Characterizations of congruence lattices of abstract algebras

By G. GRÄTZER and E. T. SCHMIDT in Budapest

INTRODUCTION

In this paper we deal with the characterization problem of the lattice $\Theta(A)$ of *all* congruence relations of an abstract algebra A (briefly, congruence lattice). In § 1 of the Introduction we summarize our results concerning the general characterization problem, the solution of which answers Problem 50 of G. BIRKHOFF [1], originally proposed by BIRKHOFF and FRINK [2]. In § 2 we show that the representation theorems of WHITMAN and JONSSON are easy consequences of our results; we also solve the problem of complete representation. Concerning congruence lattices of type 2 and 3 we are able to prove more than the results stated in § 1. These results are summarized in § 3 in the form of embedding theorems for abstract algebras. In the next section we outline the method of the paper based on the systematic study of partial abstract algebras. The contents of the paper are sketched in the same section.

§ 1. Congruence lattices

An element x of the complete lattice L is called *compact* if $x \leqq \bigvee (x_\lambda; \lambda \in \Lambda)$ implies $x \leqq \bigvee (x_\lambda; \lambda \in \Lambda')$ for some finite $\Lambda' \subseteqq \Lambda$. A lattice L is *compactly generated* if it is complete and every element of L is the complete join of compact elements.

If A is an abstract algebra, $a, b \in A$, then there is a least $\Theta \in \Theta(A)$ such that $a \equiv b(\Theta)$; this is denoted by Θ_{ab}. Every Θ_{ab} as an element of $\Theta(A)$ is easily shown to be compact and thus *every congruence lattice is compactly generated.*[1])

The question whether or not every compactly generated lattice is isomorphic to a congruence lattice was proposed by BIRKHOFF and FRINK [2], again in BIRKHOFF [1] as Problem 50. One of our principal results is to answer this problem affirmatively.

Theorem I. *To any compactly generated lattice L there corresponds an abstract algebra A for which $\Theta(A)$, the lattice of all congruence relations of A, is isomorphic to L.*

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[1]) This assertion was first observed by BIRKHOFF and FRINK [2]; the conditions they have used are equivalent to, yet different from, those used above. The notion of compact element goes back to BÜCHI [3] and NACHBIN [10]. In [7], HASHIMOTO proves that every congruence lattice is isomorphic to the lattice of all ideals of a semilattice, a statement again equivalent to the above one.

One may hope to get a stronger form of Theorem I, so as to impose further conditions on A. In order to do this, consider $\Theta, \Phi \in \Theta(A)$ and $x, y \in A$. It is known that $x \equiv y(\Theta \cup \Phi)$ if and only if there exists a sequence $x = z_0, z_1, \ldots, z_m, z_{m+1} = y$ of elements of A such that $z_i \equiv z_{i-1}(\Theta)$ or $z_i \equiv z_{i-1}(\Phi)$ $(i = 1, 2, \ldots, m+1)$. We say A is *of type n* if, for every x, y, Θ, Φ $(x \equiv y(\Theta \cup \Phi))$, the sequence $\{z_i\}$ may be chosen so that $m = n$. This means, that while in an arbitrary abstract algebra, corresponding to a fixed quadruple x, y, Θ, Φ, the least m may be arbitrarily large, in algebras of type n, m may not exceed n; e. g. a ring or a group is always of type 1.

It is easy to prove that if A is of type 1 or 2 then $\Theta(A)$ is modular. Hence, from this point of view we get the best possible result if we can replace A of Theorem I by one of type 3. This is done in

Theorem II. *Let L be a compactly generated lattice. Then there exists an abstract algebra A of type* 3 *such that L and* $\Theta(A)$ *are isomorphic.*

As we said above, if A is of type 2 then $\Theta(A)$ is modular. This raises the question: which lattices are isomorphic to such a $\Theta(A)$? This is answered by

Theorem III. *Every compactly generated modular lattice is isomorphic to the congruence lattice of a suitable abstract algebra of type* 2.

§ 2. Representations

If H is a set then the set $\mathscr{E}(H)$ of all equivalence relations of H is a complete lattice and $\mathscr{E}(H) \cong \Theta(H)$ if H is considered as an abstract algebra without operations.

By a *representation* of the lattice L we mean an ordered pair $\langle F, H \rangle$, where H is a set and

$$x \to F(x)$$

is an isomorphism of L into $\mathscr{E}(H)$. If this isomorphism preserves complete join and meet, then the representation is called *complete.*

It is well known that $\langle F, A \rangle$,

$$F(\Theta) = \Theta,$$

is a complete representation of $\Theta(A)$; this will be called the *natural representation* of $\Theta(A)$. Further, it is easily shown that a lattice having a complete representation is compactly generated. Hence Theorem I implies at once

Corollary I. 1. *A complete lattice L has a complete representation if and only if L is compactly generated.*

This is the analogue of Whitman's fundamental theorem [11], asserting that every lattice has a representation. In fact, Whitman's theorem is a trivial consequence of Corollary I. 1. Indeed, if L_1 is a lattice then we extend it to L_2 by adding a zero element. Then we define L as the lattice of all ideals of L_2. Obviously, L is compactly generated, hence by Corollary I. 1 it has a representation $\langle F, H \rangle$ which is at the same time a representation of L_1. Thus

Corollary I. 2. (Whitman [11].) *Every lattice has a representation.*

Jonsson [8] defined the concept of representation of type n. If $x, y \in L$ and if $\langle F, H \rangle$ is a representation of L, then define $F(x); F(y)$ as the relation theoretic product of $F(x)$ and $F(y)$, i. e. $u \equiv v(F(x); F(y))$ $(u, v \in H)$ if and only if there is a $w \in H$ such that $u \equiv w(F(x))$ and $w \equiv v(F(y))$. Then $F(x) \cup F(y)$ is the join of the ascending series

$$F(x); F(y), \quad F(x); F(y); F(x), \quad F(x); F(y); F(x); F(y), \ldots .$$

If this series terminates at its n-th member for all $x, y \in L$ then the representation $\langle F, H \rangle$ of L is said to be of type n.

It is obvious that *an abstract algebra A is of type n if and only if the natural representation of $\Theta(A)$ is of type n.* Thus we get

Corollary II. 1. *A complete lattice L has a complete representation of type* 3 *if and only if it is compactly generated.*

Corollary II. 2. (Jonsson [8].) *Every lattice has a representation of type* 3.

And, similarly, the consequences of Theorem III are:

Corollary III. 1. *A complete lattice L has a complete representation of type* 2, *if and only if L is modular and compactly generated.*

Corollary III. 2. (Jonsson [8].) *Every modular lattice has a representation of type* 2, *and conversely.*

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Another type of representation is obtained by means of subgroups of a group. A *subgroup representation* $\langle F, G \rangle$ of a lattice L consists of a group G and an isomorphism F of L into $L(G)$, the lattice of all subgroups of G. The subgroup representation is *complete,* if the isomorphism preserves complete joins and meets.

From Theorem I we conclude easily

Corollary I. 3. *A complete lattice L has a complete subgroup representation if and only if L is compactly generated.*

Corollary I. 4. (Whitman [11].) *Every lattice has a subgroup representation.*

§ 3. Embedding of abstract algebras

To prove Theorem II and III it is enough to construct only one abstract algebra A satisfying the hypotheses. In fact, we can prove much more. Given an arbitrary abstract algebra A we embed it in an abstract algebra B, such that $\Theta(A) \cong \Theta(B)$ and B is of type 3, or of type 2 if $\Theta(A)$ is modular. These — together with Theorem I — are much more than Theorems II and III. For a precise formulation of these new theorems we need a definition of embedding, because in these constructions A is not a subalgebra of B.

We say that the algebra B is an *extension* of the algebra A if [2])

1. $A \subseteqq B$;
2. to every operation f of A there corresponds an operation $\bar{f}$ of B (the extension of f), such that $f(a_1, a_2, \ldots, a_n) = \bar{f}(a_1, a_2, \ldots, a_n)$ if $a_1, a_2, \ldots, a_n \in A$.

If B is an extension of A and $\bar{\Theta}$ is a congruence relation of B then it includes a congruence relation Θ on A: let $a \equiv b(\Theta)$, $a, b \in A$ if and only if $a \equiv b(\bar{\Theta})$. If $\bar{\Theta} \to \Theta$

[2]) $\subseteqq$ is the set theoretical inclusion.

is an isomorphism between $\Theta(B)$ and $\Theta(A)$ then we say that $\Theta(B)$ and $\Theta(A)$ are isomorphic *in the natural way.*

Theorem II′. *Every abstract algebra A may be extended to an abstract algebra B of type* 3, *such that $\Theta(A)$ is isomorphic to $\Theta(B)$ in the natural way.*

Theorem III′. *Let A be an abstract algebra such that $\Theta(A)$ is modular. Then A has an extension B of type* 2, *such that $\Theta(A)$ is isomorphic to $\Theta(B)$ in the natural way.*

§ 4. The method and lay-out of the paper

To prove the theorems listed above we have to construct abstract algebras; to carry out these constructions seems to be rather difficult. But if we dispense with the assumption that an operation of an abstract algebra must be defined for every n-tuple (n depending on the operation), thus getting the definition of partial abstract algebra, then the task is fairly easy. The difficulty lies in the next step: we want to extend the partial abstract algebra to an abstract algebra so that the „good" properties should not be altered. E. g. such a property is that $\Theta(A)$ be isomorphic to L, where L is fixed.

We use two methods to bypass these difficulties: the first is the extension of a partial algebra to a free algebra; and the second is a procedure which identifies the „new" congruence relations of the free algebra with the congruence relations of the partial algebra.

It is not surprising that on proving theorems for abstract algebras the key role is played by partial abstract algebras, for partial algebras are nothing but generating systems considered *in abstracto*. This was kept in mind when the analogues of the notions of abstract algebras were defined for partial abstract algebras.

In the Introduction only the most important results are listed. All the theorems of the paper are numbered by arabic numerals; these are related to the results mentioned in the Introduction as follows: Theorem I is essentially Theorem 10; Theorem II is part of the Corollary to Theorem 14; Theorem II′ is part of Theorem 14; Theorem III is contained in the Corollary to Theorem 15; Theorem III′ is contained in Theorem 15.

The contents of the paper are the following: In Chapter I the notion of partial abstract algebra and the free algebra generated by a partial algebra are introduced and some of their properties are examined. The most important result of this part is Theorem5 which states that every congruence relation of a partial algebra may be extended to the free algebra generated by the partial algebra. In Chapter II contructions are developed in order to prove Theorem 10 (Theorem I). In the last section several applications of Theorem 10 are proved. In Chapter III our first task is to modify the construction in Chapter II in order to prove Theorem 14 (Theorem II). Finally, an analysis of the proof of Theorem 14 shows how to make further modifications which lead us to Theorem 15 (Theorem III).

Some open questions are mentioned in the last section of Chapter III.