On a pair of commutative contractions

By T. ANDÔ in Bloomington (Indiana, U. S. A.)

1. Introduction

Let T be a contraction on a Hilbert space $\mathfrak{H}$, i. e. $\|T\| \leqq 1$. A unitary (resp. isometric) operator $\mathbf{U}$ is called a *unitary* (resp. *isometric*) *dilation* of T if $\mathbf{U}$ acts on a Hilbert space $\mathfrak{K}$ containing $\mathfrak{H}$ as a subspace, and

$$T^n f = \mathbf{P}\mathbf{U}^n f \quad (f \in \mathfrak{H}) \qquad n = 1, 2, \ldots \tag{1}$$

where $\mathbf{P}$ is the orthogonal projection from $\mathfrak{K}$ onto $\mathfrak{H}$. SZ.-NAGY [3, 4] proved the existence of a unitary dilation of any contraction. In this paper we shall concern ourselves with a pair of commutative contractions and prove the following theorem.

Theorem. *Let T_1, T_2 be a pair of commutative contractions. Then there exists a pair of commutative unitary operators $\mathbf{U}_1, \mathbf{U}_2$ on a Hilbert space $\mathfrak{K}$ containing $\mathfrak{H}$ as a subspace such that*

$$T_1^{n_1} T_2^{n_2} f = \mathbf{P}\mathbf{U}_1^{n_1}\mathbf{U}_2^{n_2} f \qquad (f \in \mathfrak{H};\ n_1, n_2 = 1.\ 2.\ \ldots), \tag{2}$$

where $\mathbf{P}$ is the orthogonal projection from $\mathfrak{K}$ onto $\mathfrak{H}$.

This gives a partial answer to a problem raised by SZ.-NAGY [5] in which a finite number of commutative contractions comes into question.

The author would like to thank Professor SZ.-NAGY for his valuable suggestions.

2. Reduction of the problem

First of all, if the theorem is proved, replacing the word "*unitary*" by "*isometric*", the unitary operators in question can be readily obtained, because a pair of commutative isometries can be extended to a pair of commutative unitary operators on a larger Hilbert space by ITO's theorem [2] (see also BREHMER [1]). Secondly, if $\mathbf{U}_1, \mathbf{U}_2$ are isometries on $\mathfrak{K} \supset \mathfrak{H}$ such that

$$T_i f = \mathbf{P}\mathbf{U}_i f \qquad (f \in \mathfrak{H};\ i = 1, 2) \tag{3}$$

and

$$\mathbf{U}_i(\mathfrak{K} \ominus \mathfrak{H}) \subset \mathfrak{K} \ominus \mathfrak{H} \qquad (i = 1, 2) \tag{4}$$

then the condition (2) is necessarily satisfied. Thus it suffices to prove the following proposition instead of the theorem.

For any pair of commutative contractions T_1, T_2 there exists a pair of commutative isometries $\mathbf{U}_1, \mathbf{U}_2$ with the properties (3) *and* (4).

3. Proof

For the purpose, SCHÄFFER's construction [6] is used in the following modified form; $\mathfrak{K}$ is the orthogonal sum of countably many copies of $\mathfrak{H}$, indexed by all non-negative integers: the elements of $\mathfrak{K}$ are the sequences $\varphi=\{f_n\}_0^\infty$ of elements $f_n\in\mathfrak{H}$ with norm $\|\varphi\|^2=\sum_{n=0}^{\infty}\|f_n\|^2$. $\mathfrak{H}$ is embedded in $\mathfrak{K}$ by identifying $f\in\mathfrak{H}$ with the sequence $\{f_n\}$ where $f_0=f$ and $f_n=0$ for $n>0$. Then operators $\mathbf{V}_i$ $(i=1,2)$ are defined as follows: $\{g_n\}=\mathbf{V}_i\{f_n\}$ if and only if $g_0=T_if_0$, $g_1=Z_if_0$, $g_2=0$ and $g_n=f_{n-2}$ for $n>2$ where $Z_i=(I-T_i^*T_i)^{1/2}$. Since

$$\|Z_if\|^2=\|f\|^2-\|T_if\|^2 \qquad (f\in\mathfrak{H};\ i=1,2) \tag{5}$$

from the definitions of $\mathbf{V}_1, \mathbf{V}_2$ it is readily seen that they are isometries with the properties (3) and (4) for $\mathbf{V}_i$ instead of $\mathbf{U}_i$. Moreover from (5) it follows that

$$\|Z_2T_1f\|^2+\|Z_1f\|^2 = \|T_1f\|^2-\|T_2T_1f\|^2+\|f\|^2-\|T_1f\|^2 = \|f\|^2-\|T_2T_1f\|^2$$

and similarly

$$\|Z_1T_2f\|^2+\|Z_2f\|^2 = \|f\|^2-\|T_1T_2f\|^2;$$

hence the commutativity of T_1 with T_2 implies that

$$\|Z_2T_1f\|^2+\|Z_1f\|^2 = \|Z_1T_2f\|^2+\|Z_2f\|^2. \tag{6}$$

Now consider the orthogonal sum $\mathfrak{G}$ of four copies of $\mathfrak{H}$, i. e. $\mathfrak{G}=\mathfrak{H}\oplus\mathfrak{H}\oplus\mathfrak{H}\oplus\oplus\mathfrak{H}$ and let $\mathfrak{M}_1$ and $\mathfrak{M}_2$ be the subspace consisting of all the elements of the form

$$\{Z_2T_1f, 0, Z_1f, 0\} \qquad (f\in\mathfrak{H})$$

and

$$\{Z_1T_2f, 0, Z_2f, 0\} \qquad (f\in\mathfrak{H}),$$

respectively. From the relation (6) it follows that there exists an isometry W with domain $\mathfrak{M}_2$ and range $\mathfrak{M}_1$ wich assigns $\{Z_1T_2f, 0, Z_2f, 0\}$ to $\{Z_2T_1f, 0, Z_1f, 0\}$ $(f\in\mathfrak{H})$. If $\dim(\mathfrak{G}\ominus\mathfrak{M}_2)=\dim(\mathfrak{G}\ominus\mathfrak{M}_1)$, W can be extended to a unitary operator on $\mathfrak{G}$. This restriction on dimensions is actually guaranteed; in fact, in case $\mathfrak{H}$ is finite dimensional, it follows from the fact $\dim(\mathfrak{M}_1)=\dim(\mathfrak{M}_2)$, and in the contrary case, $\dim(\mathfrak{H})=\dim(\mathfrak{G})\geqq\dim(\mathfrak{G}\ominus\mathfrak{M}_i)\geqq\dim(\mathfrak{H})$ $(i=1,2)$, because each $\mathfrak{G}\ominus\mathfrak{M}_i$ contains the subspace, isomorphic to $\mathfrak{H}$, consisting of all the elements of the form $\{0, f, 0, 0\}$ $(f\in\mathfrak{H})$. The unitary operator obtained is denoted by the same symbol W.

Now $\mathfrak{K}$ can be identified with the orthogonal sum

$$\mathfrak{H}\oplus\sum_{n=1}^{\infty}\oplus\mathfrak{G}_n,$$

where each $\mathfrak{G}_n$ is a copy of $\mathfrak{G}$, under the correspondence

$$\{f_0, f_1, f_2, \ldots, f_n, \ldots\} \leftrightarrow \{f_0, \{f_1, f_2, f_3, f_4\}, \ldots, \{f_{4n-3}, f_{4n-2}, f_{4n-1}, f_{4n}\}, \ldots\}.$$

In the sequel, this identification will always be in mind.

Let $\mathbf{W}$ be the operator on $\mathfrak{K}$ defined as follows: $\{g_n\} = \mathbf{W}\{f_n\}$ if and only if $g_0 = f_0$ and $\{g_{4n-3}, g_{4n-2}, g_{4n-1}, g_{4n}\} = W\{f_{4n-3}, f_{4n-2}, f_{4n-1}, f_{4n}\}$ $(n > 0)$. Then the unitarity of $\mathbf{W}$ follows from the unitarity of W on G, and both $\mathbf{W}$ and $\mathbf{W}^*$ have the property (4). Finally the isometries $\mathbf{U}_1$, $\mathbf{U}_2$ in question are defined by

$$\mathbf{U}_1 = \mathbf{W}\mathbf{V}_1 \quad \text{and} \quad \mathbf{U}_2 = \mathbf{V}_2\mathbf{W}^* \tag{7}$$

Since all $\mathbf{W}$, $\mathbf{W}^*$, $\mathbf{V}_1$ and $\mathbf{V}_2$ are isometries with the property (4), $\mathbf{U}_1$, $\mathbf{U}_2$ are isometries with the property (4). Obviously each $\mathbf{U}_i$ has the property (3). It remains only to prove the commutativity of $\mathbf{U}_1$ with $\mathbf{U}_2$. For any $\{f_n\} \in \mathfrak{K}$ putting

$$\{g_n\} \equiv \mathbf{U}_1\mathbf{U}_2\{f_n\} \equiv \mathbf{W}\mathbf{V}_1\mathbf{V}_2\mathbf{W}^*\{f_n\}$$

and

$$\{h_n\} \equiv \mathbf{U}_2\mathbf{U}_1\{f_n\} = \mathbf{V}_2\mathbf{W}^*\mathbf{W}\mathbf{V}_1\{f_n\} = \mathbf{V}_2\mathbf{V}_1\{f_n\},$$

simple calculations using the definitions of $\mathbf{W}$ and $\mathbf{U}_i$'s show that

$$g_0 = T_1T_2f_0$$

$$\{g_1, g_2, g_3, g_4\} = W\{Z_1T_2f_0, 0, Z_2f_0, 0\}$$

$$g_n = f_{n-4} \qquad (n > 4),$$

and

$$h_0 = T_2T_1f_0$$

$$\{h_1, h_2, h_3, h_4\} = \{Z_2T_1f_0, 0, Z_1f_0, 0\}$$

$$h_n = f_{n-4} \qquad (n > 4).$$

Since $T_1T_2 = T_2T_1$ and

$$W\{Z_1T_2f_0, 0, Z_2f_0, 0\} = \{Z_2T_1f_0, 0, Z_1f_0, 0\}$$

by the definition of W, it follows that $\mathbf{U}_1\mathbf{U}_2\{f_n\} = \mathbf{U}_2\mathbf{U}_1\{f_n\}$. Thus $\mathbf{U}_1$ commutes with $\mathbf{U}_2$.

References

[1] S. Brehmer, Über vertauschbare Kontraktionen des Hilbertschen Raumes, *Acta Sci. Math.*, **22** (1961), 106–111.

[2] T. Itô, On the commutative family of subnormal operators, *J. Fac. Sci. Hokkaido Univ.*, Ser. I, **14** (1958), 1–15.

[3] B. Sz.-Nagy, Sur les contractions de l'espace de Hilbert, *Acta Sci. Math.*, **15** (1953), 87–92.

[4] B. Sz.-Nagy, *Prolongements des transformations de l'espace de Hilbert qui sortent de cet espace*, Appendice au livre „Leçons d'analyse fonctionnelle" par F. Riesz et B. Sz.-Nagy (Budapest, 1955).

[5] B. Sz.-Nagy, Einige Probleme im Zusammenhang mit Operatoren im Hilbertraum, *Abstracts of the Second Hungarian Mathematical Congress* (Budapest, 1960).

[6] J. J. Schäffer, On unitary dilations of contractions, *Proc. Amer. Math. Soc.*, **6** (1955), 322.

(Received Aug. 23, 1962)