Commutative Schreier semigroup extensions of a group

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1. Introduction

This paper continues the study of complete structures begun by WIEGANDT [5] and later extended by WIEGANDT [6] and this author [3]. It is shown here (§ 4, Theorem 5) that the group of (equivalence classes of) commutative Schreier extensions of a group S by a semigroup Q is isomorphic to and can be obtained from, the group of commutative group extensions of S by a group Q^* which is related to the semigroup Q. It is then an immediate consequence of Theorem 5 that a commutative semigroup is complete if and only if it is a divisible group (§ 5). The first part of Theorem 5 is the special case $n=2$ of Proposition 4. 1 of CARTAN—EILENBERG [2, p. 191], but an elementary proof is incuded here for the sake of completeness. The second part of Theorem 5 can probably be generalized to arbitrary dimension n, but the present paper is intended to be a study of extension theory and not of homological algebra.

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2. Background

Hereafter the term *semigroup* is used only to refer to a commutative semigroup, written additively, with identity element 0; and the term *subsemigroup* is used only to refer to a subsemigroup containing 0. A semigroup T is called a *Schreier extension* of a subsemigroup S if there is a set of cosets of S in T which partitions T and for which there is a set $\{u_a | a \in Q\}$ of coset representatives ($u_a \in T$) such that $u_a + \alpha = u_b + \beta$ with $a, b \in Q$ and $\alpha, \beta \in S$ implies $a=b$ and $\alpha=\beta$. It is easily proved that the partitioning cosets of S in T are unique; that $S = u_a + S$ for some $a \in Q$ so that we may (and shall) assume the existence of an element $0 \in Q$ such that $u_0=0$; and that coset addition induces an addition $+$ on Q so that $(Q, +)$ is a homomorphic image of T. Moreover, the representation $u_a+\alpha$, with $a \in Q$ and $\alpha \in S$, of each element of T is unique for every set $\{u_a | a \in Q\}$ of representatives of the partitioning cosets. The extension T (or, equivalently, (T, η) with η the natural homomorphism of T onto Q) is then referred to as an extension of S by Q. Two extensions (T_1, η_1) and (T_2, η_2) are called *equivalent* if there is an isomorphism

τ of T_1 onto T_2 such that $\tau|S = \iota_S$ (=the identity automorphism of S), and $\eta_1 = \eta_2\tau$. RÉDEI [4] has shown, in an analogue of the Schreier theory of group extensions, that the existence and structure of an extension of S by Q is determined to within equivalence by the existence of a function $\varphi: Q \times Q \to S$ satisfying

I. $\varphi(a, 0) = 0$,

II. $\varphi(a, b) = \varphi(b, a)$,

III. $\varphi(a+b, c) + \varphi(a, b) = \varphi(a, b+c) + \varphi(b, c)$,

for all a, b, c in Q. For a given extension T with the set $\{u_a | a \in Q\}$ of partitioning representatives, the function φ, which is called a factor-system for the extension and the factor-system corresponding to the set $\{u_a | a \in Q\}$ of partitioning representatives, is defined by $u_a + u_b = u_{a+b} + \varphi(a, b)$. We shall write $T = T(S, Q, u, \varphi)$ to indicate that T is a Schreier extension of S by Q with $\{u_a | a \in Q\}$ a set of partitioning representatives whose corresponding factor-system is φ.

If $T_1 = T_1(S, Q, u, \varphi)$ and $T_2 = T_2(S, Q, v, \psi)$, then RÉDEI has shown that T_1 and T_2 are equivalent if and only if there exists a function $f: Q \to U(S)$ (=the group of units of S; i. e.: the group of all elements of S having inverses relative to 0) such that $\varphi(a, b) + f(a+b) = \psi(a, b) + f(a) + f(b)$ for all $a, b \in Q$. Consequently, S is a direct summand of $T = T(S, Q, u, \varphi)$ if and only if φ can be expressed as $\varphi(a, b) = f(a) + f(b) - f(a+b)$ for some f: $Q \to U(S)$. A corollary to this is the assertion that S is a direct summand of $T = T(S, Q, u, \varphi)$ only if $\varphi(Q \times Q) \subseteq U(S)$. An extension of a semigroup S in which S is a direct summand is called a splitting extension.

It can be shown that a Schreier extension T of S by Q is a group or is cancellative if and only if S and Q are both groups, or cancellative, respectively.

If Q is a semigroup, then the relation $\nu = \{(a, b) \in Q \times Q | \ a + c = b + c$ for some $c \in Q\}$ is the smallest congruence relation on Q for which Q/ν is cancellative. We shall call Q/ν the maximal cancellative homomorphic image of Q. Any cancellative semigroup Q' can be embedded in an isomorphically unique smallest group Q^*, called the difference-group of Q', each element of which is expressible (not necessarily uniquely) as the difference of two elements of Q'.

3. The semigroup of Schreier semigroup extensions of S by Q

Let $Z(Q, S)$ be the set of all factor-systems φ of Schreier extensions of semigroup S by a semigroup Q. Thus $Z(Q, S)$ is the set of all functions $\varphi: Q \times Q \to S$ satisfying **I**, **II**, and **III**. Then $(\varphi + \psi)(a,b) = \varphi(a, b) + \psi(a, b)$ defines an addition under which $Z(Q, S)$ is a semigroup. If S is a group then $Z(Q, S)$ is also a group, regardless of whether Q is a group or not.

Let $B(Q, S) = \{\varphi \in Z(Q, S) | \varphi(a, b) = f(a) + f(b) = f(a+b)$ for some $f: Q \to U(S)\}$. Then $B(Q, S)$ is a subgroup of $Z(Q, S)$; and two elements φ and ψ of $Z(Q, S)$ are factor-systems for equivalent extensions of S by Q if and only if $\varphi \in \psi + B(Q, S)$. In that case φ and ψ are themselves called equivalent. Equivalence is easily shown to be a congruence relation on $Z(Q, S)$, and the factor-semigroup of equivalence-classes is denoted by $H(Q, S) = Z(Q, S)/B(Q, S)$. $H(Q, S)$ is called the semigroup of Schreier extensions of S by Q. If both S and Q are groups,

$H(Q, S)$ is exactly the second co-homology group, $H^2(Q, S) = \mathrm{Ext}^2(Q, S)$, of all abelian group extensions of S by Q.

If S and Q are semigroups and $h: Q \to Q'$ is a homomorphism of Q into a semigroup Q', then $h^{\#}$: $Z(Q', S) \to Z(Q, S)$ defined by $(h^{\#}\varphi')(a, b) = \varphi'(h(a), h(b))$ is easily shown to be a homomorphism of $Z(Q', S)$ into $Z(Q, S)$ for which $h^{\#}(B(Q', S)) \subseteq B(Q, S)$. Therefore h induces a homomorphism h^* of $H(Q', S)$ into $H(Q, S)$, defined by $h^*(\varphi' + B(Q', S)) = h^{\#}\varphi' + B(Q, S)$.

4. The group of Schreier semigroup extensions of a group by a semigroup

Lemma 1. *Let S be a cancellative semigroup, Q a semigroup, and $\varphi \in Z(Q, S)$. If a, b, x, and y are elements of Q such that $a+x = b+x$ and $a+y = b+y$, then $\varphi(a, x) + \varphi(b, y) = \varphi(a, y) + \varphi(b, x)$.*

Proof. $\varphi(a+x, y) + \varphi(a, x) + \varphi(b, y) = \varphi(x, a+y) + \varphi(a, y) + \varphi(b, y)$ (by III) $= \varphi(x, b+y) + \varphi(a, y) + \varphi(b, y) = \varphi(b+x, y) + \varphi(a, y) + \varphi(b, x)$ (by III) $=$ $= \varphi(a+x, y) + \varphi(a, y) + \varphi(b, x)$. Therefore, by cancellation in S, $\varphi(a, x) + \varphi(b, y) =$ $= \varphi(a, y) + \varphi(b, x)$. Q. E. D.

For arbitrary semigroups S and Q let $Z_0(Q, S) = \{\varphi \in Z(Q, S) \mid a+x = b+x$ implies $\varphi(a, x) = \varphi(b, x)\}$, and let $B_0(Q, S) = B(Q, S) \cap Z_0(Q, S)$.

Lemma 2. *If S is a group and Q a semigroup, then each element φ of $Z(Q, S)$ is equivalent to some element φ_0 of $Z_0(Q, S)$.*

Proof. Let $r: Q \to Q$ be any function which is constant on each ν-class in Q, for which $(a, r(a)) \in \nu$ for each $a \in Q$, and for which $r(0) = 0$. Let $g: Q \to S$ be defined by

$$(*)\quad g(a) = \begin{cases} 0 \text{ if } a = 0, \\ \text{arbitrary if } a \in r(Q) \setminus 0, \\ g(r(a)) + \varphi(r(a), x) - \varphi(a, x) \text{ otherwise, where } x \text{ is any element of} \\ Q \text{ such that } r(a) + x = a + x. \end{cases}$$

By Lemma 1, g is single-valued. Let φ_0: $Q \times Q \to S$ be defined by $\varphi_0(a, b) = \varphi(a, b) +$ $+ g(a) + g(b) - g(a+b)$. Then φ_0 is easily shown to be an element of $Z(Q, S)$ which is clearly equivalent to φ. Moreover, if $a+x = b+x$ $(a, b, x \in Q)$ then $r(a) = r(b)$ and, for some $y \in Q$, $a+y = b+y = r(a)+y$. Consequently

$$\begin{aligned}
\varphi_0(a, x) &= \varphi(a, x) + g(a) + g(x) - g(a+x) \text{ (by definition of } \varphi_0) = \\
&= \varphi(a, x) + [g(a) + \varphi(a, y)] - \varphi(a, y) + g(x) - g(a+x) = \\
&= \varphi(a, x) + [g(r(a)) + \varphi(r(a), y)] - \varphi(a, y) + g(x) - g(a+x) \text{ (by } (*)) = \\
&= \varphi(a, x) + [g(b) + \varphi(b, y)] - \varphi(a, y) + g(x) - g(b+x) \text{ (by } (*)) = \\
&= \varphi(b, x) + g(b) + g(x) - g(b+x) \text{ (by Lemma 1)} = \\
&= \varphi_0(b, x) \text{ (by definition of } \varphi_0).
\end{aligned}$$

Thus $\varphi_0 \in Z_0(S, Q)$. Q. E. D.

Lemma 3. *Let S be a group, Q be a semigroup, and h be the natural homomorphism of Q onto its maximal cancellative homomorphic image Q'. Then the induced homomorphism h^* is an isomorphism of $H(Q', S)$ onto $H(Q, S)$.*

Proof. (1) $h^{\#} Z(Q, S) = Z_0(Q, S)$: If $\varphi' \in Z(Q', S)$ and if $a+x = b+x$ (where $a, b, x \in Q$), then $h(a) = h(b)$ and $(h^{\#}\varphi')(a, x) = \varphi'(h(a), h(x)) = \varphi'(h(b), h(x)) = (h^{\#}\varphi')(b, x)$. Thus $h^{\#}(Z(Q', S)) \subseteq Z_0(Q, S)$. Conversely, if $\varphi_0 \in Z_0(Q, S)$, let $\varphi': Q' \times Q' \to S$ be defined by $\varphi'(h(a), h(b)) = \varphi_0(a, b)$. If $h(a) = h(c)$, then there exists $x \in Q$ such that $a+x = c+x$, so that $a+b+x = c+b+x$ for every $b \in Q$. By the defining property of $Z_0(Q, S)$, we then have: (a) $\varphi_0(a, x) = \varphi_0(c, x)$, (b) $\varphi_0(a+b, x) = \varphi_0(b+c, x)$, and (c) $\varphi_0(a, b+x) = \varphi_0(c, b+x)$. Consequently, $\varphi_0(a+b, x) + \varphi_0(a, b) = \varphi_0(b, a+x) + \varphi_0(a, x)$ (by III) $= \varphi_0(b, c+x) + \varphi_0(c, x)$ (by (a)) $= \varphi_0(b+c, x) + \varphi_0(b, c)$ (by III) $= \varphi_0(a+b, x) + \varphi_0(b, c)$ (by (b)). Therefore, by II and cancellation in S, $\varphi_0(a, b) = \varphi_0(c, b)$. If, also, $h(b) = h(d)$, then, by the preceding and II, $\varphi_0(a, b) = \varphi_0(c, b) = \varphi_0(c, d)$. Thus φ' is single-valued. It is easily shown that $\varphi' \in Z(Q', S)$ and that $h^{\#}\varphi' = \varphi_0$. Thus $Z_0(Q, S) \subseteq h^{\#}(Z(Q', S))$.

(2) $h^{\#-1}(B_0(Q, S)) = B(Q', S)$: It is easily shown, by an argument similar to the preceding, that $h^{\#}(B(Q', S)) = B_0(Q, S)$. Suppose $\varphi' \in Z(Q', S)$ is such that $h^{\#}\varphi' \in B_0(Q, S)$. Then for some function $f: Q \to S$, $h^{\#}\varphi'(a, b) = f(a) + f(b) - f(a+b)$ for all $a, b \in Q$. Let $g: Q' \to S$ be defined by $g(h(a)) = f(a)$ for all $a \in Q$. If $h(a) = h(b)$, then $a+c = b+c$ for some $c \in Q$. Consequently, $h^{\#}\varphi'(a, c) = h^{\#}\varphi'(b, c)$ since, by part (1) above, $h^{\#}\varphi' \in Z_0(Q, S)$. Thus $f(a) + f(c) - f(a+c) = f(b) + f(c) - f(b+c) = f(b) + f(c) - f(a+c)$; and hence $f(a) = f(b)$. Hence g is single-valued. From $\varphi'(h(a), h(b)) = h^{\#}\varphi'(a, b) = f(a) + f(b) - f(a+b) = g(h(a)) + g(h(b)) - g(h(a) + h(b))$ (where a, b are arbitrary elements of Q) we see that $\varphi' \in B(Q', S)$.

(3) h^* is an isomorphism of $H(Q', S)$ onto $H(Q, S)$: In § 3 it is indicated (and is easily proved) that h^* is a homomorphism of $H(Q', S)$ into $H(Q, S)$. By Lemma 2, each element of $H(Q, S)$ can be expressed as $\varphi_0 + B(Q, S)$ for some $\varphi_0 \in Z_0(Q, S)$; and by part (1) above, φ_0 can be expressed as $h^{\#}\varphi'$ for some $\varphi' \in Z(Q', S)$. Thus $h^*(H(Q', S)) = H(Q, S)$. If φ' and ψ' are two elements of $Z(Q', S)$ for which $h^*(\varphi' + B(Q', S)) = h^*(\psi' + B(Q', S))$, then $h^{\#}\varphi' + B(Q, S) = h^{\#}\psi' + B(Q, S)$. Therefore $h^{\#}(\varphi' - \psi') = h^{\#}\varphi' - h^{\#}\psi' \in Z_0(Q, S) \cap B(Q, S) = B_0(Q, S)$. Then, by part (2) above, $\varphi' - \psi' \in B(Q', S)$ and hence $\varphi' + B(Q', S) = \psi' + B(Q', S)$; that is, h^* is an isomorphism. Q. E. D.

If S and Q are cancellative semigroups then any Schreier extension T of S by Q is also cancellative and can be embedded in its difference-group T^*.

Lemma 4. *Let S and Q' be cancellative semigroups, $T = T(S, Q', u, \varphi)$, and S^*, Q^*, and T^* be the difference-groups of S, Q', and T, respectively. Then T^* is a Schreier extension of S^* by Q^*, and for a suitable choice of partitioning representatives the corresponding factor-system φ^* is such that $\varphi^* | Q' \times Q' = \varphi$.*

Proof. T^* is clearly an extension of S^* since T^* is an abelian group. Each element of T has a unique representation of the form $u_a + \alpha$ with $a \in Q'$ and $\alpha \in S$, and each element of T^* has a representation of the form $t_1 - t_2$ with $t_1, t_2 \in T$. Therefore, each element of T^* has a representation of the form $(u_a + \alpha) - (u_b + \beta) = (u_a - u_b) + (\alpha - \beta) = u_a - u_b + \alpha^*$ with α^* in S^*. If $u_a - u_b + S^* = u_c - u_d + S^*$ (with $a, b, c, d \in Q^*$), then for some $\alpha, \beta \in S$, $u_a - u_b = u_c - u_d + \alpha - \beta$. But then $u_a + u_d + \beta = u_b + u_c + \alpha$, so that $u_{a+d} + \varphi(a, d) + \beta = u_{b+c} + \varphi(b, c) + \alpha$, and hence $a + d = b + c$. That is, in Q^*, $a - b = c - d$. Conversely, if $a - b = c - d$ in Q^*, then

$$u_a + u_d + \varphi(b, c) = u_b + u_c + \varphi(a, d),$$

and hence, in T^*, $u_a - u_b \in u_c - u_d + S^*$. Therefore, $u_a - u_b + S^* = u_c - u_d + S^*$. Thus there is a one-to-one correspondence between the cosets of S^* in T^* and the elements of Q^*. This correspondence is easily shown to be an isomorphism between the factor-group T^*/S^* and the group Q^*, and therefore T^* is an extension of S^* by Q^*. Clearly, if a set $\{u_a^* | a \in Q^*\}$ of representatives for the cosets of S^* in T^* are chosen so that $u_a^* = u_a$ whenever $a \in Q$, then the corresponding factor-system φ^* will be such that $\varphi^* | Q' \times Q' = \varphi$. Q. E. D.

Theorem 5. *Let S be a group, Q be a semigroup, Q' be the maximal cancellative homomorphic image of Q, and Q^* be the difference-group of Q'. Then*

$$H(Q, S) \cong H(Q', S) \cong H(Q^*, S).$$

Proof. By Lemma 3, the natural homomorphism h of Q onto Q' induces the first of the indicated isomorphisms.

Let k be the embedding mapping of Q' into Q^*, so that $k(a) = a$ for each $a \in Q'$, Then $k^{\#}\varphi^* = \varphi^* | Q' \times Q'$ for each $\varphi^* \in Z(Q^*, S)$. Let $\varphi' \in Z(Q', S)$ and let $T = T(S, Q', u, \varphi')$. By Lemma 4, $T^* = T^*(S, Q^*, u^*, \varphi^*)$ with $\varphi' = \varphi^* | Q' \times Q' = k^{\#}\varphi^*$. Therefore, $k^{\#}(Z(Q^*, S)) = Z(Q', S)$. If φ^* and ψ^* are equivalent elements of $Z(Q^*, S)$, then it is easily shown that $k^{\#}\varphi^*$ and $k^{\#}\psi^*$ are also equivalent. Conversely, if $k^{\#}\varphi^*$ and $k^{\#}\psi^*$ are equivalent elements of $Z(Q', S)$, then there exists an extension T of S by Q for which $T = T(S, Q', u, k^{\#}\varphi^*) = T(S, Q', v, k^{\#}\psi^*)$. Then by Lemma 4 $T^* = T^*(S, Q^*, u^*, \varphi^*) = T^*(S, Q^*, v^*, \psi^*)$ for suitable sets $\{u_a^* | a \in Q^*\}$ and $\{v_a^* | a \in Q^*\}$ of partitioning representatives. That is, φ^* and ψ^* are equivalent. By the Induced Homomorphism Theorem, $k^{\#}$ induces an isomorphism k^* of $H(Q^*, S)$ onto $H(Q', S)$, and the theorem is proved. Q. E. D.

5. Complete semigroups

In the terminology introduced by WIEGANDT [5], a semigroup (cancellative semigroup, group) is called *complete* if it is a direct summand of every Schreier semigroup (cancellative semigroup, group) extension of itself. By the preceding discussion, a semigroup (cancellative semigroup, group) S is complete if and only if $H(Q, S) = 0$ for every semigroup (cancellative semigroup, group) Q. It is well-known that a group is complete if and only if it is divisible (that is, contains, with each element a and each positive integer n, an element x such that $nx = a$). WIEGANDT and this author proved in [6] and [3] that a cancellative semigroup is also complete if and only if it is a divisible group. That proof is superseded and supplemented by the following corollary to Theorem 5.

Theorem 6. *A semigroup (cancellative semigroup) S is complete if and only if it is a divisible group.*

Proof. Let α be any element of S and let n be any positive integer. Let Q be the additive semigroup of the integers reduced modulo n, and let $\varphi \in Z(Q, S)$ be defined by $\varphi(a, b) = \left[\frac{a+b}{n}\right]\alpha$ where $[x]$ denotes the greatest integer in x, and $a+b$ denotes the ordinary sum of a and b. Then an extension $T = T(S, Q, u, \varphi)$ is a semigroup or a cancellative semigroup according as S is a semigroup or a cancellative semigroup. In either case, if S is complete, then $T=S\oplus K$ for some subsemigroup K of T. Therefore there exist elements $\beta\in S$ and $k\in K$ such that $u_1 = \beta+k$. Then $\alpha = nu_1 = n\beta+nk\in S\oplus K$ so that $nk=0$ and $n\beta=\alpha$. Thus S must be divisible. Moreover, $\varphi(Q\times Q)\cong U(S)$ and hence $\alpha = \varphi(n-1, 1)\in U(S)$, so that S is a group.

Conversely, if S is a divisible group, then $H(Q^*, S)=0$ for every group Q^*. Hence, by Theorem 5, $H(Q, S)=0$ for every semigroup Q. That is, a divisible group S is a complete semigroup, a complete cancellative semigroup and a complete group. Q. E. D.

The factor-system φ used in the proof of Theorem 6 is due to WIEGANDT [6].

Theorem 7. *A semigroup Q is such that every extension of itself by a group S is a splitting extension if and only if the difference-group Q^* of the maximal cancellative homomorphic image Q' of Q is a free abelian group.*

Proof. By Theorem 7.2 of EILENBERG and MACLANE [2], $H(Q^*, S)=0$ for every group S if and only if Q^* is a free abelian group. By Theorem 5, $H(Q^*, S)=0$ if and only if $H(Q, S)=0$. Therefore, $H(Q, S)=0$ for every group S if and only if Q^* is a free abelian group. Q. E. D.

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