On one-parameter groups and semi-groups of operators in Hilbert space*)

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We wish to study one-parameter groups of bounded operators $\{T_s\}_{-\infty<s<\infty}$ on a complex Hilbert space H, under the assumption that one of the operators in the group (other than T_0 which is the identity operator) is spectral in the sense of DUNFORD.

Our prinicipal result is that merely uniform boundedness of $\|T_s\|$ for s in finite intervals implies each operator T_s is spectral; further, there exists a bicontinuous operator A such that AT_sA^{-1} all have normal scalar parts, or equivalently, the resolutions of the identity of all the T_s belong to a single bounded Boolean algebra of (not necessarily self-adjoint) projections. We also obtain similar, but weaker, results for semi-groups of operators. These results complement certain results of FOIAŞ [6]; our present work is inspired by this work of FOIAŞ and by the work of one of us [8]. For material on spectral operators, we refer to [2, 3] and the references given there.

An important tool will be a theorem which has been proved in many forms [1].

Theorem. *Let $G=\{g\}$ be a commutative group, and suppose $g\to T_g$ is a uniformly bounded representation of G as operators on H. Then there exists a bicontinuous operator A on H such that AT_gA^{-1} is unitary for every g.*

Our use of this theorem will be the same as FOIAŞ': we will have a bounded Boolean algebra of projections $\{E(\sigma)\colon \sigma$ a Borel set in the plane$\}$, and a uniformly bounded one-parameter group of operators $\{U_s\}$ each of which commutes with every E. Let G be the group of all pairs (s,σ) with the composition $(s,\sigma)\cdot(t,\tau) = (s+t, \sigma\cup\tau-\sigma\cap\tau)$, and the representation $(s,\sigma)\to U_s(I-2E(\sigma))$. If A is an operator given by the theorem, then each operator AU_sA^{-1} is unitary (take σ to be empty so $E(\sigma)=0$), and each $A(I-2E(\sigma))A^{-1} = I-2AE(\sigma)A^{-1}$ is unitary so each $AE(\sigma)A^{-1}$ is self-adjoint (take $s=0$ so $U_s=I$).

To the above theorem we will need the

Scholium. *Let $\mathfrak{L}$ be the class of operators which commute with every T_g. Then the operator A may be chosen so that $\|ALA^{-1}\|\leqq\|L\|$ for every L in $\mathfrak{L}$.*

Proof. In each of the proofs of the above theorem, either a new norm $|||x|||$ is defined as some sort of generalised limit of $\|T_gx\|$ for suitable $g\in G$ and A is chosen

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so that $\|Ax\| = |||x|||$, or else A^2 is found as an operator in the weak operator closed convex hull of the operators $T_g^* T_g$. In either case we choose A to be self-adjoint and positive, and we have in the first instance

$$|||Lx||| = \lim_g \|T_g Lx\| = \lim_g \|LT_g x\| \leqq \|L\| \lim_g \|T_g x\| = \|L\| \, |||x|||$$

so that if we set $y = Ax$, we have

$$\|ALA^{-1} y\| = |||Lx||| \leqq \|L\| \, |||x||| = \|L\| \, \|Ax\| = \|L\| \, \|y\|.$$

In the second instance the computations are essentially the same: for any $x \in H$ and $g \in G$,

$$(T_g^* T_g LA^{-1} x, LA^{-1} x) = (LT_g A^{-1} x, LT_g A^{-1} x) \leqq \|L\|^2 (T_g^* T_g A^{-1} x, A^{-1} x)$$

so that convex combinations of this inequality yield

$$(A^2 LA^{-1} x, LA^{-1} x) \leqq \|L\|^2 (A^2 A^{-1} x, A^{-1} x)$$

or since A is self-adjoint,

$$\|ALA^{-1} x\|^2 \leqq \|L\|^2 \|x\|^2.$$

Theorem 1. *Let $\{T_s\}$ be a one-parameter group of bounded operators on H such that T_1 is a scalar type spectral operator and $\|T_s\|$ is uniformly bounded on finite s-intervals. Then T_s is scalar for every s and there exists a bicontinuous operator A such that AT_sA^{-1} is normal for every s.*

Proof. Let $\mathscr{E} = \{E(\cdot)\}$ be the resolution of the identity for T_1 and define the operators R_s by

$$R_s = \int_{\sigma(T_1)} |\lambda|^s e^{is \arg \lambda} E(d\lambda), \qquad 0 \leqq \arg \lambda < 2\pi.$$

It is clear that $R_n = T_1^n = T_n$ for every integer n; also $\{R_s\}$ is a one-parameter group of operators:

$$R_s R_t = \int_{\sigma(T_1)} |\lambda|^s e^{is \arg \lambda} E(d\lambda) \int_{\sigma(T_1)} |\lambda|^t e^{it \arg \lambda} E(d\lambda)$$

$$= \int_{\sigma(T_1)} |\lambda|^{s+t} e^{i(s+t) \arg \lambda} E(d\lambda) = R_{s+t}.$$

Since T_1 commutes with every T_s, each E in the resolution of the identity of T_1 commutes with every T_s, and so each R_t commutes with every T_s. Thus $\{U_s\}$, defined by $U_s = R_{-s} T_s$, is a one-parameter group of operators. Notice that

$$U_s = R_{-\{s\}} R_{-[s]} T_{[s]} T_{\{s\}} = R_{-\{s\}} T_{\{s\}} = U_{\{s\}}, \qquad s = [s] + \{s\},$$

and so

$$\|U_s\| \leqq \|R_{-\{s\}}\| \, \|T_{\{s\}}\| \leqq 4 \sup_{\lambda \in \sigma(T_1)} |\lambda|^{-1} \cdot \sup_{E \in \varepsilon} \|E\| \cdot \sup_{0 \leqq s \leqq 1} \|T_s\|.$$

Thus $\{U_s\}$ is a uniformly bounded one-parameter group of operators, and each U_s commutes with every $E \in \mathscr{E}$, so there exists a bicontinuous operator A on H such that each AU_sA^{-1} is unitary and each AR_sA^{-1} is normal (because each AEA^{-1} is self-adjoint). Therefore for any s, the operator $AT_sA^{-1} = (AR_sA^{-1})(AU_sA^{-1})$ is the product of a normal operator with a commuting unitary and so is normal.

We remark that by writing s/s_0 in place of s, we could prove this theorem with the assumption that T_{s_0} is spectral for some $s_0 \neq 0$ in place of T_1 spectral. Also it is not difficult to see that uniform boundedness of $\|T_s\|$ on a non-trivial interval implies uniform boundedness on any given finite interval.

By using the same technique, we can prove an analogous theorem for a one-parameter group of operators with T_1 spectral.

Theorem 2. *Let $\{T_s\}$ be a one-parameter group of operators on H such that T_1 is spectral and $\|T_s\|$ is uniformly bounded on finite s-intervals. Then T_s is spectral for every s and there exists a bicontinuous operator A such that AT_sA^{-1} has normal scalar part for every s.*

Proof. Let $\mathcal{E} = \{E\}$ be the resolutions of the identity of T_1 and let N be the quasi-nilpotent part of T_1. We would like to form operators R_s to play the part of T_1^s as in theorem 1, through the use of SCHWARTZ's formula [7]

$$f(T_1) = \sum_{n=0}^{\infty} \frac{N^n}{n!} \int_{\sigma(T_1)} f^{(n)}(\lambda)\, E(d\lambda)$$

valid for functions f analytic on $\sigma(T_1)$. We wish to apply this formula when $f(\lambda) = \lambda^s$, but unfortunately λ^s may not have a single-valued analytic branch on $\sigma(T)$, for non-integral s. However, we proceed boldly and define

$$R_s = \sum_{n=0}^{\infty} \frac{N^n}{n!} s(s-1)\dots(s-n+1) \int_{\sigma(T_1)} |\lambda|^{s-n} e^{i(s-n)\arg\lambda} E(d\lambda), \qquad 0 \leqq \arg\lambda < 2\pi.$$

This series converges in the uniform operator topology, uniformly in any finite s-interval; in fact, if we consider the norm of the n^{th} summand of the series and take n^{th} root, we have at most

$$\|N^n\|^{\frac{1}{n}} \left(\frac{|s|\,|s-1|\dots|s-n+1|}{n!} \right)^{\frac{1}{n}} \Big(4 \sup_{\lambda \in \sigma(T_1)} |\lambda|^{s-n} \sup_{E \in \mathcal{E}} \|E\|\Big)^{\frac{1}{n}}.$$

The first term of this product tends to 0 as n becomes infinite because of the quasi-nilpotency of N. The remaining two terms are bounded in n, uniformly in any finite s-interval. The operators R_s coincide with T_s when s is an integer, and form a one-parameter group:

$$R_s R_t = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \left[\frac{N^k}{k!} s(s-1)\dots(s-k+1) \int_{\sigma(T_1)} |\lambda|^{s-k} e^{i(s-k)\arg\lambda} E(d\lambda) \right] \cdot$$

$$\cdot \left[\frac{N^{n-k}}{(n-k)!} t(t-1)\dots(t-n+k+1) \int_{\sigma(T_1)} |\lambda|^{t-n+k} e^{i(t-n+k)\arg\lambda} E(d\lambda) \right] =$$

$$= \sum_{n=0}^{\infty} \left[\frac{N^n}{n!} \int_{\sigma(T_1)} |\lambda|^{s+t-n} e^{i(s+t-n)\arg\lambda} E(d\lambda) \right] \cdot$$

$$\cdot \left[\sum_{k=0}^{n} \binom{n}{k} s(s-1)\dots(s-k+1)\, t(t-1)\dots(t-n+k+1) \right] = R_{s+t}.$$

Since every T_s commutes with T_1 and hence with N and the E's, every T_s commutes with each R_t, and so $U_s = R_{-s}T_s$ constitute a one-parameter group of operators with $U_s = U_{\{s\}}$ uniformly bounded in norm for all s, as in the proof of theorem 1. The remainder of the theorem follows exactly as before.

There are extensions of these results to the semi-group case; these results show the importance of the existence of inverses to theorems 1 and 2.

Theorem 3. *Let $\{T_s\}$ be a one-parameter semi-group of bounded operators on H such that T_1 is a spectral operator with 0 an isolated point of its spectrum and $\|T_s\|$ is uniformly bounded for $a \leqq s \leqq b$, for some $0 \leqq a < b$. Then T_s is spectral for every s and there exists a bicontinuous operator A such that AT_sA^{-1} has normal scalar part for every s.*

Proof. Let E be the idempotent in the resolution of the identity associated with the non-zero points of $\sigma(T_1)$. Then E commutes with every T_s, so EH is an invariant subspace for every T_s. The operators ET_s constitute a one-parameter semi-group of operators on EH with the property that ET_1 has an inverse, since the spectrum of ET_1 as an operator on EH consists of the non-zero points of the spectrum which are bounded away from zero by hypothesis. The relation $ET_sET_{1-s} = ET_1$ shows that all the operators ET_s, $0 < s < 1$, must have inverses, on EH, and therefore every ET_s must have an inverse. The ET_s, together with their inverses, form a group of operators on EH which has its norms uniformly bounded on finite s-intervals, since its norms are uniformly bounded on one non-degenerate s-interval. It follows from theorem 2 that all the operators ET_s are spectral and that their resolutions of the identity all belong to a uniformly bounded Boolean algebra, since under an equivalent norm on EH, the resolutions of the identity all consist of self-adjoint projections. Clearly, for $s > 0$, the operators ET_s, extended to all of H by being O in $(I-E)H$, are also spectral and their resolutions all belong to a uniformly bounded Boolean algebra.

Now for $s > 0$, $T_s = ET_s + (I-E)T_s(I-E)T_s$ is quasi-nilpotent, for if n be an integer exceeding $1/s$, $[ns] \geqq 1$,

$$[(I-E)T_s]^n = (I-E)T_{[ns]}T_{\{ns\}} = [(I-E)T_1]^{[ns]}T_{\{ns\}}.$$

This is the product of two commuting operators, one of which is quasi-nilpotent, and so is itself quasi-nilpotent; the quasi-nilpotence of $[(I-E)T_s]^n$ implies that of $(I-E)T_s$. Thus T_s is the sum of a spectral operator and a commuting quasi-nilpotent, and so is spectral. The resolution of the identity of T_s consists of the projections F, $F+(I-E)$ where F belongs to the resolution of ET_s. Let A be a bicontinuous operator such that AEA^{-1} and AFA^{-1} are all self-adjoint. Then AT_sA^{-1} all have normal scalar part.

It is possible to prove a weaker theorem, valid for more arbitrary semi-groups of operators. We use the concept of semi-similarity introduced by Feldzamen [4]. Two semi-groups $\{T_s\}$, $\{R_s\}$ on H will be called *semi-similar* if there exist two bounded Boolean algebras of projections $\mathscr{E}$, $\mathscr{F}$ both generated by their atoms $\{E_\alpha\}$, $\{F_\alpha\}$, the elements of $\mathscr{E}$ commuting with every T_s and the elements of $\mathscr{F}$ with every R_s, and there exist bicontinuous operators A_α from $E_\alpha H$ onto $F_\alpha H$ such that $A_\alpha T_s = R_s A_\alpha$ for every α and s.

Theorem 4. *Let $\{T_s\}$ be a semi-group of bounded operators on H such that T_1 is spectral and $\|T_s\|$ is uniformly bounded for $a \leqq s \leqq b$, for some $0 \leqq a < b$. Then $\{T_s\}$ is semi-similar to a semi-group of spectral operators with normal scalar parts.*

Proof. For integers $n \geqq 1$, let E_n be the projection in the resolution of the identity of T_1 associated with the set of complex numbers $\{\lambda: n^{-1} \leqq |\lambda| < (n-1)^{-1}\}$, and let $E_0 = I - \sum_{n=1}^{\infty} E_n$. Since these E's belong to the resolution of T_1, they generate a uniformly bounded Boolean algebra; let A be a bicontinuous operator such that $F_n = AE_nA^{-1}$ is self-adjoint for each n. The operators $AE_nT_sA^{-1}$ on the space F_nH satisfy the hypotheses of theorem 3, so there exist operators B_n bicontinuous from F_nH onto itself such that $B_nAE_nT_sA^{-1}B_n^{-1}$ has normal scalar part for each s. It is important to note that B_n may be so chosen that the norm of $B_nAE_nT_sA^{-1}B_n^{-1}$ is no greater than that of $AE_nT_sA^{-1}$; that is, uniformly bounded in n for each fixed s.

Set now $R_s = \sum_{n=0}^{\infty} B_nAE_nT_sA^{-1}B_n^{-1}$. This is a direct sum of operators on the mutually orthogonal subspaces F_nH with norms uniformly bounded in n; this sum therefore exists in the strong operator topology. Since the summands are spectral operators with normal scalar parts, the same must be true of R_s. $\{R_s\}$ is a one-parameter semi-group, and it is not difficult to see that the operators $A_n = B_nA$ implement the semi-similarity between $\{T_s\}$ and $\{R_s\}$.

We close with some examples to demonstrate the sharpness of our theorems.

Example 1. The underlying space cannot be L_p. In $L_p[0, 1]$, $p \neq 2$, the group of transformations $[T_sx](t) = x(\{t+s\})$ is a strongly continuous group of isometries, and T_1 is the identity. However, FIXMAN [5] has shown that T_s is spectral only for rational s and that the resolutions of the identity of T_s are not bounded uniformly in s for rational s.

Example 2. Let H be square integrable functions on $[0, 1]$ but with the norm of a function x given by $\int_0^1 |x(t)|^2(t+1)\,dt$. The group of transformations $[T_sx](t) = x(\{t+s\})$ is uniformly bounded but not a group of isometries. T_1 is the identity. Thus we have an example to show that T_1 normal does not imply T_s normal, but only scalar.

Example 3. In $L_2[0, 1]$ define the semi-group T_s by

$$[T_sx](t) = \begin{cases} x(t+s), & t+s \leqq 1 \\ 0, & t+s > 1. \end{cases}$$

T_1 is O, and so is normal, but T_s is scalar for no $s < 1$. Thus the semi-group analogue of theorem 1 is not true.

Example 4. In theorem 4, semi-similarity cannot be replaced by ordinary similarity. For $s > 0$, and n a positive integer, consider the 2×2 matrix

$$T_s^{(n)} = e^{-ns}\begin{pmatrix} 1 & n \\ 0 & 0 \end{pmatrix} + e^{-ns+2\pi is}\begin{pmatrix} 0 & -n \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e^{-ns} & ne^{-ns}(1-e^{2\pi is}) \\ 0 & e^{-ns+2\pi is} \end{pmatrix}.$$

The entries of $T_s^{(n)}$ are uniformly bounded in n, s, for $|ne^{-ns}(1-e^{2\pi is})| \leqq ne^{-ns}2\pi s \leqq \leqq 2\pi \sup_{t>0} te^{-t} = 2\pi/e$. We define H as the direct sum of two-dimensional Hilbert spaces $H^{(n)}, n>0$, and define T_s to be the direct sum of the operators $T_s^{(n)}$. T_s is a semi-group and $T_1^{(n)}$ is the 2×2 identity multiplied by e^{-n} so that T_1 is even self-adjoint. However $T_{\frac{1}{2}}$ is not spectral, for $T_{\frac{1}{2}}^{(n)}$ has distinct eigenvalues $e^{-n/2}$ and $-e^{-n/2}$, and the projections corresponding to these eigenvalues have norm greater than n. Therefore $T_{\frac{1}{2}}$ does not have a uniformly bounded resolution of the identity and so cannot be spectral.

Example 5. The condition that $\|T_s\|$ be uniformly bounded in finite s-intervals cannot be dropped. Let $\{\varrho_\alpha\}$ be a Hamel basis for the reals over the rationals. Every real number s can be written uniquely as $s = r_0 + \sum r_\alpha \varrho_\alpha$, r_α rational, where only finitely many r's are non-zero. Distinguish a countable number of the ϱ's, denoted by $\varrho_1, \ldots, \varrho_n, \ldots$. Let $r_n(s)$ denote the coefficient of the distinguished basis element ϱ_n.

Now let H be a countable direct sum of two-dimensional Hilbert spaces $H^{(n)}$, $n>0$. Define T_s to be the direct sum of the operators $T_s^{(n)}$ defined on $H^{(n)}$ by the matrix

$$\begin{pmatrix} e^{2\pi i r_n(s)} & n(e^{2\pi i r_n(s)} - e^{-2\pi i r_n(s)}) \\ 0 & e^{-2\pi i r_n(s)} \end{pmatrix}.$$

Each T_s is bounded in norm because only a finite number of the r's are non-zero; however, the norm of T_s is uniformly bounded in no interval of positive length. Also, T_1 is the identity, but the norms of the resolutions of T_s are not uniformly bounded in s.

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