Matrices of normal extensions of subnormal operators

By T. ANDÔ in Bloomington (Indiana, U. S. A.)

1. A (bounded) operator T on a Hilbert space $\mathfrak{H}$ is called *subnormal* in case there exists a normal operator $\mathbf{N}$, called a *normal extension* of T, acting on a Hilbert space $\mathfrak{K}$ containing $\mathfrak{H}$ as a subspace such that

$$\mathbf{N}f = Tf \qquad (f \in \mathfrak{H}). \tag{1}$$

A characterization of subnormality in terms of T has been obtained by HALMOS [2] and BRAM [1]; T is subnormal if and only if

$$\sum_{i,j} (T^i f_j, T^j f_i) \geqq 0 \tag{2}$$

for every finite sequence (f_i) in $\mathfrak{H}$. Their construction of the space $\mathfrak{K}$, however, depends heavily on T. It seems natural to raise a problem whether $\mathfrak{K}$ can be taken to be a fixed Hilbert space, independent of T as in SCHÄFFER's construction [4] for a unitary dilation of a contraction, and whether $\mathbf{N}$ can be constructed on $\mathfrak{K}$ along a definite line from T. In this paper this problem will be settled (Theorem 1), producing another characterization of subnormality (Theorem 2). At the same time a discussion concerning a commutative family of a subnormal operators will be made (Theorem 3).

Introduction of some notations will simplify later discussions. For any positive integer n, $\mathfrak{H}^n$ stands for the orthogonal sum of n copies of $\mathfrak{H}$, indexed by $0, 1, 2, \ldots, n-1$. In other words, the elements of $\mathfrak{H}^n$ are the n-sequences $\varphi = \{f_0, f_1, \ldots, f_{n-1}\}$ of elements $f_i \in \mathfrak{H}$ with norm $\|\varphi\|^2 = \sum_{i=0}^{n-1} \|f_i\|^2$. $\mathfrak{H}^\infty$ is similarly defined. In case $n > m$, $\mathfrak{H}^m$ is embedded into $\mathfrak{H}^n$ by identifying $\{f_0, f_1, \ldots, f_{m-1}\} \in \mathfrak{H}^m$ with $\{f_0, f_1, \ldots, f_{m-1}, 0, \ldots, 0, 0\} \in \mathfrak{H}^n$. $\mathfrak{H}$ is always identified with $\mathfrak{H}^1$. An operator $\mathbf{M}$ on $\mathfrak{H}^n$ $(1 \leqq n \leqq \infty)$ can be associated with a square n-rowed matrix each of whose entries is an operator on $\mathfrak{H}$. More precisely, if $M(i,j)$ stands for the (i,j)-th entry of $\mathbf{M}$, $\{g_i\} = \mathbf{M}\{f_i\}$ means that

$$g_i = \sum_{j=0}^{n-1} M(i,j) f_j \qquad (0 \leqq i \leqq n-1).$$

The requirement that $\mathfrak{H}$ is invariant under $\mathbf{M}$ and the restriction of $\mathbf{M}$ to $\mathfrak{H}$ coincides with T can be expressed by the requirement that $M(0,0) = T$ and $M(i,0) = 0$ for all $i > 0$. Finally we shall formulate a simple Lemma.

Lemma 1. *If T is subnormal and V is an operator from $\mathfrak{H}$ into another Hilbert space $\mathfrak{M}$ such that $V^*VT = T$, then VTV^* is subnormal on $\mathfrak{M}$.*

In fact, since $(VTV^*)^k = VT^kV^*$ $(k=1, 2, \ldots)$ by assumption, for every finite sequence (φ_i) in $\mathfrak{M}$

$$\sum_{i,j} ((VTV^*)^i \varphi_j, (VTV^*)^j \varphi_i) = \sum_{i,j} (VT^iV^* \varphi_j, VT^jV^* \varphi_i) =$$
$$= \sum_{i,j} (V^*VT^iV^* \varphi_j, T^jV^* \varphi_i) = \sum_{i,j} (T^iV^* \varphi_j, T^jV^* \varphi_i) \geqq 0$$

(the last inequality follows from (2)), hence the criterion (2) yields the subnormality of VTV^*.

2. First of all, if $\mathbf{N}$ is a normal extension of T, from (1) and the normality of $\mathbf{N}$ it follows that

$$\mathbf{N}\mathbf{N}^*f = \mathbf{N}^*\mathbf{N}f = \mathbf{N}^*Tf, \tag{3}$$

$$(\mathbf{N}^*f, g) = (T^*f, g) \qquad (f, g \in \mathfrak{H}), \tag{4}$$

$$\|Tf\| = \|\mathbf{N}f\| = \|\mathbf{N}^*f\| \geqq \|T^*f\|, \tag{5}$$

and moreover on account of Bram's theorem [1] the norm $\|\mathbf{N}\|$ may be assumed to be equal to $\|T\|$.

(5) is equivalent to the positive definiteness of $T^*T - TT^*$. Let $S = (T^*T - TT^*)^{\frac{1}{2}}$, then

$$\|(\mathbf{N}^* - T^*)f\| = \|Sf\| \qquad (f \in \mathfrak{H}), \tag{6}$$

because by (4) and (5)

$$\|(\mathbf{N}^* - T^*)f\|^2 = \|\mathbf{N}^*f\|^2 - 2\,\mathrm{Re}(\mathbf{N}^*f, T^*f) + \|T^*f\|^2 = \|Tf\|^2 - \|T^*f\|^2 = \|Sf\|.$$

From this it follows that $Sf=0$ is equivalent to $\mathbf{N}^*f = T^*f$, and the latter, in turn, is equivalent to the fact that $\mathbf{N}^*f$ is contained in $\mathfrak{H}$. Now each element φ in $\mathfrak{H} + \mathbf{N}^*\mathfrak{H}$ can be written in the form

$$\varphi = f + (\mathbf{N}^* - T^*)g \qquad \text{with} \qquad f, g \in \mathfrak{H}$$

and this decomposition is unique, because of the orthogonality of $\mathfrak{H}$ with $(\mathbf{N}^* - T^*)\mathfrak{H}$ by (4), consequently

$$\|\varphi\|^2 = \|f\|^2 + \|(\mathbf{N}^* - T^*)g\|^2. \tag{7}$$

Combining (7) with (6), it follows that the operator $\mathbf{V}$ which assigns $\{f, Sg\}$ to φ maps isometrically $\mathfrak{H} + \mathbf{N}^*\mathfrak{H}$ into $\mathfrak{H}^2$, and can be extended isometrically on the closure $\mathfrak{L}$ of $\mathfrak{H} + \mathbf{N}^*\mathfrak{H}$. On the other hand, $\mathfrak{L}$ is invariant under $\mathbf{N}$, because by (2)

$$\mathbf{N}(\mathfrak{H} + \mathbf{N}^*\mathfrak{H}) \subset T\mathfrak{H} + \mathbf{N}^*T\mathfrak{H} \subset \mathfrak{H} + \mathbf{N}^*\mathfrak{H}.$$

Therefore the restriction $\mathbf{M}$ of the normal operator $\mathbf{N}$ to the invariant subspace $\mathfrak{L}$ is subnormal with norm equal to $\|T\|$ by the definition of subnormality. Since clearly $\mathbf{V}^*\mathbf{V}\mathbf{M} = \mathbf{M}$, Lemma 1 yields the subnormality of $\mathbf{T} = \mathbf{V}\mathbf{M}\mathbf{V}^*$ and the norm $\|\mathbf{T}\|$ is equal to $\|T\|$.

In order to obtain the matrix of $\mathbf{T}$ on $\mathfrak{H}^2$ it suffices to calculate $\mathbf{T}\{f, Sg\}$ $(f, g \in \mathfrak{H})$, because $\mathbf{V}^*\{0, h\} = 0$ whenever $S^*h (= Sh) = 0$ and the orthogonal complement of the null space of S coincides with the closure of the range of S. To this effect, consider the densely defined operator S^{-1}, called the *partial inverse* of S, such that $S^{-1}S=P$ and $S^{-1}(I-P)=0$ where P denotes the orthogonal projection from $\mathfrak{H}$ onto the closure of the range of S. From (3) and the definition of $\mathbf{V}$ it follows that

$$\mathbf{T}\{f, Sg\} = \mathbf{VN}(f+(N^*-T^*)g) =$$
$$= \mathbf{V}(Tf+(T^*T-TT^*)g+(\mathbf{N}^*-T^*)Tg) = \{Tf+S^2g, STg\}$$

and this, in turn, means that the matrix in question is given by $\begin{pmatrix} T & S \\ 0 & STS^{-1} \end{pmatrix}$, *a fortiori* STS^{-1} is bounded. The bounded extension of STS^{-1} on $\mathfrak{H}$ will be denoted by the same symbol. Moreover, since $\mathbf{N}^*f\in\mathfrak{H}$ implies $\mathbf{N}^*Tf = \mathbf{NN}^*f\in\mathfrak{H}$ by (3), it follows that $Sf=0$ implies $STf=0$, i. e. $ST=STP=STS^{-1}\cdot S$.

Summing up, if T is subnormal, then T^*T-TT^* is positive definite, STS^{-1} is bounded and $ST=STS^{-1}\cdot S$, and the operator $\begin{pmatrix} T & S \\ 0 & STS^{-1} \end{pmatrix}$ on $\mathfrak{H}^2$ is subnormal with norm equal to $\|T\|$. This can be further generalized as follows:

Lemma 2. *Let T be subnormal and let R_n, S_n and T_n be defined by the following recurrent formulas:*

$$R_0=S_0=0,\ T_0=T,$$

$$R_n=S_{n-1}^2+T_{n-1}^*T_{n-1}-T_{n-1}T_{n-1}^*,\ S_n=R_n^{\frac{1}{2}},\ T_n=S_nT_{n-1}S_n^{-1} \qquad (n=1, 2, \ldots)$$

Then, in each step, R_n is positive definite, T_n is bounded and $S_nT_{n-1}=T_nS_n$, and the operator $\mathbf{N}_n$ on $\mathfrak{H}^n$ with the entries $N_n(i, i)=T_i$ $(0\leqq i\leqq n-1)$, $N_n(i, i+1)=S_{i+1}$ $(0\leqq i\leqq n-2)$, $N_n(i,j)=0$ (for all other indices), is subnormal with norm equal to $\|T\|$.

Proof by induction. The assertions for $n=1$ have been just proved above. Suppose that the assertions on R_i, S_i and T_i $(0\leqq i\leqq n-1)$ and on $\mathbf{N}_n$ have been proved. On account of the arguments preceding this lemma, $\mathbf{N}_n^*\mathbf{N}_n-\mathbf{N}_n\mathbf{N}_n^*$ is positive definite, $\mathbf{WN}_n\mathbf{W}^{-1}$ is bounded, where $\mathbf{W}=(\mathbf{N}_n^*\mathbf{N}_n-\mathbf{N}_n\mathbf{N}_n^*)^{\frac{1}{2}}$ and $\mathbf{W}^{-1}$ is its partial inverse, and $\mathbf{WN}_n=\mathbf{WN}_n\mathbf{W}^{-1}\mathbf{W}$ and the operator $\begin{pmatrix} \mathbf{N}_n & \mathbf{W} \\ 0 & \mathbf{WN}_n\mathbf{W}^{-1} \end{pmatrix}$ on the orthogonal sum $\mathfrak{H}^n\oplus\mathfrak{H}^n$ is subnormal with norm equal to $\|\mathbf{N}_n\|=\|T\|$. Putting $\mathbf{N}_n^*\mathbf{N}_n^*=\mathbf{A}$ and $\mathbf{N}_n\mathbf{N}_n^*=\mathbf{B}$, simple calculations show that

$$\begin{aligned} A(i, i-1)&=S_iT_{i-1} && (1\leqq i\leqq n-1),\\ A(i, i)&=S_i^2+T_i^*T_i && (0\leqq i\leqq n-1),\\ A(i, i+1)&=T_i^*S_{i+1} && (0\leqq i\leqq n-2),\\ A(i, j)&=0 && \text{(for all other indices)}, \end{aligned}$$

and similarly

$$\begin{aligned} B(i, i-1)&=T_iS_i && (1\leqq i\leqq n-1),\\ B(i, i)&=T_iT_i^*+S_{i+1}^2 && (0\leqq i\leqq n-2),\\ B(i, i+1)&=S_{i+1}T_{i+1}^* && (0\leqq i\leqq n-2),\\ B(n-1, n-1)&=T_{n-1}T_{n-1}^* && \\ B(i, j)&=0 && \text{(for all other indices)}. \end{aligned}$$

Since, by assumption,

$$S_iT_{i-1}=T_iS_i \qquad (1\leqq i\leqq n-1),$$

$$S_i^2+T_i^*T_i=T_iT_i^*+S_{i+1}^2 \qquad (0\leqq i\leqq n-2),$$

all the entries of $\mathbf{N}_n^*\mathbf{N}_n-\mathbf{N}_n\mathbf{N}_n^*$ are equal to 0 except the $(n-1, n-1)$th, which is equal to $S_{n-1}^2+T_{n-1}^*T_{n-1}-T_{n-1}T_{n-1}^* = R_n$ by definition. Hence the positive definiteness of $\mathbf{N}_n^*\mathbf{N}_n-\mathbf{N}_n\mathbf{N}_n^*$ implies the positive definiteness of R_n. Similarly all the entries of $\mathbf{W}\mathbf{N}_n\mathbf{W}^{-1}$ are equal to 0 except the $(n-1, n-1)$ th which is equal to $S_nT_{n-1}S_n^{-1}=T_n$ by definition and is bounded. Moreover $\mathbf{W}\mathbf{N}_n=\mathbf{W}\mathbf{N}_n\mathbf{W}^{-1}\cdot\mathbf{W}$ implies $S_nT_{n-1}=T_nS_n$. Finally considering the operator $\mathbf{V}$, with norm one, from $\mathfrak{H}^n\oplus\mathfrak{H}^n$ into $\mathfrak{H}^{n+1}$ defined by $\mathbf{V}\{\{f_0,f_1,\ldots,f_{n-1}\},\{g_0,g_1,\ldots,g_{n-1}\}\} = \{f_0,f_1,\ldots$ $\ldots,f_{n-1},g_{n-1}\}$,

$$\mathbf{V}^*\mathbf{V}\begin{pmatrix}\mathbf{N}_n & \mathbf{W}\\ 0 & \mathbf{W}\mathbf{N}_n\mathbf{W}^{-1}\end{pmatrix}=\begin{pmatrix}\mathbf{N}_n & \mathbf{W}\\ 0 & \mathbf{W}\mathbf{N}_n\mathbf{W}^{-1}\end{pmatrix} \text{ and } \mathbf{N}_{n+1}=\mathbf{V}\begin{pmatrix}\mathbf{N}_n & \mathbf{W}\\ 0 & \mathbf{W}\mathbf{N}_n\mathbf{W}^{-1}\end{pmatrix}\mathbf{V}^*$$

hence by Lemma 1 $\mathbf{N}_{n+1}$ is also subnormal with norm equal to $\|\mathbf{N}_{n+1}\|=\|T\|$. Thus induction is complete.

Inspecting the above proof, from the definitions of R_n, S_n and T_n, and of $\mathbf{N}_n$ and from the relations $S_nT_{n-1}=T_nS_n$ $(n=1, 2, \ldots)$, it follows

$$\|\mathbf{N}_{n+1}^*\varphi\|=\|\mathbf{N}_n\varphi\| \qquad (\varphi\in\mathfrak{H}^n) \tag{8}$$

where, on the right side, φ is considered as an element of $\mathfrak{H}^{n+1}$.

Now the matrix representation of a normal extension of T is near at hand, using R_n, S_n and T_n in Lemma 2.

Theorem 1. *If T is subnormal, the operator $\mathbf{N}$ on $\mathfrak{H}^\infty$ with the entries $N(i, i)=T_i$ $(i\geqq 0)$, $N(i, i+1)=S_{i+1}$ $(i\geqq 0)$, $N(i,j)=0$ (for all other indices), is a normal extension with norm equal to $\|T\|$.*

In fact, in view of Lemma 2, all $\mathbf{P}_n\mathbf{N}\mathbf{P}_n$ are bounded with norm equal to $\|T\|$ $n=0, 1, 2, \ldots$, where each $\mathbf{P}_n$ is the orthogonal projection from $\mathfrak{H}^\infty$ onto $\mathfrak{H}^n$, consequently, as readily seen, $\mathbf{N}$ itself is bounded with norm equal to $\|T\|$, and is an extension of T. Moreover from (8) it follows that

$$\|\mathbf{P}_{n+1}\mathbf{N}^*\mathbf{P}_n\varphi\|=\|\mathbf{P}_n\mathbf{N}\mathbf{P}_n\varphi\| \qquad (\varphi\in\mathfrak{H}^\infty)\ (n=0, 1, 2, \ldots)$$

hence

$$\|\mathbf{N}\varphi\|=\lim_{n\to\infty}\|\mathbf{P}_n\mathbf{N}\mathbf{P}_n\varphi\|=\lim_{n\to\infty}\|\mathbf{P}_{n+1}\mathbf{N}^*\mathbf{P}_n\varphi\|=\|\mathbf{N}^*\varphi\|.$$

This shows the normality of $\mathbf{N}$.

Lemma 2 also produces a characterization of subnormality in terms of R_n, S_n, and T_n in it.

Theorem 2. *If, for an operator T, each R_n is positive definite, each T_n is bounded and $S_nT_{n-1}=T_nS_n$ $(n=0, 1, 2, \ldots)$, then T is subnormal.*

In fact, the operator $\mathbf{N}$ on $\mathfrak{H}^\infty$ in Theorem 1 can be defined on the linear sum $\mathfrak{M}$ of all $\mathfrak{H}^n$'s, and is an extension of T. Moreover by (8)

$$\|\mathbf{N}\varphi\|=\|\mathbf{N}^*\varphi\| \qquad (\varphi\in\mathfrak{M}).$$

Since $\mathfrak{M}$ is dense in $\mathfrak{H}^\infty$, it follows that $\mathbf{N}^*\mathbf{N}\varphi=\mathbf{N}\mathbf{N}^*\varphi$ $(\varphi\in\mathfrak{M})$, in particular $\mathbf{N}^{*j}\mathbf{N}^i f=\mathbf{N}^i\mathbf{N}^{*j}f$ $(f\in\mathfrak{H})$ $(i,j=0,1,2,\dots)$. Therefore, for every finite sequence (f_i) in $\mathfrak{H}$,

$$\sum_{i,j}(T^jf_j,T^if_i)=\sum_{i,j}(\mathbf{N}^{*j}\mathbf{N}^if_j,f_i)=\sum_{i,j}(\mathbf{N}^{*j}f_j,\mathbf{N}^{*i}f_i)=\|\sum_k\mathbf{N}^{*k}f_k\|^2\geqq 0,$$

and the criterion (2) can be applied.

3. Itô [3] answered to the question when a commutative family of subnormal operators admits simultaneous commutative normal extensions. At this moment, it seems, however, difficult for us to construct matrices for these simultaneous commutative extensions along the line as that developed in § 2. Here we shall confine ourselves to a special case, namely, a doubly commutative family of subnormal operators.

Let $(T_\omega)_{\omega\in\Omega}$ be a *doubly commutative* family of subnormal operators, that is, each T_ω commutes with both T_γ and T_γ^* whenever $\omega\neq\gamma$. Let Δ denote the space of all generalized sequences $\{i_\omega\}$ such that all i_ω are non-negative integers and $\sum_{\omega\in\Omega}i_\omega<\infty$. θ denotes the element of Δ whose terms are all equal to 0. For any $\omega\in\Omega$ and $\Gamma\in\Delta$, ω_Γ is the ω-th term of Γ and $\Gamma+\omega$ stands for the element Λ such that $\omega_\Lambda=\omega_\Gamma+1$ and $\gamma_\Lambda=\gamma_\Gamma$ for all $\gamma\neq\omega$. $\mathfrak{H}^\Delta$ is the orthogonal sum of copies of $\mathfrak{H}$; indexed by all the elements in Δ; the elements of $\mathfrak{H}^\Delta$ are the generalized sequences $\varphi=\{f_\Gamma\}$ whose terms are in $\mathfrak{H}$ with norm $\|\varphi\|^2=\sum_{\Gamma\in\Delta}\|f_\Gamma\|^2$. $\mathfrak{H}$ is embedded in $\mathfrak{H}^\Delta$ by identifying $f\in\mathfrak{H}$ with $\{f_\Gamma\}$ where $f_\theta=f$ and $f_\Gamma=0$ $(\Gamma\neq\theta)$. In Theorem 3 below, $S_{\omega,n}$ and $T_{\omega,n}$ correspond to S_n and T_n respectively in Lemma 2, starting from T_ω instead of T.

Theorem 3. *A doubly commutative family of subnormal operators $(T_\omega)_{\omega\in\Omega}$ has simultaneous commutative normal extensions $(\mathbf{N}_\omega)_{\omega\in\Omega}$ on $\mathfrak{H}^\Delta$ with the entries: $N_\omega(\Gamma,\Gamma)=T_{\omega,\omega_\Gamma}$, $N_\omega(\Gamma,\Gamma+\omega)=S_{\omega,\omega_\Gamma+1}$, $N_\omega(\Gamma,\Lambda)=0$ for all other indices.*

Proof. Just as in Theorem 1, each $\mathbf{N}_\omega$ is a normal extension of T_ω $(\omega\in\Omega)$. For $\omega\neq\gamma$, putting $\mathbf{N}_\omega\mathbf{N}_\gamma=\mathbf{A}$ and $\mathbf{N}_\gamma\mathbf{N}_\omega=\mathbf{B}$, simple calculations based on the definitions of $\mathbf{N}_\omega$'s show that

$$A(\Gamma,\Gamma)=T_{\omega,\omega_\Gamma}T_{\gamma,\gamma_\Gamma},\qquad B(\Gamma,\Gamma)=T_{\gamma,\gamma_\Gamma}T_{\omega,\omega_\Gamma},$$

$$A(\Gamma,\Gamma+\omega)=S_{\omega,\omega_\Gamma+1}T_{\gamma,\gamma_\Gamma},\qquad B(\Gamma,\Gamma+\omega)=T_{\gamma,\gamma_\Gamma}\cdot S_{\omega,\omega_\Gamma+1},$$

$$A(\Gamma,\Gamma+\gamma)=T_{\omega,\omega_\Gamma}S_{\gamma,\gamma_\Gamma+1},\qquad B(\Gamma,\Gamma+\gamma)=S_{\gamma,\gamma_\Gamma+1}T_{\omega,\omega_\Gamma},$$

$$A(\Gamma,\Gamma+\omega+\gamma)=S_{\omega,\omega_\Gamma+1}S_{\gamma,\gamma_\Gamma+1},\quad B(\Gamma,\Gamma+\omega+\gamma)=S_{\gamma,\gamma_\Gamma+1}S_{\omega,\omega_\Gamma+1},$$

and all other entries of $\mathbf{A}$ and $\mathbf{B}$ are equal to 0. Therefore the commutativity of $\mathbf{N}_\omega$ with $\mathbf{N}_\gamma$ will follows from the commutativity of the family $\{S_{\omega,i},T_{\omega,i}\}_{i=0}^\infty$ with the family $\{S_{\gamma,i},T_{\gamma,i}\}_{i=0}^\infty$. In order to prove the latter commutativity, we shall show, by induction, that $T_\omega=T_{\omega,0}$ is doubly commutative with all $S_{\gamma,n}$ and $T_{\gamma,n}$, $n=0,1,2,\dots$ The assertion for $n=0$ follows directly from the assumption. Suppose that the assertion for n is proved, then T_ω commutes with $S_{\gamma,n+1}$ because, as in [2], the latter is uniformly approximated by polynomials of $S_{\gamma,n}^2+T_{\gamma,n}^*T_{\gamma,n}-T_{\gamma,n}T_{\gamma,n}^*$.

which commutes with T_ω. This, in turn, implies the commutativity of T_ω with $S_{\gamma,n+1}^{-1}$, hence with $T_{\gamma,n+1}$. Similarly T_ω commutes with $T_{\gamma,n+1}^*$. In quite a similar way it is proved that the family $\{S_{\omega,i}, T_{\omega,i}\}_{i=0}^{\infty}$ commutes with the family $\{S_{\gamma,i}, T_{\gamma,i}\}_{i=0}^{\infty}$.

References

[1] J. BRAM, Subnormal operators, *Duke Math. J.*, **22** (1955), 75—94.
[2] P. R. HALMOS, Normal dilations and extensions of operators, *Summa Brasil. Math.*, **2** (1950), 125—134.
[3] T. ITÔ, On the commutative family of subnormal operators, *J. Fac. Sci. Hokkaido Univ.*, Ser. I, **13** (1958), 1—15.
[4] J. J. SCHÄFFER, On unitary dilations of contractions, *Proc. Amer. Math. Soc.*, **6** (1955), 322.

(Received Aug. 23, 1962)