CHAPTER I

PARTIAL ABSTRACT ALGEBRAS

§ 1. Some notions and notations

Set theoretical join and meet of the sets A, B will be designated by $A \vee B$, $A \wedge B$ and by $\bigvee A_\alpha$, $\bigwedge A_\alpha$, if α runs over an index set. $A \setminus B$ stands for the set theoretical difference if $A \supseteqq B$, i. e. $B \vee (A \setminus B) = A$, $B \wedge (A \setminus B) = \emptyset$ (the void set).

Let a set A be given. A *partial operation* f on A is a function which maps a part of $A \times A \times \ldots \times A$ (n times) into A. The domain of f will be denoted by $D(f, A)$ ($\subseteqq A \times A \times \ldots \times A$). If $D(f, A) = A \times A \times \ldots \times A$, then f is an *operation*. If $D(f, A) = \emptyset$ then f is called *trivial*.

A *partial abstract algebra* (briefly: *partial algebra*) is a set A and a set $P(A)$ of partial operations defined on A. Let $P^*(A)$ denote the set of all non trivial operations of A. We say that the partial algebra B is the *homomorphic image* of the partial algebra A, if there is a many-one mapping η of A onto B and a one-to-one correspondence $f \to g$ between $P^*(A)$ and $P^*(B)$ such that the usual property

$$\eta f(a_1, a_2, \ldots, a_n) = g(\eta a_1, \eta a_2, \ldots, \eta a_n) \quad (a_1, a_2, \ldots, a_n) \in D(f, A)$$

holds true. It is an *isomorphism* if η is one-to-one. We should like to point out that in the definition of homomorphism and isomorphism the trivial operations are dispensed with. Endomorphisms and automorphisms are defined as usual.

According to the definition of homomorphism, an equivalence relation Θ of A is called a congruence relation if $(a_1, \ldots, a_n)$, $(b_1, \ldots, b_n) \in D(f, A)$, $a_i \equiv b_i(\Theta)$ $(i = 1, 2, \ldots, n)$, $f \in P(A)$ imply $f(a_1, \ldots, a_n) \equiv f(b_1, \ldots, b_n)$ (Θ). Under the usual partial ordering the congruence relations of A form a complete lattice $\Theta(A)$ called the *congruence lattice* of A.

Theorem 1. *If A is a partial algebra, then $\Theta(A)$ is a compactly generated lattice*[3]).

Proof. The proof of the similar assertion for algebras uses the well known description of the complete join in $\Theta(A)$. Although this fails to be true in case of partial algebras, the following weaker analogue is true: if $x \equiv y(\bigvee \Theta_\lambda)$ $(x, y \in A)$ then there exists a finite subset $\{\Theta_i\}$ of the $\{\Theta_\lambda\}$ such that $x \equiv y \left(\bigvee_{i=1}^{n} \Theta_i \right)$. Using this weaker assertion one can prove that the congruence relation Θ is compact if and only if it is of the form $\bigvee_{i=1}^{n} \Theta_{a_i b_i}$, where Θ_{ab} $(a, b \in A)$ denotes the least congruence relation under wich $a \equiv b$. From this the assertion of the theorem follows as usual.

Let A be a partial algebra and H a subset of A and P a subset of $P(A)$. If f is a partial operation of A beloging to P then it may be also considered as a partial operation [4]) of H: $(h_1, \ldots, h_n)$ $(h_i \in H)$ is in the domain of f if $(h_1, \ldots, h_n) \in D(f, A)$

[3]) The notion of compactly generated lattice is defined in § 1 of this Introduction.

[4]) There is no danger of confusion, therefore we do not introduce notation for the restricted operation..

and $f(h_1, \ldots, h_n) \in H$. With this definition H is a partial algebra and $P = P(H) \subseteq \subseteq P(A)$. In this case A will be called an *extension* of H. (Or we may say that H is a *restriction* of A.) Using this construction of partial algebras one says that a generating systems of an algebra may always be considered as a partial algebra. The converse of this statement is the

Theorem 2. *Every partial algebra may be extended to an algebra.*

Proof. The assertion is trivial: if A is a partial algebra then let $B = \{A, p\}$ where p is a new element and if $f \in P(A)$ and $(u_1, \ldots, u_n) \notin D(f, A)$ $(u_1, \ldots, u_n \in B)$ then define $f(u_1, \ldots, u_n) = p$. Obviously, B is an algebra and it is an extension of A.

§ 2. Free algebras

In the proof of Theorem 2 the least extension of a partial algebra to an algebra has been constructed. Nevertheless, this construction fails to have the property that every congruence relation of the partial algebra may be extended to the algebra, which is a very important property in this paper. Therefore we confine now our attention to the construction of an extension having this additional property.

It is much simpler to perform this construction if on the partial algebra only partial operations of one variable are defined. Since in this and in the next chapter only such partial algebras are dealt with we suppose that this is the case.

Let S be a partial algebra such that $P(S)$ consists of partial operations of one variable. In this case if $\varphi \in P(S)$ then $D(\varphi, S) \subseteq S$. Further, let $\varphi(H)$, $H \subseteq D(\varphi, S)$ denote the set of all $\varphi(x)$, $x \in H$. If $\varphi, \psi \in P(S)$ we put $\varphi\psi(x) = \varphi(\psi(x))$. Similarly, we use the notation $\varphi_1 \ldots \varphi_n(x)$ $(\varphi_1, \ldots, \varphi_n \in P(S), x \in S)$.

We fix a $\varphi \in P(S)$ and to every $x \in S \setminus D(\varphi, S)$ we define a new element $\bar{x}$, such that $\bar{x} \notin S$ and $x \neq y$, $x, y \in S \setminus D(\varphi, S)$ imply $\bar{x} \neq \bar{y}$. The set formed by S and all the $\bar{x}$ is denoted by $S[\varphi]$. We define partial operations on $S[\varphi]$:

1. Let every partial operation ψ of S different from φ be a partial operation of $S[\varphi]$ with an unchanged domain: $D(\psi, S) = D(\psi, S[\varphi])$;

2. φ is a partial operation of $S[\varphi]$; on $D(\varphi, S)$ it is defined as it was; if $x \in S \setminus D(\varphi, S)$ then $\varphi(x) = \bar{x}$; $\varphi(x)$ is defined for no $x \in S[\varphi] \setminus S$.

$S[\varphi]$ with the partial operations defined under 1 and 2 is a partial algebra; it is an extension of S. The element $\bar{x}$ $(x \in S \setminus D(\varphi, S))$ will be denoted by $\varphi(x)$.

To every $\varphi \in P(S)$ we construct $S[\varphi]$ such that if $\varphi \neq \psi$ then $S[\varphi] \wedge S[\psi] = S$. We define S_1 as the join of the $S[\varphi]$:

$$S_1 = \bigvee (S[\varphi];\ \varphi \in P(S)).$$

S_1 as the set theoretical join of partial algebras is itself a partial algebra. We may write also $P(S) = P(S_1)$, for every partial operation of S_1 is the extension of a partial operation of S. Thus S_1 is an extension of S. In a similar way we define

$$S_2 = \bigvee (S_1[\varphi];\ \varphi \in P(S)), \ldots, S_n = \bigvee (S_{n-1}[\varphi];\ \varphi \in P(S)).$$

The partial algebras $S_1, S_2, \ldots$ form an ascending chain, all of them are extensions of S, indeed, S_n is an extension of S_{n-1}; thus their join $\bar{S}$ is also a partial algebra and it is also an extension of S, and $P(\bar{S}) = P(S)$.

Theorem 3. *$\bar{S}$ as constructed above is an algebra, and $\bar{S}$ is generated by S. The algebra $\bar{S}$ is free in the following sense: if the algebra S^* is generated by the partial algebra S', $P(S')=P(S^*)$ and $x\to x'$ is an isomorphism between S and S' then $x\to x'$ may be extended to a homomorphism of $\bar{S}$ onto S^*.*

Proof: trivial.

§ 3. Extension of congruence relations

Let the partial algebra B be an extension of the partial algebra A. We say that the congruence relation Φ of B is the *extension* of the congruence relation Θ of A if $x\equiv y(\Theta)$ and $x\equiv y(\Phi)$ are equivalent whenever $x, y\in A$. If Θ has an extension, then it has, obviously, a least extension, which will be denoted by $\bar{\Theta}$.

Theorem 4. *Every congruence relation of S may be extended to $S[\varphi]$.*

Supplement. *If $\Theta\in\Theta(S)$ and $\bar{\Theta}$ is the least extension of Θ to $S[\varphi]$ then $\bar{\Theta}$ may be described as follows: $u\equiv v(\bar{\Theta})$ $(u, v\in S[\varphi])$ if and only if one of the following conditions hold:*

I. *$u, v\in S$ and $u\equiv v(\Theta)$;*

II. *$u, v\in S[\varphi]\setminus S$, i. e. $u=\varphi(x)$, $v=\varphi(y)$, where $x, y\in S\setminus D(\varphi, S)$ and either 1. $x\equiv y(\Theta)$ or 2. there exist $a=\varphi(x_0)$, $b=\varphi(y_0)\in S$ such that $x\equiv x_0(\Theta)$, $y\equiv y_0(\Theta)$, $a\equiv b(\Theta)$;*

III. *$u\in S$, $v\in S[\varphi]\setminus S$ (or symmetrically, interchanging u and v), i. e. $v=\varphi(y)$, $y\in S\setminus D(\varphi, S)$ and there exists an $a=\varphi(y_0)\in S$, for which $u\equiv a(\Theta)$ and $y\equiv y_0(\Theta)$.*

Fig. 1 helps to visualize case I III.

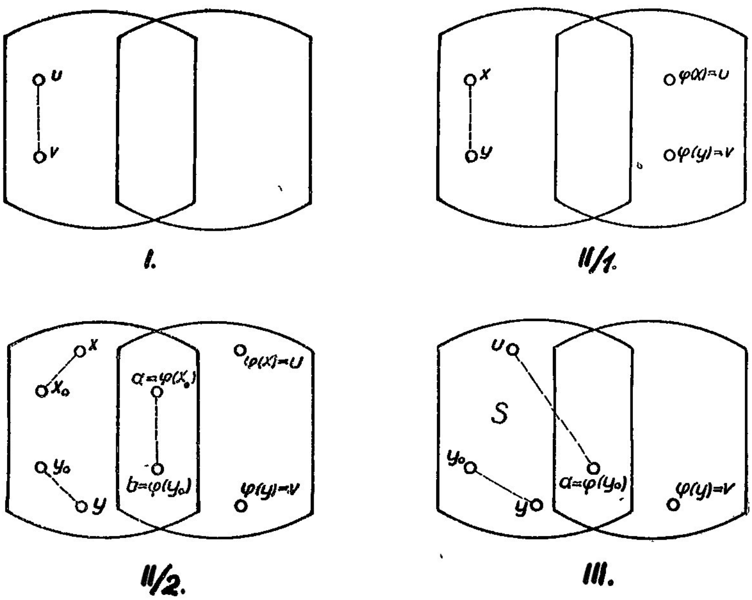

Fig. 1

In Fig. 1 a broken line connecting two elements means that the two elements are congruent modulo Θ.

Proof. Let $\Theta[\varphi]$ be the relation defined by I—III of Supplement to Theorem 4. It is enough to prove that it is a congruence relation, for the relation $\Theta[\varphi]=\overline{\Theta}$ is then obvious.

Owing to I and II/1 we get that $\Theta[\varphi]$ is reflexive and, by the symetry of I—III in u and v, it is also symmetric. The substitution property may be proved as follows: Let $\psi\in P(S[\varphi]) = P(S)$, $x\equiv y(\Theta[\varphi])$ and $x, y\in D(\psi, S[\varphi])$. We distinguish two cases:

(1) $\varphi\neq\psi$. Then $x, y\in D(\psi, S[\varphi]) = D(\psi, S)$ and by I we get $x\equiv y(\Theta)$ and so $\psi(x)\equiv\psi(y)(\Theta)$, and again by I $\psi(x)\equiv\psi(y)(\Theta[\varphi])$.

(2) $\varphi=\psi$. Then necessarily $x, y\in S$. We want to prove $\varphi(x)\equiv\varphi(y)(\Theta[\varphi])$; this follows from I if $x, y\in D(\varphi, S)$, from III with $a=\varphi(x)$ if $x\in D(\varphi, S)$, $y\notin D(\varphi, S)$ (and in the symmetrical case), from II/1 if $x, y\notin D(\varphi, S)$.

It remains to prove that $\Theta[\varphi]$ is transitive.

Let $u\equiv v(\Theta[\varphi])$, $v\equiv w(\Theta[\varphi])$; we have to prove $u\equiv w(\Theta[\varphi])$. We will distinguish 8 cases.

(α) $u, v, w\in S$. In this case $u\equiv w(\Theta[\varphi])$ is clear owing to I and the transitivity of Θ.

(β) $u, v\in S$; $w\in S[\varphi]\setminus S$; i. e. $w=\varphi(x)$, $x\in S$. By I $u\equiv v(\Theta)$; from III we conclude the existence of an $a=\varphi(x_0)\in S$ satisfying $v\equiv a(\Theta)$, $x_0\equiv x(\Theta)$. Thus $u\equiv a(\Theta)$ and $x_0\equiv x(\Theta)$, $a=\varphi(x_0)\in S$, i. e. by III we get $u\equiv w(\Theta[\varphi])$.

(β') $v, w\in S$; $u\in S[\varphi]\setminus S$. The proof is the same as under (β).

(γ) $u, w\in S$; $v\in S[\varphi]\setminus S$; i. e. $v=\varphi(x)$, $x\in S$. By III $u\equiv v(\Theta[\varphi])$ means the existence of an $a=\varphi(x_0)\in S$ such that $u\equiv a(\Theta)$, $x_0\equiv x(\Theta)$. Similarly, there exists a $b=\varphi(y_0)\in S$ with $w\equiv b(\Theta)$, $y_0\equiv x(\Theta)$. Thus $x_0\equiv y_0(\Theta)$, i. e. $a=\varphi(x_0)\equiv\varphi(y_0)= =b(\Theta)$; consequently, $u\equiv a(\Theta)$, $a\equiv b(\Theta)$, $b\equiv w(\Theta)$, so $u\equiv w(\Theta)$, and by I we get $u\equiv w(\Theta[\varphi])$.

(δ) $u\in S$; $v, w\in S[\varphi]\setminus S$; i. e. $v=\varphi(x)$, $w=\varphi(y)$. Owing to III we get that with suitable $a=\varphi(x_0)\in S$ the congruences $u\equiv a(\Theta)$, $x_0\equiv x(\Theta)$ hold. The congruence $v\equiv w(\Theta[\varphi])$ means that either

1. $x\equiv y(\Theta)$, or that
2. there exist $a'=\varphi(x_0')$ and $b=\varphi(y_0)$ such that $x_0'\equiv x(\Theta)$, $y_0\equiv y(\Theta)$, and $a'\equiv b(\Theta)$.

In the first case $x_0\equiv y(\Theta)$ and $a=\varphi(x_0)\equiv\varphi(y)=w(\Theta[\varphi])$. But $u\equiv a(\Theta)$. Thus owing to III we get $u\equiv w(\Theta[\varphi])$.

In the second case $x_0\equiv x_0'(\Theta)$, thus $a=\varphi(x_0)\equiv\varphi(x_0')=a'(\Theta)$ implying $a\equiv b(\Theta)$ and so $u\equiv b(\Theta)$. But $y_0\equiv y(\Theta)$, resulting — by III — $u\equiv w(\Theta[\varphi])$.

(δ') $w\in S$; $u, v\in S[\varphi]\setminus S$. The proof is the same as under (δ).

(ε) $v\in S$; $u, w\in S[\varphi]\setminus S$; thus $u=\varphi(x)$, $w=\varphi(y)$.

Owing to III we get the existence of $a=\varphi(x_0)$, $b=\varphi(y_0)\in S$ such that $v\equiv a(\Theta)$, $x_0\equiv x(\Theta)$, $v\equiv b(\Theta)$ and $y_0\equiv y(\Theta)$. We get from these $a\equiv b(\Theta)$, and thus owing to II/2 we get $u\equiv w(\Theta[\varphi])$.

(φ) $u, v, w\in S[\varphi]\setminus S$, thus $u=\varphi(x)$, $v=\varphi(y)$, $w=\varphi(z)$. Let $a=\varphi(x_0)$, $b=\varphi(y_0)$, $c=\varphi(z_0)$, $d=\varphi(v_0)$ be suitable elements of S. $u\equiv v(\Theta[\varphi])$ means either

a/1 $x\equiv y(\Theta)$,

or a/2 $x\equiv x_0(\Theta)$, $a\equiv b(\Theta)$, $y_0\equiv y(\Theta)$.

$v \equiv w(\Theta[\varphi])$ is equivalent to either
b/1 $y \equiv z(\Theta)$
or b/2 $y \equiv z_0(\Theta)$, $c \equiv d(\Theta)$, $v_0 \equiv z(\Theta)$.

If a/1 and b/1 hold then $x \equiv z(\Theta)$, thus — by II/1 — $u \equiv w(\Theta[\varphi])$ holds.

If a/1 and b/2 hold then $x \equiv z_0(\Theta)$, thus II implies $u \equiv w(\Theta[\varphi])$. The case when a/2 and b/1 hold is similar.

If a/2 and b/2 hold then $a \equiv b(\Theta)$, $y_0 \equiv y(\Theta)$, $y \equiv z_0(\Theta)$, $c \equiv d(\Theta)$, i. e. $a \equiv d(\Theta)$, thus $u \equiv w(\Theta[\varphi])$. The proof of Theorem 4 is finished.

Based on Theorem 4 we prove

Theorem 5. *Let S be a partial algebra and $\bar{S}$ be the free algebra generated by S* (as defined in § 2). *Every congruence relation of S may be extended to $\bar{S}$.*

Before proving this theorem, we need

Lemma 1. *Let be given a partial algebra S and a set of partial algebras $\{S_\alpha\}$, for which*

1. *S_α is an extension of S, $(P(S) = P(S_\alpha))$;*
2. *$S_\alpha \wedge S_\beta = S$ if $\alpha \neq \beta$;*
3. *$x \in S_\alpha$, $\varphi \in P(S)$, $\varphi(x) \in S_\beta$ and $\alpha \neq \beta$ imply $\varphi(x) \in S$;*
4. *every congruence relation of Θ may be extended to every S_α.*

Then $S^* = \bigvee S_\alpha$ is a partial algebra containing S, S^* is an extension of S, and every congruence relation of S may be extended to S^*.

Proof. Only the last assertion calls for proof. Let Θ_α be the extension of Θ to S_α. We define the relation Φ:

I. $x \equiv y(\Phi)$, $x, y \in S_\alpha$ is equivalent to $x \equiv y(\Theta_\alpha)$;

II. $x \equiv y(\Phi)$, $x \in S_\alpha$, $y \in S_\beta$, $\alpha \neq \beta$ if and only if with a suitable $a \in S$ we have $x \equiv a(\Theta_\alpha)$, $a \equiv y(\Theta_\beta)$.

It is routine to check that Φ is a congruence relation and, obviously, it is an extension of Θ to S^*.

Now we prove Theorem 5. Let $\Theta \in \Theta(S)$. Theorem 4 guarantees the extendability of Θ to the $S[\varphi_\alpha]$, $\varphi_\alpha \in P(S)$. The set of the $S[\varphi_\alpha]$ satisfies the hypotheses of Lemma 1, thus Θ may be extended to S_1 (which is the S^* of Lemma 1). In a similar way we get that Θ may be extended to $S_2, S_3, \ldots$ and hence to $\bar{S}$, finishing the proof of Theorem 5.

CHAPTER II

COMPACTLY GENERATED LATTICES AS CONGRUENCE LATTICES

§ 1. Preliminary constructions

Our principal aim in this chapter is to prove Theorem I (Theorem 10). This will be done in § 3 while in §§ 1 and 2 some preparations are made.

Let S be a partial algebra, $\varphi_1(x), \varphi_2(x), \varphi_3(x) \in P(S)$, $D(\varphi_1, S) = \{a\}$, $D(\varphi_2, S) = \varnothing$, $D(\varphi_3, S) = \{b\}$, $a, b \in S$ and $\varphi_1(a) = c$, $\varphi_3(b) = d$, $c, d \in S$. In the partial algebra

$\bigvee_{i=1}^{3} S[\varphi_i]$ we identify $\varphi_1(b)$ with $\varphi_2(b)$ and $\varphi_2(a)$ with $\varphi_3(a)$ getting the partial algebra T' (see Fig. 2).

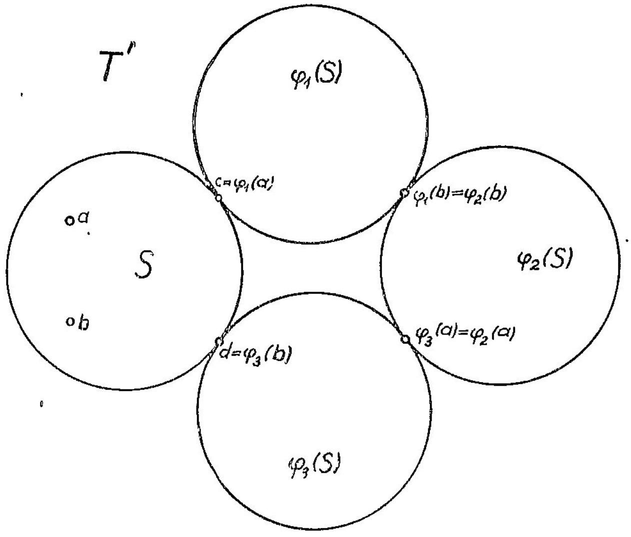

Fig. 2

T' is an extension of S but it is not necessarily true that every congruence relation of S may be extended to T'. Call a congruence relation Θ of S *admissible* if it satisfies one of the following conditions:

$$A_1 : a \not\equiv b(\Theta);$$

$$A_2 : a \equiv b(\Theta) \text{ and } c \equiv d(\Theta).$$

Roughly speaking, Θ is admissible if $a \equiv b(\Theta)$ implies $c \equiv d(\Theta)$.

Now suppose that Θ may be extended to T' and let $\overline{\Theta}$ be an extensions of Θ. If $a \equiv b(\Theta)$ then $a \equiv b(\overline{\Theta})$ and $c = \varphi_1(a) \equiv \varphi_1(b)(\overline{\Theta})$, $\varphi_2(a) \equiv \varphi_2(b)(\overline{\Theta})$, $\varphi_3(a) \equiv \varphi_3(b) = d(\overline{\Theta})$, thus the assumptions $\varphi_1(b) = \varphi_2(b)$ and $\varphi_2(a) = \varphi_3(a)$ imply that $c \equiv d(\overline{\Theta})$, consequently $c \equiv d(\Theta)$.

This proves that if a congruence relation is extensible then it is admissible. This and the converse of this statement is contained in

Theorem 6. *The congruence relation Θ of S is admissible if and only if it may be extended to T'.*

Proof. We have to prove the „only if" part of the theorem. Suppose that Θ is admissible and define a relation Θ^* of T' as follows: let $u \equiv v(\Theta^*)$ mean for $u, v \in S$ that $u \equiv v(\Theta)$ and for $u, v \in \varphi_i(S)$ ($\varphi_i(S)$ denotes the set of all $\varphi_i(x)$, $x \in S$) that $u = \varphi_i(x), v = \varphi_i(y), x, y \in S$ and $x \equiv y(\Theta)$, otherwise let $u \not\equiv v(\Theta^*)$. Then Θ^* is a symmetric and reflexive relation having the substitution property. Let $\overline{\Theta}$ denote the transitive extension of Θ^*. The relation $\bar{\Theta}$ is trivially a congruence relation of T'. We prove that on S and on the $\varphi_i(S)$ the relations $\bar{\Theta}$ and Θ^* coincide. It is enough to prove this for S, a similar reasoning applies then to $\varphi_i(S)$. *Per definitionem* $u \equiv v(\bar{\Theta})$, $u, v \in S$ if and only if there exists a sequence $u = x_0, x_1, \ldots, x_n = v$ of elements of T' such that $x_{i-1} \equiv x_i(\Theta^*)$ $(i = 1, 2, \ldots, n)$. If all the $x_i \in S$ then $u \equiv v(\Theta)$, thus $u \equiv v(\Theta^*)$ is obvious. S and a $\varphi_i(S)$ ($\varphi_i(S)$ and a $\varphi_j(S)$, $i \neq j$) have at most *one* element in common. This if we impose the natural condition on the sequence $x_0, \ldots, x_n$ that no element may occur more than once, then we see that the sequence must contain elements from all the $\varphi_i(S)$. It is easy to see that such a sequence may be substituted by the following simpler one: $u = x_0$, $x_1 = c$, $x_2 = \varphi_1(b)$, $x_3 = \varphi_2(a)$, $x_4 = d$, $x_5 = v$ (or interchanging u with v). $x_1 \equiv x_2(\Theta^*)$ implies $a \equiv b(\Theta)$, and by A_2 we get $c \equiv d(\Theta)$; thus $u \equiv v(\Theta)$ and $u \equiv v(\Theta^*)$, proving that $\bar{\Theta}$ and Θ^* are equivalent on S, finishing the proof of this theorem. We proved a little more than required; we have exhibited at the same time a well-described extension of an admissible congruence relation.

* * *

Now let S be a partial algebra; the operations of S will be denoted by $\omega^\nu(x)$ $(\nu \in \Omega_1)$ and the partial operations by $\varphi_i^\mu(x)$ $(\mu \in \Omega_2, i = 1, 2, 3)$; we suppose that $D(\varphi_1^\mu, S) = \{a^\mu\}$, $D(\varphi_2^\mu, S) = \emptyset$, $D(\varphi_3^\mu, S) = \{b^\mu\}$ and $\varphi_1^\mu(a^\mu) = c^\mu$, $\varphi_3^\mu(b^\mu) = d^\mu$ $(a^\mu, b^\mu, c^\mu, d^\mu \in S)$. To each μ the φ_i^μ are of the type described at the beginning of the section, thus the corresponding T' — which now will be denoted by T_μ — may be constructed. We also suppose that $\mu \neq \mu'$ implies $T_\mu \cap T_{\mu'} = S$. Further, let $T = \bigvee_\mu T_\mu$ and $\bar{T}$ the free algebra generated by T.

The congruence relation Θ of S is called *admissible* if it is admissible for any fixed $\mu \in \Omega_2$ (i. e. if for $\mu \in \Omega_2$ the congruence $a^\mu \equiv b^\mu(\Theta)$ holds, then $c^\mu \equiv d^\mu(\Theta)$).

Let $\Theta \in \Theta(S)$; then there exists a unique admissible congruence relation Θ' which is minimal with respect to $\Theta' \geqq \Theta$. Indeed, let Ω_2^1 denote the set of those $\mu \in \Omega_2$ for which $a^\mu \equiv b^\mu(\Theta)$, and define $\Theta_1 = \Theta \cup \bigvee(\Theta_{c^\mu d^\mu};\ \mu \in \Omega_2^1)$, if Θ_{n-1} is defined, set $\Theta_n = (\Theta_{n-1})_1$ and $\Theta' = \bigvee_{n=1}^{\infty} \Theta_n$. Obviously, Θ' is admissible and the least admissible congruence relation $\geqq \Theta$.

A central result of this paper is

Theorem 7. *The congruence relation Θ of S may be extended to $\bar{T}$ if and only if it is admissible. To every pair u, v of elements of $\bar{T}$, there exists a uniquely determined least admissible congruence relation Θ such that under $\bar{\Theta}$ (the minimal extension of Θ to $\bar{T}$) u and v are congruent.*

The first assertion of the theorem is obvious from Theorems 5 and 6 and Lemma 1. The second assertion is rather involved; as a preparation we will prove Lemmas 2 and 3.

Lemma 2. *Let S and T' be as in Theorem 6. Then to every $u, v \in T'$ there exists a least admissible $\Theta \in \Theta(S)$ such that $u \equiv v(\bar{\Theta})$.*

Proof. If $u, v \in S$ and $a \not\equiv b(\Theta_{uv})$ (resp. $a \equiv b(\Theta_{uv})$), then Θ_{uv} (resp. $\Theta_{uv} \cup \Theta_{cd}$) is the least admissible congruence relation. Θ may be found similarly if $u, v \in \varphi_i(S)$.

If u and v are not both in S or in $\varphi_i(S)$ then it is not simple to find Θ. We will show how to construct Θ in a typical case, the complete discussion will be left to the reader.

Let $u \in S$, $v \in \varphi_2(S)$; i. e. $v = \varphi_2(x)$, $x \in S$. We state $\Theta = \Theta_{ax} \cup \Theta_{bx} \cup \Theta_{uc} \cup \Theta_{cd}$. This Θ is admissible for $a \equiv b(\Theta)$ and $c \equiv d(\Theta)$. Further, $u \equiv d(\Theta)$, $b \equiv a(\Theta)$, $a \equiv x(\Theta)$, so $u \equiv d(\overline{\Theta})$, $\varphi_3(b) \equiv \varphi_3(a) = \varphi_2(a)(\overline{\Theta})$, $\varphi_2(a) \equiv \varphi_2(x)(\overline{\Theta})$; consequently $u \equiv v = \varphi_2(x)(\overline{\Theta})$. Finally, we have to prove that if $\Phi \in \Theta(S)$, Φ is admissible, and $u \equiv v(\overline{\Phi})$, then $\Phi \geqq \Theta$. Indeed, $u \equiv v(\overline{\Phi})$ (by the proof of Theorem 6) implies that either

1. $u \equiv d(\Phi^*)$, $d = \varphi_3(b) \equiv \varphi_3(a) = \varphi_2(a)(\Phi^*)$, $\varphi_2(a) \equiv \varphi_2(x)(\Phi^*)$

or

2. $u \equiv c(\Phi^*)$, $c = \varphi_1(a) \equiv \varphi_1(b) = \varphi_2(b)(\Phi^*)$, $\varphi_2(b) \equiv \varphi_2(x)(\Phi^*)$,

where Φ^* is the relation defined in the proof of Theorem 6.

Let us consider the first possibility. By the definition of Φ^* we get from the relations of 1 the congruences $u \equiv d(\Phi)$, $b \equiv a(\Phi)$, $a \equiv x(\Phi)$. Consequently, $\Theta_{ud} \cup \Theta_{ba} \cup \Theta_{ax} \leqq \Phi$. Thus $a \equiv b(\Phi)$; hence by A_2 we get $c \equiv d(\Phi)$, i. e. $\Theta_{cd} \leqq \Phi$. So $\Theta_{ud} \cup \Theta_{ab} \cup \Theta_{ax} \cup \Theta_{cd} \leqq \Phi$. But $\Theta = \Theta_{ud} \cup \Theta_{ab} \cup \Theta_{ax} \cup \Theta_{cd}$ is obvious, thus in the first case $\Theta \leqq \Phi$ is proved. The second case may be proved in the same way, thus the proof is finished.

Lemma 3. *Let S and T be as in Theorem 7. Then to every $u, v \in T$ there exists a least admissible congruence relation $\Theta \in \Theta(S)$ such that $u \equiv v(\overline{\Theta})$.*

Proof. Let $u, v \in T = \bigvee T_\mu$, it is enough to consider the case $u \in T_\mu \setminus S$, $v \in T_\nu \setminus S$, $\mu \neq \nu$, for the other cases were treated in Lemma 2.

There are nine cases to be distinguished; from these we pick out a typical one, the others may be treated similarly.

Let $u \in \varphi_3^\mu(S) \setminus S$ and $v \in \varphi_2^\nu(S)$, i. e. $u = \varphi_3^\mu(x)$, $v = \varphi_2^\nu(y)$, $x, y \in S$. Let Φ be admissible such that $u \equiv v(\overline{\Phi})$. Then one of the following conditions 1—4 holds;

1.
$$u = \varphi_3^\mu(x) \equiv \varphi_3^\mu(b^\mu) = d^\mu(\Phi^*),\ d^\mu \equiv c^\nu = \varphi_1^\nu(a^\nu)(\Phi^*),$$
$$\varphi_1^\nu(a^\nu) \equiv \varphi_1^\nu(b^\nu)(\Phi^*),\ \varphi_1^\nu(b^\nu) = \varphi_2^\nu(b^\nu) \equiv \varphi_2^\nu(y) = v(\Phi^*)$$

from which we get

$$\Theta_1 = \Theta_{xb^\mu} \cup \Theta_{d^\mu c^\nu} \cup \Theta_{a^\nu b^\nu} \cup \Theta_{b^\nu y} \leqq \Phi.$$

2.
$$u = \varphi_3^\mu(x) \equiv \varphi_3^\mu(b^\mu) = d^\mu(\Phi^*),\ d^\mu \equiv d^\nu = \varphi_3^\nu(b^\nu)(\Phi^*),$$
$$\varphi_3^\nu(b^\nu) \equiv \varphi_3^\nu(a^\nu)(\Phi^*),\ \varphi_3^\nu(a^\nu) = \varphi_2^\nu(a^\nu) \equiv \varphi_2^\nu(y) = v(\Phi^*)$$

from which we get

$$\Theta_2 = \Theta_{xb^\mu} \cup \Theta_{d^\mu d^\nu} \cup \Theta_{b^\nu a^\nu} \cup \Theta_{a^\nu y} \leqq \Phi.$$

3.-4.
$$u = \varphi_3^\mu(x) \equiv \varphi_3^\mu(a^\mu) = \varphi_2^\mu(a^\mu)(\Phi^*),\ \varphi_2^\mu(a^\mu) \equiv$$
$$\equiv \varphi_2^\mu(b^\mu)(\Phi^*);\ \varphi_2^\mu(b_\mu) = \varphi_1^\mu(b^\mu) \equiv \varphi_1^\mu(a^\mu) = c^\mu\ (\Phi^*),$$

further in case 3 $c^\mu \equiv c^\nu(\Phi^*)$, $c^\nu = \varphi^\nu(a^\nu) \equiv \varphi^\nu(b^\nu) = \varphi_2^\nu(b^\nu)(\Phi^*)$, $\varphi_2^\nu(b^\nu) \equiv \varphi_2^\nu(y) = v(\Phi^*)$ and in case 4 $c^\mu \equiv d^\nu(\Phi^*)$, $d^\nu = \varphi_3^\nu(b^\nu) \equiv \varphi_3^\nu(a^\nu) = \varphi_2^\nu(a^\nu)(\Phi^*)$, $\varphi_2^\nu(a^\nu) \equiv$

$\equiv \varphi_2^\nu(y) = v(\Phi^*)$, and so we get respectively

$$\Theta_3 = \Theta_{xa^\mu} \cup \Theta_{b^\mu a^\mu} \cup \Theta_{c^\mu c^\nu} \cup \Theta_{a^\nu b^\nu} \cup \Theta_{b^\nu y} \leqq \Phi,$$

$$\Theta_4 = \Theta_{xa^\mu} \cup \Theta_{b^\mu a^\mu} \cup \Theta_{c^\mu d^\nu} \cup \Theta_{b^\nu a^\nu} \cup \Theta_{a^\nu y} \leqq \Phi.$$

We prove that $\Theta = \Theta_1'$ (the notation was introduced before Theorem 7). It is enough to show that $\Theta_1' \leqq \Theta_i'$ for $i=2, 3, 4$. But $(\Phi')' = \Phi'$ holds for every $\Phi \in \Theta(A)$, thus it is enough to prove $\Theta_1 \leqq \Theta_i'$ $(i=2, 3, 4)$.

The case $i=2$ is trivial because of $\Theta_1' = \Theta_2'$. (This follows from the special choice of u and v.) Now we prove $\Theta_1 \leqq \Theta_3'$ as follows: obviously

$$\Theta_{xb^\mu} \leqq \Theta_{xa^\mu} \cup \Theta_{b^\mu a^\mu},$$

further
$$\Theta_{d^\mu c^\nu} \leqq (\Theta_{b^\mu a^\mu} \cup \Theta_{c^\mu d^\nu} \cup \Theta_{a^\nu b^\nu})';$$

thus the relation

$$\Theta_1 \leqq \Theta_3'$$

is obvious. The last relation $\Theta_1 \leqq \Theta_4'$ may be proved similarly finishing the proof of Lemma 3.

Now we are going to prove Theorem 7. Let $u, v \in \overline{T}$, $u = \gamma_1 \ldots \gamma_n(x)$, $v = \delta_1 \ldots \delta_m(y)$, $\gamma_i, \delta_i \in P(S)$, $x \notin D(\gamma_n, S)$, $y \notin D(\delta_m, S)$. Now we use the assumption that all the partial operations of S are either operations (the $\omega^\nu(x)$, $\nu \in \Omega_1$) or of the special type φ_i^μ. It follows that γ_n and δ_m are of type φ_i^μ.

Let T^p denote the set of all elements of $\overline{T}$ which may be represented in the form

$$\gamma_1 \ldots \gamma_n(x), \quad n \leqq p, \quad x \in S, \quad x \notin D(\gamma_n, S), \quad \gamma_1, \ldots, \gamma_n \in P(S).$$

Then
$$S = T^0 \subseteqq T^1 = T \subseteqq T^2 \ldots$$

and
$$\overline{T} = \bigcup T^i.$$

We suppose $u, v \in T^p$ and prove our assertions by induction on p.

The case $p=1$ was settled in Lemmas 2 and 3. Let us suppose that we have proved the assertion for all $k<p$. The set $T^p \setminus T^{p-1}$ is the join of sets of the form

$$H_\alpha = \bigcup_{i=1}^{3} \lambda_1 \ldots \lambda_{p-1} \varphi_i^\mu(S)$$

(α depending on $\lambda_1, \ldots \lambda_{p-1}, \mu$ and i). If both u and v are in T^{p-1} then the assertion follows from the induction hypothesis. So we may suppose that $u \notin T^{p-1}$, thus $u \in H_\alpha$ for some α.

Now we may repeat the chain of thoughts of Lemmas 2 and 3; the role of S is taken by T^{p-1}, that of T_ν by H_α. The only difference is that for S the assertion was trivial; now, for T^{p-1} it is the induction hypothesis. This is essential when we are looking for the least admissible congruence relation, under whose extension e. g. c^μ and d^ν are congruent.

§ 2. Compactly generated lattices

Before proving Theorem I we need two easy theorems on compactly generated lattices the first of which is probably well-known while the second is due to NACHBIN.

Theorem 8. *Let L be a compactly generated lattice and H a complete sublattice of L. Then H is also compactly generated.*

Proof. A principal ideal of a compactly generated lattice is obviously compactly generated. Thus we may suppose that the unit element of H is the unit element of L. Now let u be an arbitrary element of L, and define $a(u)$ as the meet of all $h \in H$ with $h \geqq u$,

$$a(u) = \bigwedge (h; h \in H, h \geqq u).$$

H is a complete sublattice, thus $a(u) \in H$; in fact $a(u)$ is the least element of H which is $\geqq u$. It is routine to check that if u is compact in L then $a(u)$ is compact in H. From this the assertion follows easily.

Let F be a semilattice with O, i. e. let be defined on F a binary operation $\cup$, which is idempotent, commutative and associative, further, $x \cup O = x$ for all $x \in F$. A subset I of F is called an ideal, if it is non-void and $x \cup y \in I (x, y \in F)$ if and only if x and $y \in F$. A natural partial ordering of F is: $x \leqq y$ if and only if $x \cup y = y$; then $x \cup y$ is the least upper bound of x and y. Now, I is an ideal if and only if 1. $x, y \in I$ imply $x \cup y \in I$; 2. $x \in I$, $y \in F$, $y \leqq x$ imply $y \in I$. The set $I(F)$ of all ideals of F form a complete lattice if the partial ordering is the set-inclusion.

Theorem 9. (NACHBIN [10].) *A lattice L is compactly generated if and only if L is isomorphic to the lattice of all ideals of a semilattice F with O. In fact, if L is compactly generated then F is isomorphic to the semilattice of all compact elements of L. Further, the compact elements of $I(F)$ are the principal ideals.*

A sketch of the proof. Let L be the compactly generated lattice and F the semilattice with zero of the compact elements of L. First, one has to prove that F is really a semilattice, i. e. the join of two compact elements is again compact. Then take an $a \in L$ and define I_a as the set of all $x \in F$ with $x \leqq a$. The correspondence $a \to I_a$ is an isomorphism between L and $I(F)$. The only non-trivial step is to prove that if I is an ideal of F and $a = \bigvee (x; x \in I)$, where the complete join is in L, then $I_a = I$. Indeed, if $y \in I_a$, then $y \leqq \bigvee (x; x \in I)$. Thus by the compactness of y we get the existence of a finite subset I' of I such that $y \leqq \bigvee (x; x \in I')$, i. e. $y \in I$. We proved $I_a \subseteq I$ while $I \subseteq I_a$ is trivial, thus $I = I_a$ as required.

§ 3. A characterization theorem

Now we are ready to prove Theorem I.

Theorem 10. *A lattice L is compactly generated if and only if there exists an abstract algebra A such that L is isomorphic to $\Theta(A)$.*

Proof. It is known that $\Theta(A)$ is compactly generated (e. g. it follows easily from Theorem 8).

Now suppose that L is a compactly generated lattice with more than 2 elements. Then there exists a semilattice F such that L is isomorphic to $I(F)$. By using this fact, we construct first a partial algebra B with $\Theta(B)\cong L$.

The elements of B are the finite subsets of $F\setminus\{O\}$. The void set is also an element of B if we identify it with the element O of F. Therefore, it will be denoted by O. We define operations and partial operations on B ($\vee$ and $\wedge$ denote the set theoretical union and intersection, i. e. the operations of B; $\cup$ denotes the only operation of F):

1. to every $u\in B$ let be assigned two operations

$$\varphi_u(x) = u\vee x \quad \text{and} \quad \psi_u(x) = u\wedge x,$$

2. to any $a, b, c\in B$ with $c\leqq a\cup b$ let a partial operation $\alpha_{abc}(x)$ be defined, whose domain is O and $\{a, b\}$: let $\alpha_{abc}(0)=0$, $\alpha_{abc}(\{a, b\})=\{c\}$.

We assert that $\Theta(B)\cong I(F)$. First observe that B is a generalized Boolean algebra endowed with the partial operations $\alpha_{abc}(x)$; in fact, the join and meet operation of B was given in such a way that one variable was fixed. Thus every congruence relation Θ is completely determined by $I(\Theta)=\{x; x\equiv 0(\Theta)\}$. Every element of B is a finite join of atoms, thus $I(\Theta)$ is completely determined by $I\{\Theta\}$, the set of atoms contained in $I(\Theta)$. The elements of $I\{\Theta\}$ are of the form $\{a\}$, where $a\in F$. Let $\bar{I}\{\Theta\}$ denote a subset of F consisting of 0 and of all a for which $\{a\}\in \in I\{\Theta\}$.

We prove that $\Theta\to\bar{I}\{\Theta\}$ is an isomorphism between $\Theta(B)$ and I(F).

First we prove that $\bar{I}\{\Theta\}$ is an ideal of F. If $a, b\in\bar{I}\{\Theta\}$ then $\{a\}$ and $\{b\}\in I\{\Theta\}$, thus $\{a, b\}\in I(\Theta)$. But applying $\alpha_{a,b\, a\cup b}$ we get $\alpha_{a,b,\, a\cup b}(\{a, b\})\equiv\alpha_{a,b,\, a\cup b}(0)(\Theta)$, i. e. $\{a\cup b\}\in I(\Theta)$ and so $a\cup b\in\bar{I}\{\Theta\}$. On the other hand, if $c\leqq a\in\bar{I}\{\Theta\}$, then $\{a\}\in \in I\{\Theta\}$; thus $\{a\}\equiv 0$ (Θ) and then $\alpha_{aac}(\{a\})\equiv\alpha_{aac}(0)$ (Θ) i. e. $\{c\}\equiv 0(\Theta)$ and we reached $c\in\bar{I}\{\Theta\}$, as required.

Now let $I\in I(F)$, we prove that there exists a $\Theta\in\Theta(B)$ such that $I=\bar{I}\{\Theta\}$. On defining Θ it is enough to give a criteria for an element x of B to be congruent to 0. This is the following: let $x\equiv 0$ (Θ) if and only if $x=0$ or x is the join of atoms $\{a\}$ such that $a\in I$. It is routine to check that Θ is a congruence relation and $\bar{I}\{\Theta\}=I$.

Thus $\Theta\to\bar{I}\{\Theta\}$ is a one-to-one order preserving correspondence between $\Theta(B)$ and $I(F)$, so this is an isomorphism.

To make possible the application of the results developed so far we change B to B'. This new partial algebra B' is essentially the same as B only every operation $\alpha_{abc}(x)$ is replaced by three operations: $\overset{i}{\alpha}_{abc}(x)$ $(i=1, 2, 3)$. Let

$$D(\overset{1}{\alpha}_{abc}, B')=\{\{a, b\}\}, \quad D(\overset{2}{\alpha}_{abc}, B')=\emptyset, \quad D(\overset{3}{\alpha}_{abc}, B')=\{0\},$$

and

$$\overset{1}{\alpha}_{abc}(\{a, b\})=\{c\}, \quad \overset{3}{\alpha}_{abc}(0)=0.$$

Obviously, B' has more congruence relations than B had, but using the notion of admissible congruence relations, as defined before Theorem 7, we see that a congruence relation Θ of B' is a congruence relation of B if and only if it is admissible.

Now we apply the construction of Theorem 7 (we may do so, for every partial operation of B' is either an operation, or one of the type φ_i^μ, $i=1, 2, 3$, $\mu\in\Omega_2$;

here Ω_2 is the set of all triples a, b, c of F, for which $c \leqq a \cup b$), leading to an algebra B_1 (which was $\overline{T}$ in Theorem 7). Now, according to Theorem 7, every admissible congruence relation Θ of B' may be extended to a congruence relation $\overline{\Theta}$ of B_1; further, to every pair u, v of elements of B_1, there exists a smallest admissible congruence relation Θ, such that $u \equiv v(\overline{\Theta})$. Denoting by Φ' the smallest admissible congruence relation $\geqq \Phi$, it is obvious that $\Theta = \Theta'_{a(u,v)\,0}$ with a suitable $a(u, v) \in B'$. But $\Theta'_{a(u,v)\,0} = \Theta_{a(u,v)\,0}$ (this is perhaps the most important property of B'!) thus we can associate with Θ an element $a(u, v)$ of B'. If we require that $a(u, v)$ be an atom, then it is uniquely determined.

Now we define for every $u, v \in B$, three partial operations $\overset{i}{\alpha}_{uv}(x)$, such that

$$D(\overset{1}{\alpha}_{uv}, B_1) = \{u\}, \quad D(\overset{2}{\alpha}_{uv}, B_1) = \emptyset, \quad D(\overset{3}{\alpha}_{uv}, B_1) = \{v\},$$

and

$$\overset{1}{\alpha}_{uv}(u) = a(u, v), \quad \overset{3}{\alpha}_{uv}(v) = 0.$$

If we consider B_1 together with these new partial operations, we get B_1'.

We assert that a congruence relation Θ of B_1 is admissible if and only if it is the extension of an admissible congruence relation of B'.

First, let Φ be an admissible congruence relation of B_1, and let Θ denote the congruence relation of B' which is induced by Φ (i. e. $x \equiv y(\Theta)$, $x, y \in B'$ if and only if $x \equiv y(\Phi)$). Let $u \equiv v(\Phi)$, $u, v \in B_1'$. Φ is admissible, so $a(u, v) \equiv 0(\Phi)$; thus $a(u, v) \equiv 0(\Theta)$. We get that in B' the relation $\Theta_{a(u,v)\,0} \leqq \Theta$ holds true. By definition

$$u \equiv v(\overline{\Theta}_{a(u,v)0}),$$

thus

$$u \equiv v(\overline{\Theta}),$$

and we see that $\overline{\Theta} = \Phi$. On the other hand, if $\Phi = \overline{\Theta}$ with a suitable $\Theta \in \Theta(B')$, and $u \equiv v(\Phi)$, then $\Theta_{a(u,v)0} \leqq \Theta$ by the definition of $a(u, v)$, and so $a(u, v) \equiv 0(\Phi)$; i. e., Φ is admissible.

Now, we construct from B_1' an algebra B_2 by the method of Theorem 7, and proceeding so we get $B_2', B_3, \ldots$ and so on.

We have constructed an ascending sequence (of type ω) of algebras

$$B' \subset B_1 \subset B_2 \subset \ldots$$

Let A be the union of these:

$$A = \bigcup_{i=1}^{\infty} B_i.$$

A is obviously an algebra. Every admissible congruence relation of B' may be extended to B_1, from B_1 to B_2 and so forth to A. We assert that A has no other congruence relation. Of course, a congruence relation Φ of A induces a congruence relation Φ_n of B_n ($n = 1, 2, \ldots$). But Φ_n may be extended to B_{n+1} (in fact, Φ_{n+1} is such an extension), thus — as we have proved above — Φ is an extension of an admissible congruence relation of B'. Thus $\Theta(A)$ is isomorphic to the lattice of all admissible congruence relations of B', which is isomorphic to L, completing the proof of Theorem 10.

§ 4. Applications

In this section we will draw some conclusions from Theorem 10.

Corollary 1. *To every finite lattice L, there corresponds an abstract algebra A such that* $L \cong \Theta(A)$.

More generally:

Corollary 2. *Let L be a lattice with zero element and satisfying the ascending chain condition.**) *Then there exists an abstract algebra A with* $L \cong \Theta(A)$.

The assertion of Corollary 2 is obvious from Theorem 10, for if L satisfies the hypotheses of Corollary 2, then every ideal of L is a principal one, thus $L \cong I(L)$; Theorem 10 gives an algebra A with $\Theta(A) \cong I(L)$; hence we get $L \cong \Theta(A)$, as asserted.

Corollary 3. *A lattice L has a complete representation if and only if L is compactly generated.*

This is now obvious, for $\mathscr{E}(H)$ (see the notation in § 2 of the Introduction) is compactly generated and by Theorem 8 every complete sublattice of a compactly generated lattice is itself compactly generated. Thus if L has a complete representation $\langle F, H \rangle$ then the sublattice of $\mathscr{E}(H)$ formed by the $F(x)$, $x \in L$ is compactly generated and so is L. Conversely, if L is compactly generated, then by Theorem 10 there exists an algebra A with $L \cong \Theta(A)$; let $\varphi : x \to x\varphi \in \Theta(A)$ be this isomorphism. If $\langle F, A \rangle$ is the natural (complete) representation of $\Theta(A)$ (see § 2 of Introduction) then $\langle F\varphi, A \rangle$ is a complete representation of L, where $F\varphi$ denotes the product of the mappings F and φ.

Corollary 4. (Whitman [11].) *Every lattice has a representation.*

* * *

We get an other type of application if we consider the special properties of the algebra A, constructed in the proof of Theorem 10.

In our paper [6] we have proved the following theorem:

To every abstract algebra C there exists an abstract algebra D such that $\Theta(C) \cong \Theta(D)$ and every compact congruence relation of D is of the form Θ_{ab}.

The question arises whether or not it is possible to choose such a D where the element a may be fixed. An answer is given in

Corollary 5. *To every abstract algebra C there exists an abstract algebra D and a fixed element o of D such that* $\Theta(C) \cong \Theta(D)$, *and every compact congruence relation of D is of the form* $\Theta_{oa} (a \in D)$.

Let $L = \Theta(C)$ and $D = A$, where A, is the algebra constructed in Theorem 10 if we start with L. Then $A = D$ has the property stated with $o = 0$. The easy proof is left to the reader.

Let $G(A)$ denote the automorphism group of A. The question arises what relation has the structure of $G(A)$ to $\Theta(A)$. We will prove that already the simplest $G(A)$ allows $\Theta(A)$ to be arbitrary.

---

*) This means that if $x_1, x_2, \ldots$ are elements of L such that $x_1 \leqq x_2 \leqq \ldots$, then there exists an integer n such that $x_n = x_{n+1} = \ldots$.

Corollary 6. *The algebra A constructed in* § 3 *has a trivial automorphism group, i. e.* $G(A)\cong 1$.

Proof. The reader should remember that there is a subset B' of A such that B' generates A; there is an operation φ_0 which is the identity operation on B', i. e. $\varphi_0(x)=x$ for all $x\in B'$. But if $x\notin B'$, then by free generation $\varphi_0(x)\neq x$; thus

(i) $x\in B'$ if and only if $\varphi_0(x)=x$, where φ_0 is a fixed operation of A and B' is a generating system of A.

Suppose $\alpha\in G(A)$ and $x\in B'$ then $\varphi_0(\alpha x)=\alpha\varphi_0(x)=\alpha x$ and thus by (i) we get $\alpha x\in B'$. On the other hand if $x\in A$ and $\alpha x\in B'$ then $x=\alpha^{-1}(\alpha x)\in B'$. We get the following result:

(ii) α (restricted to B') is an automorphism of B'.

By free generation this implies

(iii) the automorphism groups of B' and A are isomorphic.

B' is a generating system of the whole A; it follows that if $\alpha\neq\beta$ are automorphisms of A then their restrictions to B' are different automorphisms of B'; we conclude:

(iv) if $G(B')=1$ then $G(A)=1$.

Thus Corollary 6 is proved if $G(B')=1$.

Now suppose $G(B')\neq 1$, i. e. $\alpha\in G(B')$, $x\in B'$ and $\alpha x\neq x$. It is no restriction to suppose x is an atom. Obviously, there exist in B_1 elements u, v such that $a(u, v)=x$, i. e. there is a partial operation β of B_1' which is defined only at u and $\beta(u)=x$. This implies $\alpha(\beta(u))\neq\beta(u)$, i. e. $\beta(\alpha u)\neq\beta(u)$, thus $\alpha u\neq u$ and $\beta(\alpha u)=\alpha x\in B'$. But $\beta(a)$ is in B' if and only if $a=u$ or $a=v$ thus $\alpha u=v$, and we reach $\alpha x=0$, a contradiction.

* * *

Finally we mention

Corollary 7. *A complete lattice L has a complete subgroup representation if and only if L is compactly generated.*

An application of Corollary 3 shows that it is enough to prove that $\mathscr{E}(H)$, the lattice of all equivalence relations of A, has a complete subgroup representation. It is a result of G. Birkhoff that $\mathscr{E}(H)$ has a subgroup representation (see [11], where the proof is reproduced). But his proof gives, in fact, a complete subgroup representation of $\mathscr{E}(H)$, as may be easily checked. Thus Corollary 7 is proved.

CHAPTER III

ABSTRACT ALGEBRAS OF TYPE 2 AND 3

§ 1. Preliminary results

If we want to prove Theorems II′ and III′ then it is not enough to have the theory of free algebras developed only for algebras with unitary operations. Therefore we now formulate these results for arbitrary algebras.

Let S be a partial algebra and $\varphi\in P(S)$. $D(\varphi, S)$ denotes the n-tuples $(a_1,\dots,a_n)$ for which φ is defined. We assigne to every n-tuple $(u_1,\dots,u_n)\notin D(\varphi, S)$ a new element $X_{u_1,\dots,u_n}$ such that if $(u_1,\dots,u_n)\neq(v_1,\dots,v_n)$ then $X_{u_1,\dots,u_n}\neq X_{v_1,\dots,v_n}$. $S[\varphi]$ denotes the set S together with the new elements. We define operations on $S[\varphi]$:

1. $\psi(a_1,\dots,a_m)$ is defined for a $\psi\neq\varphi$ if and only if $(a_1,\dots,a_m)\in D(\psi,S)$.

2. $\varphi(a_1,\dots,a_n)$ is unchanged if $(a_1,\dots,a_n)\in D(\varphi,S)$; if all the $a_i\in S$ but $(a_1,\dots,a_n)\notin D(\varphi,S)$ then $\varphi(a_1,\dots,a_n)=X_{a_1,\dots,a_n}$; for other n-tuples φ is not defined.

Now construct $S[\varphi]$ for all $\varphi\in P(S)$ such that if $\varphi\neq\psi$ then $S[\varphi]\wedge S[\psi]=S$; define $S_1=\bigvee(S[\varphi];\ \varphi\in P(S))$, $S_2=\bigvee(S_1[\varphi];\ \varphi\in P(S))$ and so on and $\overline{S}=\bigvee_{i=1}^{\infty} S_i$.

The same proof as those of Theorem 3, 4, 5 applies to get the following result.

Theorem 11. *$\overline{S}$ is the free algebra generated by S. Every congruence relation of S may be extended to $\overline{S}$.*

* * *

Let S be a partial algebra, whose partial operations are either operations $\omega^\nu(x_1,\dots,x_n)$ $(\nu\in\Omega_1)$ or of the type $\varphi_i^\mu(x)$: $i=1,2,3$, $\mu\in\Omega_2$ and $D(\varphi_1^\mu,S)=\{a^\mu\}$, $D(\varphi_2^\mu,S)=\emptyset$, $D(\varphi_3^\mu,S)=\{b^\mu\}$. The congruence relation Θ is called admissible if for every $\mu\in\Omega_2$, $a^\mu\equiv b^\mu(\Theta)$ implies $\varphi_1^\mu(a^\mu)\equiv\varphi_3^\mu(b^\mu)\,(\Theta)$.

Theorem 12. *S may be extended to an algebra S^1 such that a congruence relation Θ of S may be extended to a congruence relation $\overline{\Theta}$ of S^1 if and only if Θ is admissible. Further, if Φ is a congruence relation of S^1 then there exists an admissible congruence relation Θ of S such that $\Phi=\overline{\Theta}$. Finally, the relations $\varphi_1^\mu(b^\mu)=\varphi_2^\mu(b^\mu)$, $\varphi_2^\mu(a^\mu)=\varphi_3^\mu(a^\mu)$, $\mu\in\Omega_2$ hold true in S^1.*

* * *

We need also a new form of the result of our paper [6].

Theorem 13. *Every abstract algebra A may be extended to an abstract algebra A_1 such that*

1. *every congruence relation Θ of A may be extended to a congruence relation $\overline{\Theta}$ of A_1;*

2. *$\Theta\to\overline{\Theta}$ is an isomorphism between $\Theta(A)$ and $\Theta(A_1)$ i. e. to every $\Phi\in\Theta(A_1)$ there exists a $\Theta\in\Theta(A)$ such that $\Phi=\overline{\Theta}$;*

3. *every compact congruence relation of A_1 is minimal;*

4. *if $a,b,c,d\in A$ then there exists $e,f,g\in A_1$ such that $\Theta_{ab}=\Theta_{ef}$, $\Theta_{cd}=\Theta_{fg}$, $\Theta_{ab}\cup\Theta_{cd}=\Theta_{eg}$.*

Remark. Conditions 1 and 2 mean that $\Theta(A)$ and $\Theta(A_1)$ are isomorphic in the natural way.

The theorem stated in [6] is weaker than our Theorem 13, but we actually proved Theorem 13 for algebras with unitary operations; a slight modification of the construction of [6] gives the result of Theorem 13.[5])

[5]) In [6] we used the fact that the algebra has only unitary operations only at the step, when we constructed A_1 from A, in § 3. If A has operations f of more than one variable, then we define its extension on A_1 as follows: $f(a_1,\dots,a_n)=f(b_1,b_2,\dots,b_n)$ where $a_i=b_i$, if $a_i\in A$, $b_i=a$ otherwise. One can easily that with this definition one can carry out the proof of the theorem.

§ 2. Abstract algebras of type 3

We will prove the following theorem:

Theorem 14. *To every abstract algebra A there corresponds an abstract algebra B such that the following conditions are satisfied:*

1. *B is an extension of A;*
2. *every congruence relation Θ of A may be extended to a congruence relation $\overline{\Theta}$ of B;*
3. *$\Theta \to \overline{\Theta}$ sets up an isomorphism between $\Theta(A)$ and $\Theta(B)$;*
4. *B is of type 3;*
5. *every compact congruence relation of B is minimal.*

Remark. Conditions 2 and 3 mean that $\Theta(A)$ and $\Theta(B)$ are isomorphic in the natural way.

One can see that Theorem 14 contains Theorem II′ of § 3 of the Introduction. Further, according to Theorem 10, for every compactly generated lattice L there exists an algebra A with $L \cong \Theta(A)$. Now if we construct the algebra B of Theorem 14 corresponding to this algebra A, then we get that there exists an algebra B with $L \cong \Theta(B)$ and B is of type 3. Summing up we get the following.

Corollary. *The following conditions on a lattice L are equivalent:*

1. *L is compactly generated;*
2. *L has a complete representation;*
3. *L has a complete representation of type 3;*
4. *there exists an abstract algebra A with $L \cong \Theta(A)$;*
5. *there exists an abstract algebra A of type 3 with $L \cong \Theta(A)$.*

Now we are going to prove Theorem 14. We start with the algebra $A_0 = A$ and we extend A_0 to A_0^1 according to Theorem 13. Let $x, y, u, v \in A_0^1$ such that $x \equiv y(\Theta_{uv})$; then we define three partial operations $\varphi_1, \varphi_2, \varphi_3$ on A_0^1:

$$D(\varphi_1, A_0^1) = \{u\}, \quad D(\varphi_2, A_0^1) = \emptyset, \quad D(\varphi_3, A_0^1) = \{v\}$$

and $\varphi_1(u) = x$, $\varphi_3(v) = y$. Let A_0^2 be defined as the partial algebra which we get if the φ_i are defined on A_0^1 for every quadruple $x, y, u, v (x \equiv y(\Theta_{uv}))$.

Every congruence relation of A_0^2 is admissible; it further satisfies all the assumptions we have made in Theorem 12, therefore we can extend A_0^2 to an algebra A_1, such that A_1 already satisfies conditions 1, 2, 3 of Theorem 14. Now we construct A_2 from A_1, A_3 from A_2, and so on, in the same way as A_1 has been constructed from A_0, The algebras $A_0, A_1, \ldots$ form an ascending chain, therefore $B = \bigvee_{i=1}^{\infty} A_1$ is an algebra. Since all the A_i satisfy 1, 2, 3, and 5 of Theorem 14, therefore so does B. It remains only to verify condition 4. Let $x \equiv y(\Theta \cup \Phi)$, then there exist compact congruence relations $\Theta_1 \leqq \Theta$ and $\Phi_1 \leqq \Phi$ such that $x \equiv y(\Theta_1 \cup \Phi_1)$. By condition 5 $\Theta_1 = \Theta_{ab}$ and $\Phi_1 = \Theta_{cd}$ with suitable elements a, b, c, d of B. There exists an integer n with $x, y, a, b, c, d \in A_n$. By condition 4 of Theorem 13, there exist elements e, f, g of A_n^1 such that $\Theta_{ab} = \Theta_{ef}$, $\Theta_{cd} = \Theta_{fg}$ and $\Theta_{ab} \cup \Theta_{cd} = \Theta_{eg}$. Thus $x \equiv y(\Theta_{eg})$. Therefore A_n^2 has operations $\varphi_1, \varphi_2, \varphi_3$ such that $\varphi_1(e) = x$, $\varphi_1(g) = \varphi_2(g)$, $\varphi_2(e) = \varphi_3(e)$, $\varphi_3(g) = y$.[6] Then $z_0 = x$, $z_1 = \varphi_1(f)$, $z_2 = \varphi_2(f)$,

[6]) See the construction in § 1 of Ch. II and Theorem 12.

$z_3=\varphi_3(f)$, $z_4=y$ is a sequence of elements such that $z_0\equiv z_1(\Theta_{ab})$, $z_1\equiv z_2(\Theta_{cd})$, $z_2\equiv z_3(\Theta_{ab})$, $z_3\equiv z_4(\Theta_{cd})$. Indeed, $e\equiv f(\Theta_{ab})$ (for $\Theta_{ab}=\Theta_{ef}$), thus $z_0=\varphi_1(e)\equiv\equiv\varphi_1(f)=z_1(\Theta_{ab})$. Similarly, $z_1=\varphi_1(f)\equiv\varphi_1(g)(\Theta_{cd})$ (for $\Theta_{fg}=\Theta_{cd}$) and $\varphi_1(g)==\varphi_2(g)\equiv\varphi_2(f)=z_2(\Theta_{cd})$ thus $z_1\equiv z_2(\Theta_{cd})$, and so on.

To sum up, whenever $x\equiv y(\Theta\cup\Phi)$ $(x, y\in A,\ \Theta, \Phi\in\Theta(A))$ we can find elements $x=z_0, z_1, z_2, z_3, z_4=y$ such that $z_0\equiv z_1(\Theta)$, $z_1\equiv z_2(\Phi)$, $z_2\equiv z_3(\Theta)$, $z_3\equiv z_4(\Phi)$ (we take into consideration that $\Theta_{ab}\leqq\Theta_1\leqq\Theta$, $\Theta_{cd}\leqq\Phi_1\leqq\Phi$), wihch is the definition of algebra of type 3, Thus condition 4 of Theorem 14 is also verified.

§ 3. Abstract algebras of type 2

The analogue of Theorem 14 for modular lattices is the following:

Theorem 15. *Let A be an abstract algebra such that $\Theta(A)$ is modular. Then there exists an abstract algebra B such that the following conditions are satisfied:*

1. *B is an extension of A;*
2. *every congruence relation Θ of A may be extended to a congruence relation $\overline{\Theta}$ of B;*
3. *$\Theta\to\overline{\Theta}$ sets up an isomorphism between $\Theta(A)$ and $\Theta(B)$;*
4. *B is of type 2;*
5. *every compact congruence relation of B is minimal.*

Remark. Conditions 2 and 3 mean that $\Theta(A)$ and $\Theta(B)$ are isomorphic in the natural way.

Of course in Theorem 15 the essential conditions are that $\Theta(A)\cong\Theta(B)$ and that B is of type 2.

Again, combining Theorem 15 with Theorem 10 we get the

Corollary. *The following conditions on a lattice L are equivalent:*

1. *L is compactly generated and modular;*
2. *L has a complete representation of type 2;*
3. *there exists an abstract algebra A of type 2 such that $L\cong\Theta(A)$.*

For the Corollary the only thing we must verify is that condition 2 implies condition 1; it is enough to prove that if L has a representation of type 2 then L is modular; this is a theorem of [8][7].

For the proof of Theorem 15 we need some preliminary results. The proof of Theorem 15 will be given after Theorem 18.

The crucial point of the proof of Theorem 14 was the following: we can prove that B is of type 3 because the construction given at the beginning of § 1 of Chapter II and which is perfomed in the construction of B several times gives rise to a sequence of elements which guarantee that B is of type 3. In the construction in ques-

[7]) For completeness' sake we prove this. Let L have a representation $\langle F, A\rangle$ of type 2, $a, b, c\in L$, $a\geqq c$. Then $a\cap(b\cup c)\geqq(a\cap b)\cup c$ holds always, hence it is enough to prove that $p, q\in A$. $p\equiv q(F(a\cap(b\cup c)))$ imply $p\equiv q(F((a\cap b)\cup c))$. Indeed, if $p\equiv q(F(a\cap(b\cup c)))$ then $p\equiv q(F(a)\cap F(b\cup c))$, thus $p\equiv q(F(b\cup c))$ and $p\equiv q(F(a))$. We have a representation of type 2, thus $p\equiv q(F(b\cup c))$ implies the existence of r and s such that $p\equiv r(F(c))$, $r\equiv s(F(b))$, $s\equiv q(F(c))$. Then $c\leqq a$ implies that $r\equiv p\equiv q\equiv s$ $(F(a))$, thus $r\equiv s(F(a\cap b))$. We get $p\equiv q(F(a\cap b)\cup F(c))$ that is $p\equiv q(F((a\cap b)\cup c))$, which was to be proved.

tion we start from a partial algebra S and we take three further copies of S, and we identify some elements. One can easily seen that if we want to get an algebra of type 2 then we must reduce the number of new copies of S to 2. This is the main difficulty. Of course, the analogue of Theorem 6 for this modified construction may be proved easily, but Theorem 7 is already not true. We have to introduce some new operations — using the modularity of $\Theta(A)$ — to enforce the existence of the least admissible congruence relation, the existence of which is the main statement of Theorem 7.

So first we modify the construction of § 1 of Chapter II. Let S be a partial algebra, $\varphi_1(x), \varphi_2(x) \in P(S)$, $D(\varphi_1, S) = \{a\}$, $D(\varphi_2, S) = \{b\}$, $\varphi_1(a) = c$, $\varphi_2(a) = d$. We identify in $S[\varphi_1] \cup S[\varphi_2]$ the elements $\varphi_1(b)$ and $\varphi_2(b)$ (see Fig. 3), getting the partial algebra T'. The congruence relation Θ of S is called *admissible* again if either $a \not\equiv b(\Theta)$ or if $a \equiv b(\Theta)$ and $c \equiv d(\Theta)$ (i. e. if $a \equiv b(\Theta)$ „implies” $c \equiv d(\Theta)$). Then

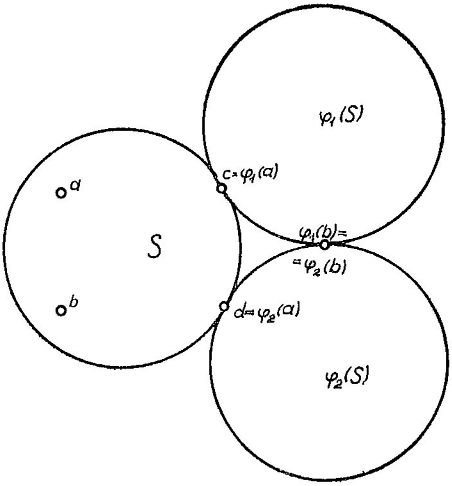

Fig. 3

Theorem 16. *The congruence relation Θ of S is admissible if and only if it may be extended to T'. The minimal extension $\overline{\Theta}$ of Θ is the transitive extension of Θ^*, where Θ^* is identical with Θ on S, and $\varphi_i(x) \equiv \varphi_i(y)$ (Θ^*), if and only if $x \equiv y(\Theta)$ $(x, y \in S)$. The relations Θ^* and $\overline{\Theta}$ are identical on S, on $\varphi_1(S)$ and on $\varphi_2(S)$.*

Proof. Copy the proof of Theorem 6.

Now we want to see what can be said about the congruence relation Θ of S for which $u \equiv v(\overline{\Theta})$, with $u, v \in T'$ fixed. To do this we make three assumptions on S: 1. $\Theta(S)$ is modular, 2. the compact congruence relations of S are minimal; 3. every congruence relation of S is admissible. We distinguish several cases.

A. $u, v \in S$. Obviously[8]), $\Theta = \Theta'_{uv}$ is the smallest admissible congruence relation for which $u \equiv v(\overline{\Theta})$.

B. $u \in S$, $v \in \varphi_1(S)$, i. e. $v = \varphi_1(x)$, $x \in S$. Let Θ be admissible, $u \equiv v(\overline{\Theta})$. Then either

$$(a) \quad u \equiv c(\Theta), \quad a \equiv x(\Theta),$$

or

$$(b) \quad u \equiv d(\Theta), \quad a \equiv b(\Theta), \quad b \equiv x(\Theta).$$

Thus the two congruences

$$\Theta_1 = \Theta_{uc} \cup \Theta_{ax}, \quad \Theta_2 = \Theta_{ud} \cup \Theta_{ab} \cup \Theta_{bx}$$

[8]) The reader should remember that if Θ is a congruence relation of S then Θ' denotes the least admissible congruence relation $\geqq \Theta$ (see the text of § 1 of Chapter II, before Theorem 7).

have the property that either $\Theta_1' \leqq \Theta$ or $\Theta_2' \leqq \Theta$. If we prove $\Theta_1' \leqq \Theta_2'$, then we are through. Indeed, $a \equiv x(\Theta_{ab} \cup \Theta_{bx})$, thus $\Theta_{ax} \leqq \Theta_2'$; further $a \equiv b(\Theta_2)$, thus $c \equiv d(\Theta_2')$; we get $\Theta_{uc} \leqq \Theta_{ud} \cup \Theta_{dc} \leqq \Theta_2'$. Hence $\Theta_1 \leqq \Theta_2'$, so $\Theta_1' \leqq (\Theta_2')' = \Theta_2'$, q. e. d.

C. $u \in S$, $v \in \varphi_2(S)$. Proof as in case B.

D. $u, v \in \varphi_1(S)$, i. e. $u = \varphi_1(x)$, $v = \varphi_1(y)$, $x, y \in S$. Then $u \equiv v(\overline{\Theta})$ implies either

$$(a) \quad x \equiv y(\Theta),$$

or

$$(b) \quad x \equiv a(\Theta), \quad c \equiv d(\Theta), \quad a \equiv b(\Theta), \quad b \equiv y(\Theta),$$

or

$$(c) \quad y \equiv a(\Theta), \quad c \equiv d(\Theta), \quad a \equiv b(\Theta), \quad b \equiv x(\Theta).$$

This shows that, obviously, $\Theta = \Theta_{xy}'$ is the smallest admissible congruence relation under which $u \equiv v(\overline{\Theta})$.

E. $u, v \in \varphi_2(S)$. Proof as in case D.

We see that the three conditions imposed on S have not yet been used.

F. $u \in \varphi_1(S)$, $v \in \varphi_2(S)$ i. e. $u = \varphi_1(x)$, $v = \varphi_2(y)$, $x, y \in S$ (or symmetrically, interchanging u and v). If $u \equiv v(\overline{\Theta})$, then either

$$(a) \quad x \equiv b(\Theta), \quad y \equiv b(\Theta),$$

or

$$(b) \quad x \equiv a(\Theta), \quad c \equiv d(\Theta), \quad a \equiv y(\Theta).$$

Let $\Theta_1 = \Theta_{xb} \cup \Theta_{yb}$, $\Theta_2 = \Theta_{xa} \cup \Theta_{cd} \cup \Theta_{ya}$. Then, if any, Θ_1' or Θ_2' should be the smallest admissible congruence relation Θ such that $u \equiv v(\overline{\Theta})$. But it turns out that neither $\Theta_1' \leqq \Theta_2'$ nor $\Theta_2' \leqq \Theta_1'$ hold in general. Now we use conditions 1—3.

Let $\Theta_3 = \Theta_{xa} \cup \Theta_{ya}$. Then $\Theta_1 \cup \Theta_2 = \Theta_1 \cup \Theta_3$ and $\Theta_3 \leqq \Theta_2$. Thus by the modularity of $\Theta(S)$ we get

$$\Theta_2 = \Theta_2 \cap (\Theta_1 \cup \Theta_3) = (\Theta_2 \cap \Theta_1) \cup \Theta_3.$$

Θ_2 and Θ_3 are compact congruence relations, therefore we can find a $\Theta_4 \leqq \Theta_2 \cap \Theta_3$ such that Θ_4 is compact and $\Theta_4 \cup \Theta_3 = \Theta_2$. Because of $\Theta_{xy} \leqq \Theta_1 \cap \Theta_2$ we may choose Θ_4 such that $\Theta_{xy} \leqq \Theta_4$ is true.

Every compact congruence relation is minimal, therefore $\Theta_4 = \Theta_{ef}$ $(c, f \in S)$. Of course, e and f are not uniquely determined by u and v; already Θ_4 is not unique, but if it were, we could, in general, choose several e and f. But let us fix a pair e, f; we may write $e = e(u, v)$, $f = f(u, v)$.

Suppose that to every $u \in \varphi_1(S)$, $v \in \varphi_2(S)$ we have found e and f. Then we assign to every u, v a new pair of partial operations $\alpha_1(x)$ and $\alpha_2(x)$ such that

$$D(\alpha_1, T') = \{e\}, \quad D(\alpha_2, T') = \{f\}, \quad \alpha_1(e) = u, \quad \alpha_2(f) = v.$$

Let T'' denote the partial algebra T' endowed with these new operations.

Theorem 17. *T'' is an extension of S. A congruence relation of S may be extended to T'' if and only if it is admissible. To every $u, v \in T''$ there exists a least admissible congruence relation Θ of S such that $u \equiv v(\overline{\Theta})$.*

Proof. Let Ψ be an admissible congruence relation of S. It is in general not true that $\overline{\Psi}$ (the extension of Ψ to T') is a congruence relation of T''. The extendability of Ψ to T'' means that extending Ψ to T'' we do not get new congruence relations in S. The extension Ψ_1 of Ψ to T'' may be defined as the transitive extension of Ψ^*, where Ψ^* is a relation equivalent to Ψ on S, $\varphi_i(x) \equiv \varphi_i(y)$ (Ψ^*) $(x, y \in S)$ if and only if $x \equiv y(\Psi)$, and $u = \varphi_1(x) \equiv \varphi_2(y) = v(\Psi^*)$ if and only if $\Theta_{ef} \leqq \Psi$ $(e = e(u, v), f = f(u, v))$.

We have the following remark: let $u = \varphi_i(x)$ $(i = 1$ or $2)$ $v = \varphi_j(y)$ $(j = 1$ or $2)$ and $u \equiv v(\Psi^*)$. Then $x \equiv y(\Psi)$. Indeed, if $i = j$, then this is true by definition. If $i \neq j$ then $\Theta_{ef} \leqq \Psi$. But e and f were chosen so that $\Theta_{xy} \leqq \Theta_{ef}$. Thus $\Theta_{xy} \leqq \Psi$ is obvious. The transitive extension Ψ_1 of Ψ^* gives rise to new congruences in S if and only if $c \equiv d(\Psi_1)$ while $c \not\equiv d(\Psi)$. We prove that this is impossible. Indeed $c \equiv d(\Psi_1)$ means the existence of a sequence $c = z_0, z_1, \ldots, z_n = d$, all the z_i being in $\varphi_1(S) \vee \vee \varphi_2(S)$, such that $z_{i-1} \equiv z_i(\Psi^*)$, $i = 1, 2, \ldots, n$. Let $z_i = \varphi_j(u_i)$ where j is either 1 or 2. Then by the remark of the last but one paragraph we have $u_0 \equiv u_1(\Psi)$, $u_1 \equiv u_2(\Psi), \ldots, u_{n-1} \equiv u_n(\Psi)$ i. e. $u_0 \equiv u_n$ (Ψ). But $\varphi_1(u_0) = c$, $\varphi_2(u_n) = d$; thus $u_0 = a$, $u_n = b$ and we have $a \equiv b(\Psi)$. Now we use that Ψ is admissible, therefore $c \equiv d(\Psi)$, contrary to the hypothesis. Q. e. d.

Now we generalize Theorem 17.

Theorem 18. *Let S be a partial abstract algebra with the following properties: the partial operations of S are $\varphi_i^\mu(x)$, $i = 1, 2$ $\mu \in \Omega$, where $D(\varphi_1^\mu, S) = \{a^\mu\}$, $D(\varphi_2^\mu, S) = \{b^\mu\}$, $\varphi_1^\mu(a^\mu) = c^\mu$, $\varphi_2^\mu(a^\mu) = d^\mu$; all other partial operations of S are operations; if Θ is a compact congruence relation then so is*[9] *Θ'; every compact congruence relation of S is minimal; the admissible congruence relations of S form a modular lattice*[10]*.*

Then there exists an abstract algebra S^* such that

I. *S^* is an extension of S;*

II. *every admissible congruence relation Θ of S may be extended to a congruence relation $\overline{\Theta}$ of S^*;*

III. *$\Theta \to \overline{\Theta}$ is an isomorphism between the lattice of admissible congruence relations of S and $\Theta(S^*)$.*

Proof. Copy the proof of Theorem 7 and use the construction of Theorem 18 rather than that of Theorem 6.

Now we are ready to prove Theorem 15. We apply the same procedure as in the proof of Theorem 14, the only difference is that we use Theorem 18 rather than Theorem 16. The algebra B will be of type 2 because the construction given before Theorem 16 uses only two new copies of S, therefore whenever $x \equiv y$ $(\Theta \cup \Phi)$ we can find a sequence $x = z_0, z_1, z_2, z_3 = y$ such that $z_0 \equiv z_1(\Theta)$, $z_1 \equiv z_2(\Phi)$, $z_2 \equiv z_3(\Theta)$. The construction of the z_i is also the same as in the proof of Theorem 14.

---

[9]) Θ' denotes the least admissible congruence relation $\geqq \Theta$. Now a congruence relation Φ is admissible if for every $\mu \in \Omega$ the relations $a^\mu \equiv b^\mu(\Phi)$, $c^\mu \equiv d^\mu(\Phi)$ are equivalent.

[10]) The admissible congruence relations of S always form a complete lattice, which is in general not a sublattice of $\Theta(S)$.

§ 4. Problems

The first main result of this paper is that to every compactly generated lattice L there exists an abstract algebra A such that $L \cong \Theta(A)$. But the algebra A which is constructed in the proof is pathological. Therefore the problem arises as to whether or not it is possible to construct an A which belongs to certain known classes.

Problem 1. *Is it true that to every compactly generated lattice there coresponds an abstract algebra A such that $L \cong \Theta(A)$ and every operation of A is binary and assoticative (A is a superposition of semigroups)?* Or the same problem, requiring A to be a semi-group.

In other words, characterize the congruence lattices of semigroups.

* * *

If L is finite the construction used gives rise to a countable A.

Problem 2. *Is it possible to represent every finite lattice in the form $\Theta(A)$, where A is a finite abstract algebra?*

This problem seems to be an extremely difficult one. Its solution should imply an answer in affirmative to Problem 48 of [1] asking whether or not every finite lattice is embeddable in a finite partition lattice. A variant of our Problem 2, the solution of which does not imply the solution of BIRKHOFF's problem, is the following.

Problem 2'. *Let $\mathfrak{A}_1$ be the class of all (finite) lattices which may be represented as $\Theta(A)$, where A is a finite abstract algebra; let $\mathfrak{A}_2$ be the class of all (finite) lattices which may be represented as sublattices of finite partition lattices. Is $\mathfrak{A}_1 = \mathfrak{A}_2$ true?*

* * *

Let $\mathfrak{A}_L$ be the class of all compactly generated lattices, $\mathfrak{A}_G$ the class of all lattices which are isomorphic to the lattice of all subgroups of a group, $\mathfrak{A}^G$ the class of lattices which are isomorphic to a complete sublattice of a lattice from $\mathfrak{A}_G$; similarly let $\mathfrak{A}_R$ be the class of lattices which are isomorphic to the lattice of all subrings of a ring and $\mathfrak{A}^R$ the class of lattices which are complete sublattices of a lattice from $\mathfrak{A}_R$. The relations $\mathfrak{A}_G \supseteq \mathfrak{A}^G$ and $\mathfrak{A}_R \supseteq \mathfrak{A}^R$ are trivial. We have proved $\mathfrak{A}_L = \mathfrak{A}^G$.

Problem 3. *Find the proper relations between $\mathfrak{A}_L (= \mathfrak{A}^G)$, $\mathfrak{A}_G$, $\mathfrak{A}_R$ and $\mathfrak{A}^R$. Are all identical?*

* * *

In this paper we have completed the argument of [6] to show that every abstract algebra A may be extended to an abstract algebra B such that $\Theta(A) \cong \Theta(B)$ and every compact congruence relation of B is of the form Θ_{ab}. And we proved that for every abstract algebra A there exists an abstract algebra B such that $\Theta(A) \cong \Theta(B)$, and every compact congruence relation of B is of the form Θ_{oa}, where o is a *fixed element* of B. Can these two results be combined?

Problem 4.[11]) *Prove that every abstract algebra can be extended to an abstract*

---

[11]) *Added in proof* (May 9, 1963): We have proved the following result.

Theorem. *Every algebra A can be extented to an algebra B such that $\Theta(A)$ and $\Theta(B)$ are isomorphic in the natural way, further, any compact congruence relation Θ is of the form Θ_{oa} where o is an arbitrary element of B (a depending on Θ and o).*

algebra B such that every compact congruence relation of B is of the form Θ_{oa}, *where o is a fixed element of B.*

* * *

The two main results of Chapter III may be formulated as follows: If L is compactly generated and L has a representation of type $i (i=2, 3)$ then $L \cong \Theta(A)$ where A is of type i. We could not prove (or disprove) the similar result for $i=1$. It is the following:

Problem 5. *Prove that to every compactly generated lattice L which has a representation of type* 1, *there exists an abstract algebra A such that* $L \cong \Theta(A)$ *and any two congruence relations of A are permutable* (*i. e. if* $x \equiv y(\Theta)$, $y \equiv z(\Phi)$ *then there exists a w such that* $x \equiv w(\Phi)$, $w \equiv z(\Theta)$).

* * *

G. Birkhoff has proved that to every group G there corresponds an abstract algebra A such that G is isomorphic to the group of all automorphisms of A. Let A be an abstract algebra; we assigne to A a couple $(G^{(A)}, L^{(A)})$, where $G^{(A)}$ is the automorphism group of A and $L^{(A)}$ the congruence lattice of A. Birkhoff's result states that every G occurs in the first place in a couple (G, L). We have proved that a lattice L occurs in the second place if and only if it is compactly generated. And what is more, we showed that if this is the case, then L already occurs in a couple $(1, L)$ where 1 denotes the group of one element. These results suggest that the first and second components of a couple are independent. More precisely:

Problem 6. *Let G be an arbitrary group and L a compactly generated lattice. Prove that there exists an abstract algebra such that* $(G^{(A)}, L^{(A)})$ *is identical with* (G, L).

Bibliography

[1] G. Birkhoff, *Lattice theory* (New York, 1948).
[2] G. Birkhoff and O. Frink, Representation of lattices by sets, *Trans. Amer. Math. Soc.*, **64** (1948), 299—316.
[3] J. R. Büchi, Representation of complete lattices by sets; *Port. Math.*, **11** (1952), 151—167.
[4] K. H. Diener, Über zwei Birkhoff—Frinksche Struktursätze der allgemeinen Algebra, *Arch. Math.*, **7** (1956), 339—345.
[5] G. Grätzer and E. T. Schmidt, Two notes on lattice congruences, *Annales Univ. Sci. Budapest*, sectio math., **1** (1958), 83—87.
[6] G. Grätzer and E. T. Schmidt, On inaccessible and minimal congruence relations, I., *Acta Sci. Math.*, **21** (1960), 337—342.
[7] J. Hashimoto, Direct, subdirect decompositions and congruence relations, *Osaka Math.*, **9** (1957), 87—117.
[8] B. Jónsson, On the representation of lattices, *Math. Scand.*, **1** (1953), 193—206.
[9] A. Komatu, On a characterization of join-homomorphic transformation lattices, *Proc. Imp. Acad. Tokyo*, **19** (1943), 119—124.
[10] L. Nachbin, On a characterization of the lattice of all ideals of a Boolean ring, *Fund. Math.*, **36** (1949), 137—142.
[11] P. M. Whitman, Lattices, equivalence relations and subgroups, *Bull. Amer. Math. Soc.*, **52** (1946), 507—522.

MATHEMATICAL INSTITUTE,
HUNGARIAN ACADEMY OF SCIENCES

(Received March 17, 1962)