Ergodic theorems for gages

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To Professor Béla Szőkefalvi-Nagy on his 50th birthday

Introduction

The theory of "non-commutative integration" which summarizes various analogies between the theory of measures and the theory of von Neumann algebras has been investigated by several authors in the last decade (especially *cf.* [3], [8] and [10]).

The purpose of the present work is to extend some of the notions and results of ergodic theory to the case of non-commutative integration.

§ 1 is devoted to general preliminaries. In § 2 a special case of the Riesz convexity theorem is extended to non-commutative L^p-spaces. This result is applied in § 3 where a non-commutative analogue of the concept of measurable transformation is introduced and a non-commutative extension of the von Neumann—Dunford—Miller mean ergodic theorem is given. In § 4 an ergodicity concept for "gages" on a von Neumann algebra A with respect to a group of *-automorphisms of A is introduced, and it is shown that the extreme points of the convex set formed by the probability gages on a von Neumann algebra A, which are invariant under a group of *-automorphisms of A, are precisely the ergodic ones.

The proofs are modelled on the corresponding proofs in the ordinary integration theory supplemented by some devices necessitated by the non-commutative character of the situation. The key role in the course of our proofs is played by a method of J. DIXMIER used in § 3 of [3].

The results of this paper were announced in [6].

§ 1. Definitions and preliminaries

1. Let $\mathfrak{H}$ be a complex Hilbert space. A *von Neumann algebra*[1]) on $\mathfrak{H}$ will mean a self-adjoint algebra of bounded, every-where defined linear operators on $\mathfrak{H}$, which is closed in the weak (or strong) operator topology, and contains the identity operator $I_{\mathfrak{H}}$ of $\mathfrak{H}$[2]). In what follows, A_P will denote the set of the projections of the von Neumann algebra A.

[1]) For the theory of von Neumann algebras *cf.* [4], chap. I, §§ 1—6. Reference to this book in each particular case will be omitted.

[2]) For any Hilbert space $\mathfrak{H}$, $I_{\mathfrak{H}}$ will denote its identity operator.

Let A be a von Neumann algebra. A non-negative valued function φ on A^+ [3]) is called a *trace* on A^+, if it has the following properties:

(i) if $S, T \in \mathsf{A}^+$ and $\lambda, \mu \geqq 0$, then $\varphi(\lambda S + \mu T) = \lambda\varphi(S) + \mu\varphi(T)$;

(ii) for every $T \in \mathsf{A}^+$ and for every unitary operator U in A: $\varphi(UTU^{-1}) = \varphi(T)$.

A trace φ on A^+ is said to be *a) faithful* if the conditions $T \in \mathsf{A}^+$, $\varphi(T) = 0$ imply $T = 0$; *b) normal* if, for every increasing directed set $\mathsf{F} \in \mathsf{A}^+$ with $\sup_{S\in\mathsf{F}} S = T \in \mathsf{A}^+$, we have $\varphi(T) = \sup_{S\in\mathsf{F}} \varphi(S)$; *c) finite* if $\varphi(T) < +\infty$ for every $T \in \mathsf{A}^+$; *d) semi-finite* if, for every $T \in \mathsf{A}^+$, $T \neq 0$ there exists $S \in \mathsf{A}^+$, $S \neq 0$ such that $S \leqq T$ and $\varphi(S) < +\infty$.

Let A be a von Neumann algebra, and let φ be a trace on A^+. The set of elements T of A^+ for which $\varphi(T) < +\infty$, is the positive portion of a two-sided ideal $\mathfrak{m}_\varphi$, called the *two-sided ideal associated with* φ. φ can be uniquely extended to a positive linear form $\dot\varphi$ on $\mathfrak{m}_\varphi$, and for every $S \in \mathfrak{m}_\varphi$, $T \in \mathsf{A}$, we have $\dot\varphi(ST) = \dot\varphi(TS)$. If φ is normal, then for every $S \in \mathfrak{m}_\varphi$ the linear form $T \to \varphi(ST)$ $(T \in \mathsf{A})$ is strongly continuous on the unit sphere of A. If φ is finite, evidently we have $\mathfrak{m}_\varphi = \mathsf{A}$ (in this case $\dot\varphi$ is a positive linear form on A).

Let now φ be a semi-finite faithful normal trace on A^+. For any $S, T \in \mathfrak{m}_\varphi^{\frac{1}{2}}$ [4]), we define $\langle S|T\rangle_{\dot\varphi} = \dot\varphi(T^*S)$. Then $\mathfrak{m}_\varphi^{\frac{1}{2}}$ becomes a unitary algebra [5]) with inner product $\langle S|T\rangle_{\dot\varphi}$. Let $\mathfrak{H}_{\mathfrak{m}_\varphi^{\frac{1}{2}}}$ be the completion of the pre-Hilbert space $\mathfrak{m}_\varphi^{\frac{1}{2}}$. For any $R \in \mathfrak{m}_\varphi^{\frac{1}{2}}$, the mapping $S \to RS$ (resp. $S \to SR$) can be uniquely extended to a bounded linear operator $\Phi(R)$ [resp. $\Psi(R)$] on $\mathfrak{H}_{\mathfrak{m}_\varphi^{\frac{1}{2}}}$. Φ (resp. Ψ) is a *-isomorphism [6]) (resp. *-antiisomorphism), called *canonical *-isomorphism* (resp. **-antiisomorphism*) between A and the left ring $\mathbf{R}^g$ (resp. right ring $\mathbf{R}^d$) of $\mathfrak{m}_\varphi^{\frac{1}{2}}$.

2. Under a *non-commutative measurable space* we shall mean a system $(\mathfrak{H}, \mathsf{A})$ composed of a complex Hilbert space $\mathfrak{H}$ and a von Neumann algebra A on $\mathfrak{H}$. A *gage-space* $(\mathfrak{H}, \mathsf{A}, m)$ is a non-commutative measurable space $(\mathfrak{H}, \mathsf{A})$ with a non-negative valued function m on A_P which is completely additive, unitarily invariant and such that every projection in A is the supremum of the projections on which m is finite. (We say that m is *completely additive*, if $m(P) = \sum_{\iota\in I} m(P_\iota)$ for any set $(P_\iota)_{\iota\in I}$ of mutually orthogonal projections in A with $\sum_{\iota\in I} P_\iota = P$, and we say that m is *unitarily invariant* if for every unitary operator $U \in \mathsf{A}$ and projection $P \in \mathsf{A}_P$, we have $m(UPU^{-1}) = m(P)$.) The function m is called a "gage" (a "non-commutative

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[3]) For any set M of linear operators in a Hilbert space $\mathfrak{H}$, M^+ denotes the *positive portion* of M, i. e. the set of all non-negative symmetric elements of M.

[4]) Let $\mathfrak{m}$ be a two-sided ideal in a von Neumann algebra A. If T runs over $\mathfrak{m}^+$ then T^α $(0 < \alpha < +\infty)$ runs over the positive portion of a uniquely determined two-sided ideal of A: it will be denoted by $\mathfrak{m}^\alpha$ (*cf.* [2]).

[5]) A *unitary algebra* $\mathbf{R}$ is an algebra over the complex numbers, on which an involutive anti-automorphism $x \to x^*$ and an inner product $\langle x|y\rangle$ are defined, such that $\mathbf{R}$ becomes a pre-Hilbert space satisfying the following axioms: *(i)* $\langle x|y\rangle = \langle y^*|x^*\rangle$; *(ii)* $\langle xy|z\rangle = \langle y|x^*z\rangle$; *(iii)* the mapping $x \to xy$ with fixed y is continuous; *(iv)* the set of elements of the form xy is dense in $\mathbf{R}$ (x, y, z arbitrary in $\mathbf{R}$). Let $\mathfrak{H}_\mathbf{R}$ be the completion of the pre-Hilbert space $\mathbf{R}$. For every $x \in \mathbf{R}$ there exists a bounded operator U_x (resp. V_x) on $\mathfrak{H}_\mathbf{R}$ satisfying $U_x y = xy$ (resp. $V_x y = yx$) for every $y \in \mathbf{R}$. The weak (or strong) closure of the operators U_x (resp. V_x) is a von Neumann algebra $\mathbf{R}^g$ (resp. $\mathbf{R}^d$), called the *left* (resp. *right*) *ring* of $\mathbf{R}$. The commutant $(\mathbf{R}^g)'$ of $\mathbf{R}^g$ is identical with $\mathbf{R}^d$ (*cf.* [4], chap. I, § 5).

[6]) A **-isomorphism* is an isomorphism (in algebraical sense) preserving the adjunction.

measure") of A. It is evident that the restriction on A_P of a semi-finite normal trace on A^+ is a gage of A. Conversely, one can show (cf. [1]) that every gage of A can be uniquely extended to a semi-finite normal trace on A^+. For any gage m, φ_m will denote this extension.

A gage space $(\mathfrak{H}, A, m)$ is said to be *finite* (resp. *regular*) if φ_m is finite (resp. faithful).

In any gage space $(\mathfrak{H}, A, m)$ there exists, by virtue of the complete additivity of m, a maximal among those projections of A on which m vanishes; let it be denoted by F_m. It belongs to the centre of A. $I_{\mathfrak{H}} - F_m$ is called the *support* of m. In the following it will be denoted by E_m. Then for every $P \in A_P$ we have $m(E_m P) = m(P)$. $(\mathfrak{H}, A, m)$ is regular if and only if $E_m = I_{\mathfrak{H}}$.

Let $(\mathfrak{H}, A)$ be a non-commutative measurable space. A closed linear operator T on $\mathfrak{H}$ is said to be "measurable" with respect to A if:

(i) T is affiliated[7]) with A;

(ii) there exists a sequence $\{P_n\}_{n=1}^{\infty}$ of projections of A such that, for every n, $P_n \mathfrak{H} \subset \mathfrak{D}_T$ ($\mathfrak{D}_T$ denotes the domain of T), $I_{\mathfrak{H}} - P_n$ is algebraically finite[8]), and $I_{\mathfrak{H}} - P_n \downarrow 0$ strongly $(n \to \infty)$. It is evident that $A \subset \mathfrak{B}(A)$. Defining the "strong sum" and "strong product" of any two $S, T \in \mathfrak{B}(A)$ by the closure of their usual sum and product, respectively, $\mathfrak{B}(A)$ is a selfadjoint algebra relative to the strong sum and product, the usual operation of multiplication by scalars, and the adjunction. In what follows, when sum or product of measurable operators occurs, always the strong sum or strong product is understood, respectively.

Let $(\mathfrak{H}, A, m)$ be a gage space. For every $T \in \mathfrak{B}(A)^+$, we put

$$m(T) = \sup_{S \in \mathfrak{m}_{\varphi_m}^+,\, S \leqq T} \varphi_m(S).$$

Then m can be uniquely extended to a complex (possibly infinite) valued linear form on $\mathfrak{B}(A)$ (identical with φ_m on $\mathfrak{m}_{\varphi_m}$), designated by the same letter m. An element $T \in \mathfrak{B}(A)$ is said to be *integrable* (with respect to m) if $m(|T|) < +\infty$[9]). An element $T \in \mathfrak{B}(A)$ is said to be p^{th} *power integrable* if $|T|^p$ is integrable. Let $\mathbf{L}^p(m)$ $(1 \leqq p < +\infty)$ denote the set of all p^{th} power integrable operators of $\mathfrak{B}(A)$. The $\mathbf{L}^p$-norm of $T \in \mathbf{L}^p(m)$ is defined as $[m(|T|^p)]^{1/p}$, and denoted by $\|T\|_p$.

Let $(\mathfrak{H}, A, m)$ be a regular gage space. Then, for every $1 \leqq p < +\infty$, $\mathbf{L}^p(m)$ is a Banach space with the $\mathbf{L}^p$-norm defined above. Further we have:

(i) $\mathfrak{m}_{\varphi_m}^{1/p}$ is dense in $\mathbf{L}^p(m)$ $(1 \leqq p < +\infty)$;

(ii) if $1 < p < +\infty$ and $\dfrac{1}{p} + \dfrac{1}{q} = 1$, there is an isometric isomorphism between the dual space $[\mathbf{L}^p(m)]^*$ of $\mathbf{L}^p(m)$ and $\mathbf{L}^q(m)$ in which corresponding elements

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[7]) A closed, densely defined linear oparator T in a Hilbert space $\mathfrak{H}$ is said to be *affiliated* with a von Neumann algebra A on $\mathfrak{H}$ (in sign $T\eta A$) if it commutes with every operator of A′.

[8]) A projection $P \in A$ is called *algebraically finite* if there exists no partially isometric operator $V \in A$ such that $V^*V = P$, $VV^* = Q < P$.

[9]) Every closed densely defined operator T in a Hilbert space can be uniquely written as a product of a partially isometric operator with the closure of the range of $|T| = (T^*T)^{\frac{1}{2}}$ as initial domain and the closure of the range of T as final domain. The decomposition $T = W|T|$ is called the *polar decomposition* of T. If $T\eta A$ (A being a von Neumann algebra), then $W \in A$, $|T|\eta A$. Hence, if $T \in \mathfrak{B}(A)$, we have $|T| \in \mathfrak{B}(A)$.

$F \in [\mathbf{L}^p(m)]^*$ and $S_F \in \mathbf{L}^q(m)$ are related by the identity

$$F(T) = m(TS_F), \quad T \in \mathbf{L}^p(m).$$

The dual space $\mathbf{L}^\infty(m)$ of $\mathbf{L}^1(m)$ is identical with the Banach space A considered with the usual operator norm;

(iii) if $\frac{1}{p} + \frac{1}{q} = 1$, where $1 \leqq p,\ q \leqq +\infty$, then $m(ST) = m(TS)$ for $S \in \mathbf{L}^p(m)$, $T \in \mathbf{L}^q(m)$;

(iv) $|m(T_1 T_2 \dots T_n)| \leqq m(|T_1 T_2 \dots T_n|) \leqq \|T_1\|_{p_1} \|T_2\|_{p_2} \dots \|T_n\|_{p_n}$, $T_i \in \mathbf{L}^{p_i}(m)$ with $\sum_{i=1}^{n} \frac{1}{p_i} = 1$, $p_i \geqq 1$ $(i = 1, 2, \dots, n)$.

For the enumerated facts concerning the theory of the non-commutative integration, we refer the reader to [3], [8] and [10].

Let $(\mathfrak{H}, A, m)$ be a gage space. An element T of A is said to be *quasi-simple* if it has the form $T = VT_0$ where T_0 is a finite linear combination of mutually orthogonal projections in $\mathfrak{m}_{\varphi_m}$: $T_0 = \sum_{j=1}^{n} \lambda_j P_j$, $P_i P_k = 0$ $(i \neq k)$, $P_j \in \mathfrak{m}_{\varphi_m}$, and V is a partially isometric operator in A whose initial domain contains the subspace $(P_1 + \dots + P_n)\mathfrak{H}$. It is easy to see that for a quasi-simple element $T = VT_0 = V \sum_{j=1}^{n} \lambda_j P_j$ we have

$$|T| = \sum_{j=1}^{n} |\lambda_j| P_j;$$

if $1 \leqq p < +\infty$

$$|T|^p = \sum_{j=1}^{n} |\lambda_j|^p P_j \quad \text{and} \quad \|T\|_p = \left[\sum_{j=1}^{n} |\lambda_j|^p m(P_j) \right]^{\frac{1}{p}};$$

further

$$\|T\|_\infty = \|T\| = \sup(|\lambda_1|, \dots, |\lambda_n|).$$

In what follows the terms and symbols introduced here will be used without further references.

§ 2. A convexity theorem for finite regular gage spaces

The following lemma which will be often applied throughout this paper is due to J. Dixmier (*cf.* [3], § 3). For the convenience of the reader we recall its proof.

Lemma 2. 1. *Let $(\mathfrak{H}, A, m)$ be a regular gage space. Then the set of the quasi-simple elements of A is dense in $\mathbf{L}^p(m)$ for $1 \leqq p < +\infty$.*

Proof. As $\mathfrak{m}_{\varphi_m}$ is dense in $\mathbf{L}^p(m)$ for $1 \leqq p < +\infty$ (*cf.* § 1.), it is enough to show that every element of $\mathfrak{m}_{\varphi_m}$ is the limit in $\mathbf{L}^p$-norm of a sequence of quasi-simple elements of A.

Let T be an arbitrary element of $\mathfrak{m}_{\varphi_m}$. Let $T = W|T|$ be the polar decomposition of T. Using the spectral representation of $|T|$, we can determine a sequence $\{T_n\}_{n=1}^{\infty}$

of elements of A^+ commuting with T such that: 1) $0 \leqq T_n \leqq I_{\mathfrak{H}}$; 2) TT_n is quasi-simple for every $n=1, 2, \ldots$; 3) $T_n \uparrow I_{\mathfrak{H}}$ strongly as $n \to \infty$.

By the uniqueness of the polar decomposition, we can see that $|T-TT_n| = |T(I_{\mathfrak{H}}-T_n)| = |T|(I_{\mathfrak{H}}-T_n)$. Therefore $\|T-TT_n\|_p^p = m(|T|^p(I_{\mathfrak{H}}-T_n)^p) = \varphi_m(|T|^p(I_{\mathfrak{H}}-T_n)^p)$. As $|T|^p \in \mathfrak{m}_{\varphi_m}$, $0 \leqq (I_{\mathfrak{H}}-T_n)^p \leqq I_{\mathfrak{H}}$ and $(I_{\mathfrak{H}}-T_n) \downarrow 0$ strongly, we have $\|T-TT_n\|_p \to 0$ as $n \to \infty$ (*cf.* § 1.)

To facilitate the statement of the next lemma which is a companion result to Lemma 7 of VI. 10 of [5], it will be convenient to introduce the following notations.

Definition 2. 1. Let $(\mathfrak{H}, A, m)$ be a finite regular gage space. If $a \in R^1$ [10]) and $a>0$, we define $A(a)$ to be the set of quasi-simple elements T of A for which

$$(*) \qquad m(|T|^{\frac{1}{a}}) \leqq 1.$$

If $a=0$, the condition $(*)$ is replaced by

$$(**) \qquad \|T\| \leqq 1.$$

Definition 2. 2. Let $(\mathfrak{H}^{(j)}, A^{(j)}, m^{(j)})$ be a finite regular gage space for each $j=1, 2$. Let $\mathscr{A}$ be the product of $A^{(1)}$ and $A^{(2)}$: $\mathscr{A} = A^{(1)} \times A^{(2)}$. If $\mathbf{a}=(a_1, a_2) \in R^2$ with $a_1 \geqq 0$, $a_2 \geqq 0$, we define $\mathscr{A}(\mathbf{a})$ to be the set of all elements $\mathbf{T}=(T_1, T_2)$ of $\mathscr{A}$ with $T_j \in A^{(j)}(a_j)$.

Lemma. 2. 2. *With the notations of the preceding definitions, let F be a complex valued bilinear form on $\mathscr{A} = A^{(1)} \times A^{(2)}$ and let*

$$(1) \qquad M(\mathbf{a}) = \sup_{\mathbf{S}\in\mathscr{A}_1, \mathbf{T}\in\mathscr{A}(\mathbf{a})} |F(\mathbf{ST})|. \text{ [11])}$$

Then $\log M(\mathbf{a})$ *is a convex function*[12]) *of* $\mathbf{a}=(a_1, a_2)$ *for* $0 \leqq a_1 \leqq 1$, $0 \leqq a_2 \leqq 1$.

Proof. Let $\mathscr{A}^+(\mathbf{a})$ denote the totality of all $\mathbf{T}=(T_1, T_2)$ in $\mathscr{A}(\mathbf{a})$ for which $T_1 \geqq 0$, $T_2 \geqq 0$. First we prove that

$$(2) \qquad M(\mathbf{a}) = \sup_{\mathbf{S}\in\mathscr{A}_1, \mathbf{T}\in\mathscr{A}^+(1)} |F(\mathbf{ST}^{\mathbf{a}})|,$$

where $\mathbf{T}^{\mathbf{a}} = (T_1^{a_1}, T_2^{a_2})$ and $\mathscr{A}(\mathbf{1}) = \mathscr{A}(1, 1)$. To see this we have to show that the sets $\mathfrak{M} = \{\mathbf{ST}\}_{\mathbf{S}\in\mathscr{A}_1, \mathbf{T}\in\mathscr{A}(\mathbf{a})}$ and $\mathfrak{N} = \{\mathbf{ST}^{\mathbf{a}}\}_{\mathbf{S}\in\mathscr{A}_1, \mathbf{T}\in\mathscr{A}(1)}$ are identical. Let $\mathbf{T}=(T_1, T_2) = \left(\sum_{j=1}^{n_1} \lambda_j^{(1)} P_j^{(1)}, \sum_{j=1}^{n_2} \lambda_j^{(2)} P_j^{(2)}\right)$ be an arbitrary element of $\mathscr{A}^+(\mathbf{1})$. Then, for every $\mathbf{a}=(a_1, a_2)$ with $a_1 \geqq 0, a_2 \geqq 0$, $\mathbf{T}^{\mathbf{a}} = (T_1^{a_1}, T_2^{a_2}) = \left(\sum_{j=1}^{n_1} (\lambda_j^{(1)})^{a_1} P_j^{(1)}, \sum_{j=1}^{n_2} (\lambda_j^{(2)})^{a_2} P_j^{(2)}\right)$ is an element of $\mathscr{A}^+(\mathbf{a}) \subset \mathscr{A}(\mathbf{a})$, and it follows that $\mathfrak{N} \subset \mathfrak{M}$. Let now $\mathbf{T}' = (T_1', T_2') =$

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[10]) R^k $(k=1, 2)$ denotes the k-dimensional real Euclidean space.

[11]) For a von Neumann algebra A, A_1 denotes its unit sphere.

[12]) Let C be a convex subset of R^2, and let M be a function defined on C having values which are either real or $+\infty$. M is said to be convex if for any $u, v \in C$

$$M(\alpha u + (1-\alpha)v) \leqq \alpha M(u) + (1-\alpha) M(v)$$

whenever $0 \leqq \alpha \leqq 1$.

$=\left(V_1 \sum_{j=1}^{n_1} \lambda_j^{(1)} P_j^{(1)}, V_2 \sum_{j=1}^{n_2} \lambda_j^{(2)} P_j^{(2)}\right)$ be arbitrary in $\mathscr{A}(\mathbf{a})$. First suppose that $a_1>0, a_2>0$. It is evident that

$$T_1'=V_1\left(\sum_{j=1}^{n_1} e^{i \arg \lambda_j^{(1)}} P_j^{(1)}\right)|T_1'|, \quad T_2'=V_2\left(\sum_{j=1}^{n_2} e^{i \arg \lambda_j^{(2)}} P_j^{(2)}\right)|T_2'|.$$

Putting

$$\mathbf{S}=(S_1, S_2)=\left(V_1 \sum_{j=1}^{n_1} e^{i \arg \lambda_j^{(1)}} P_j^{(1)}, V_2 \sum_{j=1}^{n_2} e^{i \arg \lambda_j^{(2)}} P_j^{(2)}\right),$$

$$\mathbf{T}=(T_1, T_2)=(|T_1'|^{\frac{1}{a_1}}, |T_2'|^{\frac{1}{a_2}}),$$

we have $\mathbf{S}\in\mathscr{A}_1, \mathbf{T}\in\mathscr{A}^+(\mathbf{1})$ and $\mathbf{T}'=\mathbf{S}\mathbf{T}^{\mathbf{a}}$. If $a_1=a_2=0$, then we have $\mathbf{T}'\in\mathscr{A}_1$ and $\mathbf{T}'=\mathbf{T}'\mathbf{T}^{\mathbf{0}}$ for every $\mathbf{T}\in\mathscr{A}^+(\mathbf{1})\,(\mathbf{0}=(0,0))$. As $\mathscr{A}_1=\mathscr{A}_1\mathscr{A}_1$, it follows that $\mathfrak{M}\subset\mathfrak{N}$. The cases when either $a_1>0,\ a_2=0$ or $a_1=0,\ a_2>0$, can be treated by a similar way. Hence $\mathfrak{M}=\mathfrak{N}$ which proves (2).

Let now $\mathbf{b}=(b_1, b_2)\in R^2$ and $\mathbf{T}=(T_1, T_2)\in\mathscr{A}^+(\mathbf{1})$ be arbitrary. Put $\mathbf{T}^{i\mathbf{b}}=(T_1^{ib_1}, T_2^{ib_2})$. Then for every $\mathbf{a}=(a_1, a_2)\in R^2$ with $a_1\geqq 0, a_2\geqq 0$, we have

$$\sup_{\mathbf{S}\in\mathscr{A}_1, \mathbf{T}\in\mathscr{A}^+(\mathbf{1})} |F(\mathbf{S}\mathbf{T}^{\mathbf{a}+i\mathbf{b}})| = \sup_{\mathbf{S}\in\mathscr{A}_1, \mathbf{T}\in\mathscr{A}^+(\mathbf{1})} |F(\mathbf{S}\mathbf{T}^{i\mathbf{b}}\mathbf{T}^{\mathbf{a}})| \leqq M(\mathbf{a}).$$

Therefore,

$$\sup_{\mathbf{b}\in R^2}\{\sup_{\mathbf{S}\in\mathscr{A}_1, \mathbf{T}\in\mathscr{A}^+(\mathbf{1})} |F(\mathbf{S}\mathbf{T}^{\mathbf{a}+i\mathbf{b}})|\} \leqq M(\mathbf{a}).$$

On the other hand, it is clear that

$$\sup_{\mathbf{b}\in R^2}\{\sup_{\mathbf{S}\in\mathscr{A}_1, \mathbf{T}\in\mathscr{A}^+(\mathbf{1})} |F(\mathbf{S}\mathbf{T}^{\mathbf{a}+i\mathbf{b}})|\} \geqq M(\mathbf{a}).$$

Hence

$$\sup_{\mathbf{b}\in R^2}\{\sup_{\mathbf{S}\in\mathscr{A}_1, \mathbf{T}\in\mathscr{A}^+(\mathbf{1})} |F(\mathbf{S}\mathbf{T}^{\mathbf{a}+i\mathbf{b}})|\} = M(\mathbf{a}).$$

Let now $\mathbf{T}=(T_1, T_2)\in\mathscr{A}^+(\mathbf{1})$ be arbitrary. Then, for every $\mathbf{b}=(b_1, b_2)\in R^2$ and $\mathbf{a}=(a_1, a_2)\in R^2$ with $a_1\geqq 0, a_2\geqq 0$, $\mathbf{T}^{\mathbf{a}+i\mathbf{b}}=(T_1^{a_1+ib_1}, T_2^{a_2+ib_2})$ belongs to $\mathscr{A}(\mathbf{a})$. In fact, if $T_k=\sum_{j=1}^{n_k} \lambda_j^{(k)} P_j^{(k)}$ $(k=1,2)$, then $T_k^{a_k+ib_k}=\sum_{j=1}^{n_k} (\lambda_j^{(k)})^{a_k+ib_k} P_j^{(k)}$ which means that $T_k^{a_k+ib_k}$ is a quasi-simple element of $A^{(k)}$. Further, $m^{(k)}(|T_k^{a_k+ib_k}|^{\frac{1}{a_k}})\leqq \leqq \|e^{ib_k\log T_k}\|\, m^{(k)}(|T_k|)\leqq 1$ if $a_k>0$, and $\|T_k^{a_k+ib_k}\|=\|e^{ib_k\log T_k}\|=1$ if $a_k=0$ [13]). Consequently,

$$|F(\mathbf{S}\mathbf{T}^{\mathbf{a}+i\mathbf{b}})|\leqq \sup_{\mathbf{b}\in R^2}|F(\mathbf{S}\mathbf{T}^{\mathbf{a}+i\mathbf{b}})|\leqq M(\mathbf{a}).$$

It follows that

$$M(\mathbf{a})=\sup_{\mathbf{b}\in R^2}\{\sup_{\mathbf{S}\in\mathscr{A}_1, \mathbf{T}\in\mathscr{A}^+(\mathbf{1})} |F(\mathbf{S}\mathbf{T}^{\mathbf{a}+i\mathbf{b}})|\}\leqq$$

$$\leqq \sup_{\mathbf{b}\in R^2}\{\sup_{\mathbf{S}\in\mathscr{A}_1, \mathbf{T}\in\mathscr{A}^+(\mathbf{1}), \mathbf{b}\in R^2} [\sup |F(\mathbf{S}\mathbf{T}^{\mathbf{a}+i\mathbf{b}})|]\} =$$

$$= \sup_{\mathbf{S}\in\mathscr{A}_1, \mathbf{T}\in\mathscr{A}^+(\mathbf{1})} \{\sup_{\mathbf{b}\in R^2} |F(\mathbf{S}\mathbf{T}^{\mathbf{a}+i\mathbf{b}})|\}\leqq M(\mathbf{a}).$$

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[13]) We may suppose that $T_k>0$ for $k=1, 2$.

Hence we have

(3) $$M(\mathbf{a}) = \sup_{S\in\mathscr{A}_1, T\in\mathscr{A}^+(1)} \{\sup_{\mathbf{b}\in R^2} |F(\mathbf{ST}^{\mathbf{a}+i\mathbf{b}})|\}.$$

Let $\mathbf{z}=(z_1, z_2)=(a_1+ib_1, a_2+ib_2)$ and let $\mathbf{T}=\left(\sum_{j=1}^{n_1}\lambda_j^{(1)}P_j^{(1)}, \sum_{j=1}^{n_2}\lambda_j^{(2)}P_j^{(2)}\right)\in$ $\in\mathscr{A}^+(\mathbf{1})$. We may suppose that every $\lambda_j^{(k)}>0$ ($j=1, \dots, n_k$; $k=1,2$). Then, for every $S=(S_1, S_2)\in\mathscr{A}_1$, we have

$$\mathbf{ST}^{\mathbf{z}} = \left(\sum_{j=1}^{n_1}(\lambda_j^{(1)})^{z_1}S_1P_j^{(1)}, \sum_{j=1}^{n_2}(\lambda_j^{(2)})^{z_2}S_2P_j^{(2)}\right) =$$

$$= \left(\sum_{j=1}^{n_1} e^{z_1\log\lambda_j^{(1)}}S_1P_j^{(1)}, \sum_{j=1}^{n_2} e^{z_2\log\lambda_j^{(2)}}S_2P_j^{(2)}\right).$$

As F is bilinear, $F(\mathbf{ST}^{\mathbf{z}})$ can be written as a finite sum $\sum_n f_n(z_1, z_2)F(\mathbf{B}_n)$ with $\mathbf{B}_n\in\mathscr{A}$, where $f_n(z_1, z_2)$ is an analytic[14]) function of the complex variables (z_1, z_2) and is bounded on the strip $0\leqq a_j\leqq 1$ ($i=1,2$). Hence, by VI. 10. 4 and VI. 10. 2 of [5], and the increasing nature of the logarithm, we obtain that

$$\log M(\mathbf{a}) = \sup_{S\in\mathscr{A}_1, T\in\mathscr{A}^+(1)} \log\{\sup_{\mathbf{b}\in R^2}|F(\mathbf{ST}^{\mathbf{a}+i\mathbf{b}})|\}$$

is a convex function of $\mathbf{a}=(a_1, a_2)$ for $0\leqq a_1\leqq 1$, $0\leqq a_2\leqq 1$. Hence Lemma 2. 2 is proved.

The next theorem can be considered as a non-commutative extension of a special case of the Riesz convexity theorem (*cf.* [5], VI. 10. 11).

Theorem 2. 1. *Let* $(\mathfrak{H}, \mathsf{A}, m)$ *be a finite regular gage space, and let* Φ *be a linear mapping of* A *into itself. If for a given* p $(1\leqq p\leqq +\infty)$ Φ *has an extension to a bounded linear mapping of the Banach space* $\mathbf{L}^p(m)$ *into itself, let* $\|\Phi\|_p$ *denote the norm of this extension; if no such extension exists, let* $\|\Phi\|_p=+\infty$. *Then* $\log\|\Phi\|_{1/a}$ *is a convex function of* a *for* $0\leqq a\leqq 1$.

Proof. It is evident that

$$F(\mathbf{T}) = F(T_1, T_2) = m(\Phi(T_1)T_2)$$

is a complex valued bilinear form on $\mathscr{B}=\mathsf{A}\times\mathsf{A}$. Let $a=\dfrac{1}{p}$ [$(1\leqq p\leqq +\infty$; $a=0$ if $p=+\infty$), and let]

$$M(a, 1-a) = \sup_{S\in\mathscr{B}_1, T\in\mathscr{B}(a,1-a)} |F(\mathbf{ST})|.$$

Now we are going to show that $\|\Phi\|_{1/a}\equiv M(a, 1-a)$. If both of $M(a, 1-a)$ and $\|\Phi\|_{1/a}$ are identically infinite, our assertion is trivial. Therefore, we show that

[14]) Let G be an open set in the space of the complex variables (z_1, z_2). A complex valued function f defined on G is said to be analytic in G if f is continuous and the first partial derivates $\partial f/\partial z_i$ $(i=1,2)$ exist at each point of G.

$M(a, 1-a)$ is finite if and only if $\|\Phi\|_1$ is finite and in this case we have $M(a, 1-a) = \|\Phi\|_{1/a}$.

For any $a \in [0, 1]$ we have

$$M(a, 1-a) = \sup_{\mathbf{S}\in\mathscr{B}_1,\ \mathbf{T}\in\mathscr{B}(a,1-a)} |F(\mathbf{ST})| = \sup_{\mathbf{S}\in\mathscr{B}_1,\ \mathbf{T}\in\mathscr{B}(a,1-a)} |m(\Phi(S_1T_1)S_2T_2)| \leqq$$

$$\leqq \sup_{\mathbf{S}\in\mathscr{B}_1,\ \mathbf{T}\in\mathscr{B}(a\ 1-a)} \|S_2\|\,\|T_2\|_{p'}\,\|\Phi(S_1T_1)\|_p \leqq$$

$$\leqq \sup_{\mathbf{S}\in\mathscr{B}_1,\ \mathbf{T}\in\mathscr{B}(a,1-a)} \|\Phi(S_1T_1)\|_p \qquad \left(p = \frac{1}{a}\right).$$

As $\|S_1T_1\|_p \leqq \|S_1\|\|T_1\|_p \leqq 1$ and A_1 contains the identity operator, by virtue of Lemma 2.1 we have

$$\|\Phi\|_p = \sup_{T_1\in\mathscr{A}(a)} \|\Phi(T_1)\|_p \leqq \sup_{\mathbf{S}\in\mathscr{B}_1,\ \mathbf{T}\in\mathscr{B}(a,1-a)} \|\Phi(S_1T_1)\|_p \leqq$$

$$\leqq \sup_{T\in\mathscr{A},\ \|T\|_p\leqq 1} \|\Phi(T)\|_p = \|\Phi\|_p,$$

which implies $\displaystyle \sup_{\mathbf{S}\in\mathscr{B}_1,\ \mathbf{T}\in\mathscr{B}(a,1-a)} \|\Phi(S_1T_1)\|_p = \|\Phi\|_p.$

Hence $\displaystyle M(a, 1-a) \leqq \sup_{\mathbf{S}\in\mathscr{B}_1,\ \mathbf{T}\in\mathscr{B}(a\ 1-a)} \|\Phi(S_1T_1)\| = \|\Phi\|_p.$

It follows that if $\|\Phi\|_{1/a}$ is finite for a given $a \in [0, 1]$, then $M(a, 1-a)$ is finite and $M(a, 1-a) \leqq \|\Phi\|_{1/a}$. Conversely, suppose that $M(a, 1-a)$ is finite for some a in $[0, 1]$. Let $p = \frac{1}{a}$ ($p = +\infty$ if $a = 0$), and let $\frac{1}{p} + \frac{1}{p'} = 1$. Then for every quasi-simple element T_1' of A, the linear form

$$H_{T_1'}(R) = m(\Phi(T_1')R) \quad (R \in \mathsf{A})$$

is bounded in $\mathbf{L}^{p'}$-norm on a dense subset of $\mathbf{L}^{p'}(m)$, namely on $\mathsf{A} \subset \mathbf{L}^{p'}(m)$. In fact,

$$\|H_{T_1'}\|_{p'} = \sup_{R\in\mathscr{A}(1-a)} |m(\Phi(T_1')R)| = \sup_{R\in\mathscr{A}(1-a)} \|T_1'\|_p \left| m\left(\Phi\left(\frac{T_1'}{\|T_1'\|_p}\right)R\right)\right| \leqq$$

$$\leqq \|T_1'\|_p \sup_{\mathbf{S}\in\mathscr{B}_1,\ \mathbf{T}\in\mathscr{B}(a,1-a)} |m(\Phi(S_1T_1)S_2T_2)| = \|T_1'\|_p M(a, 1-a)$$

$(\mathbf{S} = (S_1, S_2),\ \mathbf{T} = (T_1, T_2))$. Consequently, $H_{T_1'}$ can be uniquely extended to all $\mathbf{L}^{p'}(m)$, i. e. $H_{T_1'} \in (\mathbf{L}^{p'}(m))^*$. Hence there exists an element $Q \in \mathbf{L}^p(m)$ with $\|Q\|_p \leqq M(a, 1-a)\,\|T_1'\|_p$ (*cf.* § 1) such that

$$H_{T_1'}(R) = m(\Phi(T_1')R) = m(QR)$$

for all $R \in \mathbf{L}^{p'}(m)$. It follows (*cf.* § 1) that $Q = \Phi(T_1)$. Hence

$$\|\Phi(T_1')\|_p \leqq M(a, 1-a)\,\|T_1'\|_p$$

for all quasi-simple elements T_1' in A. Hence

$$\|\Phi\|_p \leqq M(a, 1-a)$$

and we can conclude that $\|\Phi\|_{1/a} \equiv M(a, 1-a)$ for $0 \leqq a \leqq 1$. By Lemma 2. 2,

$$\log M(\mathbf{a}) = \log [\sup_{S \in \mathscr{B}_1,\, T \in \mathscr{B}(\mathbf{a})} |F(ST)|]$$

is a convex function of $\mathbf{a} = (a_1, a_2)$ for $0 \leqq a_1 \leqq 1$, $0 \leqq a_2 \leqq 1$, therefore $\log \|\Phi\|_{1/a}$ is also convex for $0 \leqq a \leqq 1$, and the proof is completed.

§ 3. The non-commutative mean-ergodic theorem

We begin this section by giving a non-commutative analogue of the concept of measurable transformation.

Let (X, S) be a "measurable space", i. e. a set X and a σ-algebra S of subsets of X. Denote by $\mathfrak{B}(X)$ the algebra of all complex valued functions $f(x)$ defined on X which are measurable with respect to S. Let T be a measurable transformation of (X, S), i. e. a mapping of X into itself such that the inverse image of every element of S by T belongs to S. By $f(x) \to \theta(f(x)) = f(Tx)$, T defines an endomorphism θ of $\mathfrak{B}(X)$. By the nature of the theory of gage spaces as a non-commutative extension of the classical theory of integration over an abstract measure space, it will be natural to define a non-commutative measurable trasformation as a mapping of the set of all measurable operators into itself with analogous algebraical and topological properties as θ.

Definition 3. 1. Let $(\mathfrak{H}, A)$ be a non-commutative measurable space. A *measurable transformation* of $(\mathfrak{H}, A)$ is a *-endomorphism (homomorphism into itself which preserves the adjunction) θ of $\mathfrak{B}(A)$ with the following properties:

(i) $\theta(I_{\mathfrak{H}}) = I_{\mathfrak{H}}$;

(ii) the restriction of θ to A is a normal[15]) *-endomorphism of A sending the set of all algebraically finite projections of A into itself. An invertible measurable transformation of $(\mathfrak{H}, A)$ is a *-automorphism of $\mathfrak{B}(A)$, whose restriction to A is a *-automorphism of A.

It follows immediately from the preceding definition the

Proposition 3. 1. *Let $(\mathfrak{H}, A)$ be a non-commutative measurable space and let θ be a measurable transformation of $(\mathfrak{H}, A)$. If a sequence $\{T_n\}_{n=1}^{\infty}$ of elements of $\mathfrak{B}(A)$ converges nearly everywhere[16]) (relative to A) to an element T of $\mathfrak{B}(A)$ then $\{\theta(T_n)\}_{n=1}^{\infty}$ converges nearly everyhere to $\theta(T)$.*

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[15]) A *-endomorphism θ of A is said to be *normal* if it has the following property: if $T \in A^+$ is the supremum of an increasing directed set F of elements in A^+, then we have $\theta(T) = \sup_{S \in F} S$.

[16]) A sequence $\{T_n\}_{n=1}^{\infty}$ of elements of $\mathfrak{B}(A)$ is said to be convergent nearly everywhere (relative to A) to an element T of $\mathfrak{B}(A)$) if for every $\varepsilon > 0$ there exists a sequence $\{P_n\}_{n=1}^{\infty}$ of projections in A such that $P_n \uparrow I_{\mathfrak{H}}$ as $n \to \infty$, $\|(T - T_n)P_n\| < \varepsilon$ $(n = 1, 2, \ldots)$ and $I_{\mathfrak{H}} - P_n$ is algebraically finite for every $n = 1, 2, \ldots$ (*cf.* [9], def. 23).

The next proposition can be proved by the same way as Theorem 1 in [8], hence the details are omitted.

Proposition 3. 2. *Let* $(\mathfrak{H}, \mathsf{A})$ *be a non-commutative measurable space, and let* θ_0 *be a normal *-endomorphism of* A *with the following properties:* (i) $\theta_0(I_{\mathfrak{H}})=I_{\mathfrak{H}}$; (ii) θ_0 *sends the set of all algebraically finite projections of* A *into itself.*
Then θ_0 *can be uniquely extended to a measurable transformation* θ *of* $(\mathfrak{H}, \mathsf{A})$.

The preceding propositions imply

Proposition 3. 3. *Let* θ *be a measurable transformation of the non-commutative measurable space* $(\mathfrak{H}, \mathsf{A})$. *Then* θ *is uniquely determined by its restriction to* A.

Now we formulate our main result which can be considered as a non-commutative extension of the von Neumann—Dunford—Miller mean ergodic theorem (*cf.* [5], VIII. 5. 9.).

Theorem 3. 1. *Let* $(\mathfrak{H}, \mathsf{A}, m)$ *be a finite regular gage space, and let* θ *be a measurable transformation of* $(\mathfrak{H}, \mathsf{A})$. *Suppose that, for every projection* $P\in\mathsf{A}_P$ *and for every* $n=1, 2, \ldots,$ θ *satisfies the inequality*

$$\frac{1}{n}\sum_{j=0}^{n-1} m(\theta^j(P)) \leqq M\cdot m(P) \tag{3. 1}$$

with a constant M *independent of* P *and* n. *Then, for every* p *with* $1\leqq p<+\infty$, $T\to\theta(T)$ *is a continuous linear mapping of* $\mathbf{L}^p(m)$ *into itself and the sequence of operators* $\left\{\frac{1}{n}\sum_{j=0}^{n-1}\theta^j(\cdot)\right\}_{n=1}^{\infty}$ *is strongly convergent in the Banach space* $\mathbf{L}^p(m)$.

The following lemmas are required for the proof.

Lemma 3. 1. (*cf.* [5], VIII. 5. 3). *Let* T *be a bounded operator in an arbitrary complex Banach space* $\mathfrak{X}$. *If the sequence* $\left\{\frac{1}{n}\sum_{j=0}^{n-1} T^j\right\}_{n=1}^{\infty}$ *is bounded (in norm), then it converges strongly in* $\mathfrak{X}$ *if and only if* $\frac{1}{n}T^n x\to 0$ *as* $n\to\infty$ *for* x *in a fundamental set*[17]) *in* $\mathfrak{X}$ *and the sequence* $\left\{\frac{1}{n}\sum_{j=0}^{n-1} T^j x\right\}_{n=1}^{\infty}$ *is weakly*[18]) *sequentially compact*[19]) *for* x *in a fundamental set in* $\mathfrak{X}$.

Lemma 3. 2 (*cf.* [12], th. 3). *Let* $(\mathfrak{H}, \mathsf{A}, m)$ *be a finite regular gage space, and let* $\mathbf{K}$ *be a bounded subset of* $\mathbf{L}^1(m)$. *If, for any sequence of projections* $\{P_n\}_{n=1}^{\infty}$ *in* A *with* $P_n\downarrow 0$ $(n\to\infty)$, $m(TP_n)$ *converges to zero uniformly with respect to* $T\in\mathbf{K}$, *then* $\mathbf{K}$ *is weakly sequentially compact.*

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[17]) A subset $\mathfrak{E}$ of a Banach space $\mathfrak{X}$ is said to be fundamental in $\mathfrak{X}$ if the linear subspace spanned by $\mathfrak{E}$ is equal to $\mathfrak{X}$.

[18]) By the weak topology of $\mathfrak{X}$ is understood the weak topology induced by the dual of $\mathfrak{X}$.

[19]) A subset $\mathfrak{K}$ of $\mathfrak{X}$ is said to be *sequentially compact* if every sequence of points in $\mathfrak{K}$ has a subsequence converging to a point of $\mathfrak{X}$.

Lemma 3.3. *Let* $(\mathfrak{H}, \mathsf{A}, m)$ *be a finite regular gage space, and let* θ *be a measurable transformation of* (H, A). *Suppose that, for every* $P \in \mathsf{A}_p$, m *satisfies the inequality*

$$m(\theta(P)) \leqq Km(P) \tag{3.2}$$

with a constant K *independent of* P. *Then for every* p *with* $1 \leqq p < +\infty$, $T \to \theta(T)$ *is a continuous linear mapping of* $\mathbf{L}^p(m)$ *into itself.*

Proof. For the sake of brevity, denote by A_0 the set of all quasi-simple elements of A. It is not hard to see that θ maps A_0 into itself. Further, for every $T \in \mathsf{A}_0$ with $T = V \sum_{j=1}^{n} \lambda_j P_j$ we have

$$|\theta(T)|^p = \theta(|T|^p) = \sum_{j=1}^{n} |\lambda_j|^p \theta(P_j),$$

and

$$\|\theta(T)\|_p = \left[\sum_{j=1}^{n} |\lambda_j|^p m(\theta(P_j))\right]^{\frac{1}{p}}.$$

Hence, by (3.2), we have

$$\|\theta(T)\|_p \leqq K^{\frac{1}{p}} \left[\sum_{j=1}^{n} |\lambda_j|^p m(P_j)\right]^{\frac{1}{p}} = K^{\frac{1}{p}} \|T\|_p \qquad (1 \leqq p < +\infty). \tag{3.3}$$

Let now $T \in \mathsf{A}$ be arbitrary. As in the proof of Lemma 2.1, we can determine a sequence $\{T_n\}_{n=1}^{\infty}$ of elements of A^+ commuting with $|T|$ such that: 1) $0 \leqq T_n \leqq I_{\mathfrak{H}}$; 2) $TT_n \in \mathsf{A}_0$; 3) $T_n \uparrow I_{\mathfrak{H}}$ strongly as $n \to \infty$. It follows that $|TT_n|^p = |T|^p T_n \uparrow |T|^p$ strongly as $n \to \infty$. As φ_m is normal, we have $\|TT_n\|_p = m(|TT_n|^p)^{\frac{1}{p}} = \varphi_m(|TT_n|^p)^{\frac{1}{p}} \to \varphi_m(|T|^p)^{\frac{1}{p}} = m(|T|^p)^{\frac{1}{p}} = \|T\|_p$ $(n \to \infty)$. Further, $|\theta(TT_n)|^p = \theta(|TT_n|^p)$ for every $1 \leqq p < +\infty$. Since θ is normal on A (*cf.* Def. 3.1), $|\theta(TT_n)|^p \uparrow |\theta(T)|^p$ as $n \to \infty$ and thus $\lim_{n\to\infty} \|\theta(TT_n)\|_p = \lim_{n\to\infty} m(|\theta(TT_n)|^p)^{\frac{1}{p}} = m(|\theta(T)|^p)^{\frac{1}{p}} = \|\theta(T)\|_p$. Since $TT_n \in \mathsf{A}_0$, by (3.3) we have

$$\|\theta(TT_n)\|_p \leqq K^{\frac{1}{p}} \|TT_n\|_p.$$

Thus, we obtain

$$\|\theta(T)\|_p = \lim_{n\to\infty} \|\theta(TT_n)\|_p \leqq K^{\frac{1}{p}} \lim_{n\to\infty} \|TT_n\|_p = K^{\frac{1}{p}} \|T\|_p,$$

i. e.

$$\|\theta(T)\|_p \leqq K^{\frac{1}{p}} \|T\|_p \qquad (1 \leqq p < +\infty) \tag{3.4}$$

for every $T \in \mathsf{A}$. The inequality (3.4) shows that the restriction of θ to A, denoted by θ_0, is a continuous linear mapping of A into itself with respect to the $\mathbf{L}^p$-norm. Since A is dense in $\mathbf{L}^p(m)$, θ_0 can be uniquely extended to a continuous linear mapping θ_1 of $\mathbf{L}^p(m)$ into itself. Now, using the fact that every sequence $\{T_n\}_{n=1}^{\infty}$ of elements of A which converges in $\mathbf{L}^p$-norm to a measurable operator T contains a subsequence converging nearly everywhere to T (*cf.* [8] and [10]), it can be seen as in the classical case that $\theta_1(T)=\theta(T)$ for every $T\in \mathbf{L}^p(m)$. Thus Lemma 3. 2 is proved.

Proof of Theorem 3. 1. If $n=2$, the inequality (3. 2) gives $m(\theta(P))\leqq \leqq(2M-1)m(P)$ for any projection P in A. Hence, by Lemma 3. 3, θ is a bounded linear operator in $\mathbf{L}^p(m)$ $(1\leqq p<+\infty)$. To complete the proof of Theorem 3. 1, we have only to show the following (*cf.* Lemma 3. 1):

a) for every $T\in\mathbf{L}^p(m)$, $\left\|\frac{1}{n}\sum_{j=0}^{n-1}\theta^j(T)\right\|_p \leqq M^{\frac{1}{p}}\|T\|_p$ $(n=1, 2, \ldots)$;

b) $\frac{1}{n}\theta^n(T)$ converges strongly to zero as $n\to\infty$ for T in a fundamental set in $\mathbf{L}^p(m)$;

c) the sequence $\left\{\frac{1}{n}\sum_{j=1}^{n-1}\theta^j(T)\right\}_{n=1}^{\infty}$ of elements of $\mathbf{L}^p(m)$ is weakly sequentially compact for T in a fundamental set in $\mathbf{L}^p(m)$.

Let us prove a). First we show that

$$\left\|\frac{1}{n}\sum_{j=0}^{n-1}\theta^j(T)\right\|_1 \leqq M\|T\|_1 \qquad (n=1, 2, \ldots), \tag{3. 5}$$

for every $T\in$ A. The reasoning in the proof of Lemma 3. 3 shows that it is enough to prove (3. 5) for the quasi-simple elements of A. Let $T=VT_0$ be an arbitrary element of A_0 with $T_0=\sum_{i=1}^{N}\lambda_i P_i$. Then we have

$$\left\|\frac{1}{n}\sum_{j=0}^{n-1}\theta^j(T)\right\|_1 = \left\|\frac{1}{n}\sum_{j=0}^{n-1}\theta^j(V)\theta^j(T_0)\right\|_1 \leqq \frac{1}{n}\sum_{j=0}^{n-1}\|\theta^j(V)\|\,\|\theta^j(T_0)\|_1 \leqq$$

$$\leqq \frac{1}{n}\sum_{j=1}^{n-1}\sum_{i=1}^{N}|\lambda_i| m(\theta^j(P_i)) = \sum_{i=1}^{N}\left[|\lambda_i|\frac{1}{n}\sum_{j=1}^{n-1} m(\theta^j(P_i))\right] \leqq M\|T\|_1,$$

which proves (3. 5). Further, for every $T\in A$ we have

$$\left\|\frac{1}{n}\sum_{j=0}^{n-1}\theta^j(T)\right\| \leqq \frac{1}{n}\sum_{j=0}^{n-1}\|\theta^j(T)\| \leqq \frac{1}{n}\sum_{j=0}^{n-1}\|T\| = \|T\|. \tag{3. 6}$$

Putting $\Phi_n(\cdot)=\frac{1}{n}\sum_{j=0}^{n-1}\theta^j(\cdot)$ for every $n=1, 2, \ldots$, we have obtained

$$\|\Phi_n\|_1 \leqq M, \quad \|\Phi_n\|_\infty \leqq 1.$$

As $\Phi_n(\cdot)$ is a linear mapping of A into itself, Theorem 2. 1 now gives

$$\log\|\Phi_n\|_p \leqq \left(1-\frac{1}{p}\right)\log\|\Phi_n\|_\infty + \frac{1}{p}\log\|\Phi_n\|_1 \leqq \log\|\Phi_n\|_1^{\frac{1}{p}} \leqq \log M^{\frac{1}{p}},$$

and so
$$\|\Phi_n\|_p = M^{\frac{1}{p}} \qquad (n=1, 2, \ldots),$$

which gives a).

To prove b), we note that the set A_P is fundamental in every $\mathbf{L}^p(m)$ for $1 \leqq p < +\infty$ (*cf.* [9], [11]). Now, if $P \in A_P$, we have

$$\left\| \frac{1}{n} \theta^n(P) \right\|_p = \frac{1}{n} \left[m(|\theta^n(P)|^p) \right]^{\frac{1}{p}} = \frac{1}{n} \left[m(\theta^n(P)) \right]^{\frac{1}{p}} \leqq \frac{1}{n} \left[m(I_{\mathfrak{H}}) \right]^{\frac{1}{p}} \to 0 \text{ as } n \to \infty,$$

whence b).

Finally, c) follows from Lemma 3. 2. Indeed, let $P_n \in A_P$ such that $P_n \downarrow 0$ strongly as $n \to \infty$. Then, for every $Q \in A_P$,

$$\left| m\left(P_n \frac{1}{k} \sum_{j=0}^{k-1} \theta^j(Q) \right) \right| \leqq \left(\frac{1}{k} \sum_{j=0}^{k-1} \|\theta^j(Q)\| \right) \|P_n\|_1 \leqq \|P_n\|_1 = \varphi_m(P_n) \to 0 \text{ as } n \to \infty$$
$$(k = 1, 2, \ldots)$$

independently from Q, and this completes the proof of Theorem 3. 1.

§ 4. An ergodicity concept for gages

In his paper [11], H. UMEGAKI introduced a concept of ergodicity for "traces" of a D*-algebra R (a normed *-algebra over the complex number, with an approximative identity) which are "stationary" (i. e. invariant) concerning a group of *-automorphisms **G** of A. He called a stationary trace of R *ergodic* if it is not a linear combination with positive coefficients of two other linearly independent stationary traces of R, and he characterized the ergodic traces with the aid of the two-sided representations corresponding to them. The ergodicity concept for gages introduced by us is analogous to that for measures in the ordinary integration theory[20]). We shall show the relation between our definition of ergodicity and UMEGAKI's, the latter definition being considered in the case when R is supposed to be a von Neumann algebra.

Let $(\mathfrak{H}, A, m)$ be a gage space, **G** a group of invertible measurable transformations of $(\mathfrak{H}, A)$ (*cf.* Def. 3. 1). In what follows, an element $T \in \mathfrak{B}(A)$ is said to be $(m, \mathbf{G})$-*invariant* if for every $\theta \in \mathbf{G}$ we have $E_m\theta(T) = E_m T$ (E_m is the support of m). T is said simply to be **G**-*invariant* if for every $\theta \in \mathbf{G}$ we have $\theta(T) = T$.

Our ergodicity concept for gages is given by the following

Definition 4. 1. Let $(\mathfrak{H}, A, m)$ be a gage space, **G** a group of invertible measurable transformations of $(\mathfrak{H}, A)$. m is called **G**-*ergodic* if for every $(m, \mathbf{G})$-invariant projection P of $A \cap A'$ we have either $m(P) = 0$ or $m(I_{\mathfrak{H}} - P) = 0$.

Analogously to the classical case we have

Theorem 4. 1. *Let $(\mathfrak{H}, A, m)$ be a gage space, **G** a group of invertible measurable transformations of $(\mathfrak{H}, A)$. In order that to m be **G**-ergodic it is necessary and sufficient*

[20]) Let (X, S, μ) be a measure space, and let **G** be a group of one-to-one mappings of X onto itself and which at the same time is a group of automorphisms of S. We recall that μ is said to be **G**-ergodic if for any $E \in S$ such that $[E \cup \theta(E)] - [E \cap \theta(E)]$ has μ-measure 0 for every $\theta \in \mathbf{G}$, we have either $\mu(E) = 0$ or $\mu(X - E) = 0$. μ is **G**-ergodic if and only if every S-measurable function $f(x)$ such that, for every $\theta \in \mathbf{G}$, $f(\theta(x)) = f(x)$ almost everywhere, is equal to a constant almost everywhere.

that every $(m, \mathbf{G})$*-invariant element* T *of* $\mathfrak{B}(\mathsf{A})$ *affiliated with* $\mathsf{A} \mathring{\cap} \mathsf{A}'$ *be a scalar multiple of* E_m.

Proof. If the condition of Theorem 4 is fulfilled, then every $(m, \mathbf{G})$-invariant projection P in $\mathsf{A} \cap \mathsf{A}'$ satisfies the equality $E_m P = \lambda E_m$ with some scalar λ. Since $E_m \in (\mathsf{A} \cap \mathsf{A}')$, $E_m P$ is a projection, so we have either $\lambda = 1$ or $\lambda = 0$. Hence either $m(P) = m(E_m P) = 0$ or $m(I_{\mathfrak{H}} - P) = m(E_m(I_{\mathfrak{H}} - P)) = m(E_m) - m(E_m P) = 0$. This means that m is $\mathbf{G}$-ergodic. Conversely, suppose that m is $\mathbf{G}$-ergodic. Let $T \in (\mathsf{A} \cap \mathsf{A}')$ be a self-adjoint $(m, \mathbf{G})$-invariant operator with $T = \int \lambda \, dE_\lambda$. Since $E_m \in (\mathsf{A} \cap \mathsf{A}')$, we have $E_m T = \int \lambda \, d(E_m E_\lambda)$, and $E_m \theta(T) = \int \lambda \, d(E_m \theta(E_\lambda))$ for every $\theta \in \mathbf{G}$. As $E_m(\theta(T)) = E_m T$, it follows from the uniqueness of the spectral representation that $E_m \theta(E_\lambda) = E_m E_\lambda$ for every λ and $\theta \in \mathbf{G}$. Since $E_\lambda \in (A \cap A')$, we obtain, by the $\mathbf{G}$-ergodicity of m, that for every λ either $m(E_\lambda) = 0$ or $m(I_{\mathfrak{H}} - E_\lambda) = 0$, i. e. either $E_m E_\lambda = 0$ or $E_m E_\lambda = E_m$. This means that the spectral family of $E_m T$ contains only two projections, namely 0 and E_m. Hence we have $E_m T = \lambda_0 E_m$. Let now $T \in (\mathsf{A} \cap \mathsf{A}')$ be an arbitrary $(m, \mathbf{G})$-invariant operator. It is easy to see that T can be written as a linear combination of two self-adjoint $(m, \mathbf{G})$-invariant operators in $\mathsf{A} \cap \mathsf{A}'$. Hence T is also a scalar multiple of E_m. Finally, let T be an arbitrary $(m, \mathbf{G})$-invariant operator in $\mathfrak{B}(\mathsf{A})$ affiliated with $\mathsf{A} \cap \mathsf{A}'$. Let $T = W|T|$ be the polar decomposition of T with $T = \int \lambda \, dE_\lambda$. It is known that $W \in (\mathsf{A} \cap \mathsf{A}')$, and $E_\lambda \in (\mathsf{A} \cap \mathsf{A}')$ for every λ. Further, as $E_m \theta(T) = E_m \theta(W) \theta(|T|) = (E_m \theta(W))(E_m \theta(|T|) = E_m T = E_m W|T| = (E_m W)(E_m |T|)$, it follows from the uniqueness of the polar decomposition that $E_m \theta(W) = E_m W$ and $E_m \theta(|T|) = E_m |T|$ for every $\theta \in G$. Since $W \in (\mathsf{A} \cap \mathsf{A}')$, we have $E_m W = \alpha E_m$. Since $\theta(|T|) = \int \lambda \, d\theta(E_\lambda)$ (*cf.* [9]), $E_m \theta(|T|) = \int \lambda \, d(E_m \theta(E_\lambda))$, similarly as above, it may be seen that the spectral family of $E_m |T|$ contains only two projections: 0 and E_m. Thus we obtain $E_m |T| = \beta E_m$, which proves Theorem 4. 1.

Definition 4. 2. Let $(\mathfrak{H}, \mathsf{A}, m)$ be a gage space, $\mathbf{G}$ a group of invertible measurable transformations of $(\mathfrak{H}, \mathsf{A})$. m is said to be $\mathbf{G}$-invariant if for every projection P of A and for every $\theta \in \mathbf{G}$ we have $m(\theta(P)) = m(P)$.

Let now A be a von Neumann algebra, $\mathbf{G}$ a group of $*$-automorphisms of A. Let $\mathsf{P}^{\mathbf{G}}$ denote the set of all $\mathbf{G}$-invariant probability[21]) gages on $(\mathfrak{H}, \mathsf{A})$, and $\dot{\mathsf{P}}^{\mathbf{G}} = \{\dot{\varphi}_m : m \in \mathsf{P}^{\mathbf{G}}\}$. It is evident that $\dot{\mathsf{P}}^{\mathbf{G}}$ is a convex subset of A^*[22]). The next theorem characterizes the $\mathbf{G}$-ergodic elements of $\mathsf{P}^{\mathbf{G}}$ as follows

Theorem 4. 2. *$m \in \mathsf{P}^{\mathbf{G}}$ is $\mathbf{G}$-ergodic if and only if $\dot{\varphi}_m$ is an extreme[23]) point of $\dot{\mathsf{P}}^{\mathbf{G}}$.*

Proof. First we note that if $m \in \mathsf{P}^{\mathbf{G}}$ then E_m is $\mathbf{G}$-invariant. Indeed, for every $\theta \in \mathbf{G}$ we have $m(I_{\mathfrak{H}} - \theta(E_m)) = m(I_{\mathfrak{H}}) - m(\theta(E_m)) = m(E_m) - m(E_m) = 0$. This means that $I_{\mathfrak{H}} - \theta(E_m) \leqq I_{\mathfrak{H}} - E_m$ $(\theta \in \mathbf{G})$. It follows that $I_{\mathfrak{H}} - E_m \leqq I_{\mathfrak{H}} - \theta^{-1}(E_m)$ for every

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[21]) A gage m of A is said to be a *probability gage* if $m(I_{\mathfrak{H}}) = 1$.

[22]) For a von Neumann algebra A, A^* denotes the dual space of A when A is considered as a Banach space with $\|T\|$ as its norm.

[23]) $\dot{\varphi}_m$ is an *extreme* point of $\dot{\mathsf{P}}^{\mathbf{G}}$ if it is not a middle point of any segment belonging to $\dot{\mathsf{P}}^{\mathbf{G}}$.

$\theta\in\mathbf{G}$. As the mapping $\theta\to\theta^{-1}$ of $\mathbf{G}$ onto itself is one-two-one, we have $I_{\mathfrak{H}}-E_m\leqq\leqq I_{\mathfrak{H}}-\theta(E_m)$ $(\theta\in\mathbf{G})$. Thus we have $E_m=\theta(E_m)$ for every $\theta\in\mathbf{G}$.

Further, if $m\in\mathrm{P}^{\mathrm{G}}$ then for every $T\in\mathrm{A}$ and $\theta\in\mathbf{G}$ we have $\dot{\varphi}_m(\theta(T))=\dot{\varphi}_m(T)$. In fact, let T be an arbitrary element of A^+. As in the proof of Lemma 2. 1, we can choose a sequence $\{T_n\}_{n=1}^{\infty}$ of elements of A^+ such that: 1) $0\leqq T_n\leqq I_{\mathfrak{H}}$; 2) $T_n\uparrow I_{\mathfrak{H}}$ strongly; 3) TT_n is a finite linear combination of elements of A_P with positive coefficients. The $\mathbf{G}$-invariance of m implies that $\varphi_m(\theta(TT_n))=\varphi_m(TT_n)$. As θ is a *-automorphism, it follows that $\theta(T_n)\uparrow I_{\mathfrak{H}}$. Thus $\theta(TT_n)=\theta(T)\theta(T_n)\uparrow\theta(T)$. By the normality of φ_m, we have $\varphi_m(\theta(T))=\lim_{n\to\infty}\varphi_m(\theta(TT_n))=\lim_{n\to\infty}\varphi_m(TT_n)=\varphi_m(T)$. Since every element of A can be written as a finite linear combination of elements in A^+, our assertion follows.

For $m\in\mathrm{P}^{\mathrm{G}}$, consider the von Neumann sub-algebra $\mathrm{A}_{E_m}=\{T\in\mathrm{A}: TE_m=T\}$ of A. We note that the restriction of φ_m to $\mathrm{A}^+_{E_m}$, denoted by the same letter φ_m, is a finite faithful normal trace on $\mathrm{A}^+_{E_m}$. Let $\mathbf{R}_m$ be the unitary algebra associated with $\dot{\varphi}_m$, and let Φ_m be the canonical *-isomorphism between A_{E_m} and the left ring $\mathbf{R}^g_m$ of $\mathbf{R}_m$. Since $\theta(E_m)=E_m$ for every $\theta\in\mathbf{G}$, it is easy to see that the mapping $T\to\theta'(T)==\Phi_m[\theta(\Phi_m^{-1}(T))]$ defines a *-automorphism θ' of $\mathbf{R}^g_m$ for every θ, and so $\mathbf{G}$ induces through Φ_m a group of *-automorphisms $\mathbf{G}'$ of $\mathbf{R}^g_m$. Further, it is not hard to see that an element $T\in\mathbf{R}^g_m$ is $\mathbf{G}'$-invariant if and only if $\Phi_m^{-1}(T)$ is $\mathbf{G}$-invariant.

Let now $\mathbf{V}_m$ be the set of all bounded linear operators V on $\mathfrak{H}_{\mathbf{R}_m}$ for which

$$\langle V\theta(S)\,|\,\theta(T)\rangle_{\dot{\varphi}_m}=\langle VS\,|\,T\rangle_{\dot{\varphi}_m}$$

for all $S,T\in\mathbf{R}_m$, $\theta\in\mathbf{G}$. It is easy to see that $\mathbf{V}_m$ is a von Neumann algebra on $\mathfrak{H}_{\mathbf{R}_m}$. By a theorem of H. UMEGAKI (*cf.* [11], Th. 5), $\dot{\varphi}_m$ is an extreme point of P^{G} if and only if $(\mathbf{V}_m\cap\mathbf{R}^g_m\cap\mathbf{R}^d_m)=\{\alpha I_{\mathfrak{H}_{\mathbf{R}_m}}\}$. Hence we have to prove that $m\in\mathrm{P}^{\mathrm{G}}$ is $\mathbf{G}$-ergodic if and only if $(\mathbf{V}_m\cap\mathbf{R}^g_m\cap\mathbf{R}^d_m)=\{\alpha I_{\mathfrak{H}_{\mathbf{R}_m}}\}$.

First we show that for an element $T\in(\mathbf{R}^g_m\cap\mathbf{R}^d_m)$ we have $T\in\mathbf{V}_m$ if and only if T is $\mathbf{G}'$-invariant. Suppose that $T\in(\mathbf{R}^g_m\cap\mathbf{R}^d_m)$ is $\mathbf{G}'$-invariant. Then $\Phi_m^{-1}(T)$ is $\mathbf{G}$-invariant. Thus, for every $\theta\in\mathbf{G}$ and $R,S\in\mathbf{R}_m$ we have

$$\langle T\theta(R)\,|\,\theta(S)\rangle_{\dot{\varphi}_m}=\dot{\varphi}_m(\theta(S^*)\Phi_m^{-1}(T)\theta(R))=\dot{\varphi}_m(\theta(S^*\Phi_m^{-1}(T)R))=$$
$$=\dot{\varphi}_m(S^*\Phi_m^{-1}(T)R)=\langle TR\,|\,S\rangle_{\dot{\varphi}_m},$$

which gives that $T\in\mathbf{V}_m$. Conversely, suppose that $T\in(\mathbf{V}_m\cap\mathbf{R}^g_m\cap\mathbf{R}^d_m)$. Then for every $\theta\in\mathbf{G}$ and $R,S\in\mathbf{R}_m$, we have

$$\dot{\varphi}_m(\Phi_m^{-1}(T)\theta(R)\theta(S^*))=\dot{\varphi}_m(\theta(S^*)\Phi_m^{-1}(T)\theta(R))=\langle T\theta(R)\,|\,\theta(S)\rangle_{\dot{\varphi}_m}=\langle TR\,|\,S\rangle_{\dot{\varphi}_m}=$$
$$=\dot{\varphi}_m(S^*\Phi_m^{-1}(T)R)=\dot{\varphi}_m(\Phi_m^{-1}(T)RS^*)=\dot{\varphi}(\theta(\Phi_m^{-1}(T))\theta(R)\theta(S^*)).$$

In particular, for $S=E_m\in\mathbf{R}_m$ we have

$$\dot{\varphi}_m(\theta(\Phi_m^{-1}(T))\theta(R))=\dot{\varphi}_m(\Phi_m^{-1}(T)\theta(R)).$$

Thus for every $\theta\in\mathbf{G}$ and $R\in\mathbf{R}_m$,

$$\dot{\varphi}_m([\Phi_m^{-1}(T)-\theta(\Phi_m^{-1}(T)]\theta(R))=0.$$

It follows that $\Phi_m^{-1}(T)=\theta(\Phi_m^{-1}(T))$ for every $\theta\in\mathbf{G}$, which gives that T is $\mathbf{G}'$-invariant.

Suppose now that $m \in P^G$ is $\mathbf{G}$-ergodic, and let $T \in (\mathbf{V}_m \cap \mathbf{R}_m^g \cap \mathbf{R}_m^d)$ be arbitrary. Then $\Phi_m^{-1}(T)$ as an element of $A \cap A'$ is $(m, \mathbf{G})$-invariant. By Theorem 4. 1, $\Phi_m^{-1}(T) = \alpha E_m$. Hence $T = \alpha \Phi_m(E_m) = \alpha I_{\mathfrak{H}_{\mathbf{R}_m}}$, which gives that $(\mathbf{V}_m \cap \mathbf{R}_m^g \cap \mathbf{R}_m^d) = \{\alpha I_{\mathfrak{H}_{\mathbf{R}_m}}\}$. Conversely, suppose that $(\mathbf{V}_m' \cap \mathbf{R}_m^g \cap \mathbf{R}_m^d) = \{\alpha I_{\mathfrak{H}_{\mathbf{R}_m}}\}$, and let $T \in (A \cap A')$ be $(m, \mathbf{G})$-invariant. Then TE_m is a $\mathbf{G}$-invariant element of $A_{E_m} \cap A'_{E_m}$. It follows that $\Phi_m(TE_m) \in (\mathbf{V}_m \cap \mathbf{R}_m^g \cap \mathbf{R}_m^d)$, therefore $\Phi_m(TE_m) = \alpha I_{\mathfrak{H}_{\mathbf{R}_m}} = \alpha \Phi_m(E_m)$. Thus $TE_m = \alpha E_m$, which completes the proof of Theorem 4. 2.

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