Approximation of continuous functions on compact metric space by linear methods

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Dedicated to professor L. Kalmár in occasion of his 60th birthday

§ 1

We refer to the following theorem, due to J. P. NATANSON [2]:

Let $\{K_n(t)\}$ be a sequence of 2π-periodic L-integrable functions, for which the relations

$$\int_{-\pi}^{+\pi} K_n(t)\,dt = 1, \quad \int_{-\pi}^{\pi} |K_n(t)|\,dt = O(1)$$

and

$$\int_{-\pi}^{+\pi} |tK_n(t)|\,dt = O(\lambda_n) \qquad (\lambda_n \downarrow 0)$$

are satisfied, and with the aid of $\{K_n(t)\}$ define for arbitrary 2π-periodic continuous functions $f(t)$ the sequence of linear transformations

$$A_n(f; x) = \int_{-\pi}^{\pi} f(t+x) K_n(t)\,dt.$$

Then

$$A_n(f; t) - f(t) = O(1)\cdot\omega(f; \lambda_n),$$

where

$$\omega(f; \delta) = \max_{\substack{|h| \leqq \delta \\ x \in (-\pi, +\pi)}} |f(x+h) - f(x)|$$

is the continuity modulus of $f(x)$.

The aim of this paper is to extend this theorem to a rather general case. An example, where the generalized theorem is needed, is contained in § 5.

§ 2

Let K be a compact metric space, with the distance function $\varrho(x, y)$ $(x, y \in K)$, let further C_K be the space of real valued continuous functions $f(x)$ over K with the usual norm

(1) $$\|f(x)\| = \operatorname*{Max}_{x\in K} |f(x)|.$$

Let $\alpha(f)$ be a bounded linear functional over C_K,

(2) $$\sup_{\|f\|\leqq 1} |\alpha(f)| = A$$

and

(3) $$\sup |\alpha\{f(x)\varrho(x,\xi)\}| = A_\xi.$$

Theorem 1. *Let $\varphi(\delta)$ be a not decreasing function with $\varphi(0)=0$ and*

(4) $$\varphi(2\delta)\leqq 2\varphi(\delta) \qquad (\delta>0).$$

Then for every $f\in C_K$ the condition

(5) $$|f(x)-f(\xi)| \leqq \varphi\{\varrho(x,\xi)\} \qquad (x\in K,\ \xi \text{ fixed})$$

implies for every $\sigma>0$

(6) $$|\alpha(f)-f(\xi)\alpha(1)| \leqq (A+3\sigma^{-1}A_\xi)\varphi(\sigma).^{1})$$

Before proving our theorem, we deduce some of its consequences, the proof itself is postponed to § 3. We call K convex if for every pair $x_1, x_2\in K$ there is at least one $x_{12}\in K$ such that

(7) $$\varrho(x_i, x_{12})=\tfrac{1}{2}\varrho(x_1,x_2) \qquad (i=1,2).$$

For convex K the modulus of continuity

$$\omega(f;\delta) = \operatorname*{Max}_{\substack{x_1,x_2\in K\\ \varrho(x_1,x_2)\leqq\delta}} |f(x_1)-f(x_2)|$$

of an arbitrary function $f(x)$ satisfies

$$\omega(f;2\delta)\leqq 2\omega(f;\delta).$$

Let[2])

$$A^{(1)} = \sup_{\xi\in K} A_\xi.$$

Putting $\varphi=\omega$ we obtain from (6)

(6a) $$|\alpha(f)-f(\xi)\alpha(1)| \leqq (A+3\sigma^{-1}A^{(1)})\omega(f;\sigma)$$

for every $\sigma>0$.

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[1]) We use the notation $\alpha(1)=\alpha(f_0)$, $f_0\equiv 1$.

[2]) As a consequence of the compactness of K

$$\sup_{x_1,x_2\in K} \varrho(x_1,x_2) = R<\infty$$

so that $A_\xi\leqq RA$. From this we conclude $A^{(1)}<\infty$.

§ 3

Let K be convex and let Λ be a bounded linear transformation of C_K into itself, transforming $f \in C_K$ into $\Lambda(f; x) \in C_K$ $(x \in K)$, with the norm

$$\sup_{\|f\| \leqq 1} \|\Lambda(f)\| = \|\Lambda\|.$$

For each fixed $\xi \in K$ we consider the linear transformation

$$\Lambda_\xi(f) = \Lambda\{\varrho(x, \xi) f(x)\}$$

and set[3])

$$\Lambda^{(1)} = \sup_{\xi \in K} \|\Lambda_\xi\|.$$

Putting $\alpha(f) = \Lambda(f; \xi)$, $A \leqq \|\Lambda\|$, $A^{(1)} \leqq \Lambda^{(1)}$ in (6a), we obtain

(6b) $$|\Lambda(f; \xi) - \Lambda(1; \xi) f(\xi)| \leqq (\|\Lambda\| + 3\Lambda^{(1)}\sigma^{-1})\omega(f; \sigma)$$

for every $\sigma > 0$ and $\xi \in K$.

Now let us consider a sequence $\{\Lambda_n\}$ of bounded linear transformations over C_K, such that

(8) $$\|\Lambda_n\| = O(1) \quad \text{and} \quad \Lambda_n^{(1)} = O(\lambda_n),$$

where $\lambda_n \downarrow 0$.

Substituting $\Lambda = \Lambda_n$, $\sigma = \lambda_n$ in (6b), we obtain[4])

$$\Lambda_n(f; \xi) - f(\xi)\Lambda_n(1; \xi) = O\{\omega(f; \lambda_n)\}$$

and the constant in the O-estimate does not depend on the choice of $f \in C_K$ and $\xi \in K$.

This gives the announced generalization of NATANSON's theorem:

Theorem 2. *Let K be convex, and let the sequence $\{\Lambda_n\}$ of linear transformations over C_K satisfy* (8). *Then*

(9) $$|\Lambda_n(f; \xi) - f(\xi)\Lambda_n(1; \xi)| \leqq K_1\omega(f; \lambda_n)$$

where K_1 is neither depending on ξ nor on the choice of $f \in C_K$.

§ 4

We turn to the proof of Theorem I.

Lemma 1. *For every $\sigma > 0$ and $\vartheta > 1$ we have*

(10) $$\varphi(\vartheta\sigma) < 2\vartheta\varphi(\sigma).$$

Proof. From (4) it follows by induction

$$\varphi(2^m\delta) \leqq 2^m\varphi(\delta) \qquad (m = 1, 2, \ldots).$$

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[3]) From the inequality $\|\Lambda_\xi\| \leqq R\|\Lambda\|$ (see [1])) we conclude $\Lambda^{(1)} < \infty$.

[4]) We use the notations $\Lambda_n(1; \xi) = \Lambda_n(f_0; \xi)$, $f_0 \equiv 1$.

Let now m be the integer for which

$$2^m < \vartheta \leqq 2^{m+1}.$$

Since $\varphi(\delta)$ is monotone, we obtain

$$\varphi(\vartheta\sigma) \leqq \varphi(2^{m+1}\sigma) \leqq 2^{m+1}\varphi(\sigma) < 2\vartheta\varphi(\sigma). \qquad \text{Q. e. d.}$$

Lemma 2. *If, for a fixed $\xi \in K$, (5) and (4) are satisfied, then for arbitrary $\sigma > 0$ there is an $f_1 \in C_K$ and an $f_2 \in C_K$ such that*

(11) $$f(x) = f(\xi) + [f_1(x) + 3\sigma^{-1}\varrho(x, \xi) f_2(x)]\,\varphi(\sigma),$$

where

(12) $$\|f_1\| \leqq 1 \quad \text{and} \quad \|f_2\| \leqq 1.$$

Proof. We consider the function $F(x) = f(x) - f(\xi)$ on the closed set $\Sigma = \{x\colon \varrho(x, \xi) \leqq \sigma\}$. According to the theorem of TIETZE F can be extended as a continuous function to K, so that

$$\max_{x \in K} |F(x)| = \max_{x \in \Sigma} |F(x)| \leqq \varphi(\sigma).$$

We put $F(x) = f_1(x)\varphi(\sigma)$, $f_1 \in C_K$, $\|f_1\| \leqq 1$, and define $f_2 \in C_K$ by (11). Then $f_2(x) = 0$ for $x \in \Sigma$, and for $x \notin \Sigma$ (i. e. $\varrho(x, \xi) > \sigma$) we have by Lemma 1 with $\vartheta = \sigma^{-1}\varrho(x, \xi)$

$$|f(x) - f(\xi)| \leqq \varphi\{\varrho(x, \xi)\} \leqq 2\sigma^{-1}\varrho(x, \xi)\,\varphi(\sigma).$$

For $x \notin \Sigma$ we have

$$|F(x)| \leqq \varphi(\sigma) \leqq \sigma^{-1}\varrho(x, \xi)\varphi(\sigma),$$

so that (11) gives

$$|f_2(x)| \leqq 1, \quad x \notin \Sigma.$$

For $x \in \Sigma$ we had $f_2(x) = 0$, so that

$$\|f_2\| \leqq 1. \qquad \text{Q. e. d.}$$

Proof of Theorem 1. From the representation (11) we conclude

$$\alpha(f) - f(\xi)\alpha(1) = [\alpha(f_1) + 3\sigma^{-1}\alpha\{\varrho(x, \xi) f_2(x)\}]\varphi(\sigma),$$

hence by (12), (2) and (3) we obtain

$$|\alpha(f_1)| \leqq A$$

and

$$|\alpha\{\varrho(x, \xi) f_2(x)\}| \leqq A_\xi,$$

so that

$$|\alpha(f) - f(\xi)\alpha(1)| \leqq (A + 3\sigma^{-1}A_\xi)\varphi(\sigma). \qquad \text{Q. e. d.}$$

§ 5

We mention a typical application of our theorem. In [1] we applied the following lemma: We consider a matrix of real functions, $\{\varphi_{kn}(x)\}$ $(k=1, 2, \ldots, n;$ $n=1, 2, \ldots)$ defined over a finite interval $[a, b]$, $a<0<b$. For $x \in [a, b]$ let

$$\sum_{k=1}^{n} \varphi_{kn}(x) = 1 + O\left(\frac{1}{n}\right), \tag{13}$$

$$\sum_{k=1}^{n} |x - x_{kn}|\, |\varphi_{kn}(x)| = O\left(\frac{1}{n}\right), \tag{14}$$

and

$$\sum_{k=1}^{n} |\varphi_{kn}(x)| = O(1). \tag{15}$$

Then for every $g \in C[a, b]$

$$g(0) + \sum_{k=1}^{n} \varphi_{kn}(x)[g(x_{kn}) - g(0)] - g(x) = O(1)\,\omega\left(g; \frac{1}{4n}\right).$$

Setting

$$\Lambda_n(f; x) = \frac{\sum_{k=1}^{n} \varphi_{kn}(x) f(x_{kn})}{\sum_{k=1}^{n} \varphi_{kn}(x)},$$

$f(x) = g(x) - g(0)$, and $\lambda_n = \frac{1}{4n}$, the conditions of Theorem 2 are satisfied and we obtain as its conclusion

$$\Lambda_n(f; x) - f(x) = O(1) \cdot \omega\left(f; \frac{1}{4n}\right),$$

i. e.

$$g(0) + \sum_{k=1}^{n} \varphi_{kn}(x)[g(x_{kn}) - g(0)] - g(x) =$$

$$= O(1)\,\omega\left(g; \frac{1}{n}\right) + \max|g(x) - g(0)|\, O\left(\frac{1}{n}\right) = O(1)\,\omega\left(g; \frac{1}{4n}\right),$$

so that this lemma appears to be a consequence of Theorem 2, though it would not follow from NATANSON's theorem.

References

[1] G. FREUD, Über ein Jacksonsches Interpolationsverfahren, *Über Approximationstheorie*, ISNM, 5 (Basel—Stuttgart, 1964), 227—232.

[2] И. П. Натансон, О точности представления непрерывных периодических функций сингулярными интегралами, *Доклады Акад. Наук СССР*, **73** (1950), 273—276.

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