

$$h_i=[a_{1-2^i}, a_0],$$

then

$$h_i(b)=g_{i1}, \quad h_i(b^n)=1 \text{ when } n\neq 1;$$

hence the subgroup of H generated by $\{h_i\}_{i\in I}$ is isomorphic to G_1 and thus to G.

Finally we put $K_\lambda=H\cap N_\lambda$ for $0\leqq\lambda\leqq\sigma$. Then, as $\{N_\lambda\}_{0\leqq\lambda\leqq\sigma}$ is an ascending soluble invariant series of Q, also $\{K_\lambda\}_{0\leqq\lambda\leqq\sigma}$ is an ascending soluble invariant series of H. Adding $K_{\sigma+1}=A$ and $K_{\sigma+2}=H$ to this series, we obtain an ascending soluble invariant series that terminates with H itself; for as we have just seen, $A'\leqq N_\sigma$ and thus also $K'_{\sigma+1}\leqq K_\sigma$; and obviously also $H'\leqq A$. It follows that H is an SI^*-group of length $\sigma+2$, and the Theorem is proved.

References

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