Elementary divisors in von Neumann rings

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1. Introduction

1. 1. Terminology. In this paper L *will always denote a complemented modular lattice* and $\mathfrak{R}$ will denote an *associative regular ring with unit element.*

We will call L an $\aleph$*-geometry* if:

(1. 1. 1) Whenever $x_\alpha \in L$ for each $\alpha \in I$ with cardinal power of $I \leqq \aleph$, the union $x = \bigcup_\alpha (x_\alpha)$ and intersection $x' = \bigcap_\alpha (x_\alpha)$ exist and for each y: if $y \cap \bigcap (\bigcup (x_\alpha | \alpha \in F)) = 0$[1]) for every *finite* subset F of I then $y \cap x = 0$; if $y \cup (\bigcap (x_\alpha | \alpha \in F)) = 1$ for every *finite* subset F of I then $y \cup x' = 1$.

If (1. 1. 1) holds for *all* $\aleph$, we will call L a *von Neumann geometry.*[2])

In every von Neumann geometry there exists a unique normalized dimension function D, vector-valued with $0 \leqq D(x) \leqq 1$ for all x in L such that $x \sim y$[3]) if and only if $D(x) = D(y)$ [9, 6]. When L is irreducible D is numerical-valued and its range of values is either $0, \frac{1}{n}, \ldots, \frac{n}{n}$ for some integer n (then L is called a *finite dimensional* or *discrete* geometry of von Neumann) or all real numbers $0 \leqq t \leqq 1$ (then L is called a *continuous*[4]) geometry of von Neumann) [9, Part I, Theorem 7. 3].

$\overline{R}_{\mathfrak{R}}$, $\overline{L}_{\mathfrak{R}}$ will denote the set of principal right (respectively, left) ideals of $\mathfrak{R}$, ordered by inclusion; $\overline{R}_{\mathfrak{R}}$ and $\overline{L}_{\mathfrak{R}}$ are complemented modular lattices [9, Part II, Theorem 2 .4]. $\mathfrak{R}$ will be called an $\aleph$*-ring* or a *von Neumann ring* if $\overline{R}_{\mathfrak{R}}$ (hence also $\overline{L}_{\mathfrak{R}}$) is an $\aleph$-geometry, respectively a von Neumann geometry.

In a von Neumann ring $\mathfrak{R}$ there exists a unique, normalized rank-function $R(a)$, vector-valued with $0 \leqq R(a) \leqq 1$ for all a in $\mathfrak{R}$, defined by: $R(a) = D((a)_r)$.[5]) If $\mathfrak{R}$ is irreducible, R is numerical-valued and $\overline{R}_{\mathfrak{R}}$ must be discrete or continuous; then $\mathfrak{R}$ will be called a *discrete ring*, respectively a *continuous ring* (of von Neumann).

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[1]) $\{u | \varphi(u)\}$ will denote the class of u for which $\varphi(u)$ holds.

[2]) Thus L satisfies von Neumann's axioms I—V; his axiom VI (irreducibility) is not postulated [9, pages 1, 2].

[3]) In any lattice, $x \sim y$ means: x is perspective to y (that is, for some w, $x \cup w = y \cup w$ and $x \cap w = y \cap w$; $x \precsim y$ means $x \sim w$ for some $w \leqq y$.

[4]) In our terminology a continuous geometry is always *irreducible.*

[5]) $(a)_r$ and $(a)_l$ denote the principal right and the principal left ideal generated by a, respectively (since $\mathfrak{R}$ is a regular ring, $(a)_r = a\mathfrak{R}$).

A discrete ring must be of the form $\mathfrak{D}_n$[6]) with $\mathfrak{D}$ a (possibly non-commutative) division ring [9, Part II, Theorem 14.1 and page 292].

If $n>1$, $\mathfrak{R}_n$ must also be a regular ring [9, Part II, Theorem 2. 13] but if $\mathfrak{R}$ is a von Neumann ring, $\mathfrak{R}_n$ need not be a von Neumann ring (the union of a countable subset of $\overline{R}_{\mathfrak{R}_n}$ may not exist[7])); but if $\mathfrak{R}$ is an *irreducible* von Neumann ring then $\mathfrak{R}_n$ is also a von Neumann ring (see the Corollary to Lemma 3. 2 below).

The centre of $\mathfrak{R}$ will be denoted Z (if $\mathfrak{R}$ is a von Neumann ring, Z will be a commutative von Neumann ring); Z will be a division ring if and only if Z is irreducible and if and only if $\mathfrak{R}$ is irreducible.

A non-zero element x in a lattice L will be called minimal if $y_1 \leqq x$, $y_2 \leqq x$, $y_1 \sim y_2$ together imply $y_1 = y_2$.

By P we shall denote the set of all polynomials

$$p(t) = t^m + z_{m-1}t^{m-1} + \ldots + z_0$$

with $m \geqq 1$ and all z_i central.[8]) p, q in P will be called *relatively prime* if $h(t)p(t) + k(t)q(t) = 1$ for some h, k of the form $t^m + z_{m-1}t^{m-1} + \ldots + z_0$ with $m \geqq 0$ and all z_i central[8]). p will be called *irreducible* if p cannot be expressed as a product $p = p_1 p_2$ with p_1, p_2 in P and each of degree less than the degree of p.

If Z is not a field, Z contains a non-zero non-invertible[9]) z_0 and $p \equiv t$, $q \equiv t + z_0$ are irreducible, different but not relatively prime. This motivates the following definition.

Call p in P *pure irreducible* if for every non-zero central idempotent e, ep is irreducible in the ring $e\mathfrak{R}$. If $\mathfrak{R}$ is a von Neumann ring then for each p in P there is (obviously) a set of orthogonal non-zero central idempotents, $\{e_\lambda\}$ with $\bigcup_\lambda (e_\lambda)_r = \mathfrak{R}$ and with the property that for each λ: $e_\lambda p = e_\lambda \Pi_i q_{\lambda,i}$ with a finite set of i and with each $q_{\lambda,i}$ in P and $e_\lambda q_{\lambda,i}$ pure irreducible in $e_\lambda \mathfrak{R}$.

Let P_1 be a subset of P and let $\overline{P_1}$ consist of all $p = p_1 \ldots p_m$ (all p_i in P_1). We shall call an element a *P_1-algebraic* if $p(a) = 0$ for some p in $\overline{P_1}$, *P_1-almost-algebraic* if $\bigcap((p(a))_r | p \in \overline{P_1}) = 0$. When P_1 coincides with P we omit it in this nomenclature [10, 4].

a and b are called *similar* or *conjugate* in $\mathfrak{R}$ if $b = dad^{-1}$ for some invertible d. Then for each p in P, $p(b) = dp(a)d^{-1}$, $(p(b))_r = (dp(a))_r$ and we shall show in Corollary 1 to Lemma 2. 1 below that in a von Neumann ring $(p(b))_r \sim (p(a))_r$ and hence $R(p(a) = R(p(b))$ for each p in P.

1. 2. Elementary divisors. When $\mathfrak{R} = \mathfrak{D}_n$ with $\mathfrak{D}$ a *commutative* division ring (we shall call this the *classical case*) it is known [1, page 283], [11, pages 120—124], [7, pages 92—98] that:

(1. 2. 1) *a and b are similar if and only if they have the same elementary divisors.*

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[6]) The ring of $n \times n$ matrices with entries in $\mathfrak{S}$ will be denoted $\mathfrak{S}_n$.

[7]) This failure occurs in KAPLANSKY's example [8] where $\mathfrak{R}$ is the ring of sequences of complex numbers $a = \{a_m | m \geqq 1\}$ with all but a finite number of a_m real, with componentwise ring addition and multiplication.

[8]) An element of $\mathfrak{R}$ is called *central* if it is in the centre Z; a, b are *orthogonal* if $ab = ba = 0$.

[9]) In any ring $\mathfrak{S}$ with unit, d is called *invertible* if for some c in $\mathfrak{S}$, $dc = cd = 1$; c is called the *reciprocal* of d and denoted d^{-1}.

We give now a definition for „elementary divisors of a" in terms of the rank function, applicable in any von Neumann ring.

Note first that for b in $\mathfrak{R}$ and integer $s \geqq 0$, $(b^{s+1})_r = (b^s b)_r \leqq (b^s)_r$ and[10])

$$(1.2.2) \qquad [(b^{s+1})_r - (b^{s+2})_r] \precsim [(b^s)_r - (b^{s+1})_r]$$

((1. 2. 2) will be proved in Lemma 2. 2 below). Thus $R(b^s) \geqq R(b^{s+1})$ and $R(b^s) - R(b^{s+1}) \geqq R(b^{s+1}) - R(b^{s+2})$.

We set $R_a(p) = R(p(a))$ for each p in $eP = P(e\mathfrak{R})$ for arbitrary non-zero central idempotent e. For each integer $s \geqq 1$ we define

$$f_a(p, s) = s((R_a(p^{s-1}) - R_a(p^s)) - (R_a(p^s) - R_a(p^{s+1}))).$$

Then $f_a(p, s) \geqq 0$. The function $f_a(p, s)$ is determined by the function $R_a(p)$; the converse also holds since

$$R_a(p^{s-1}) - R_a(p^s) = \sum_{t=s}^{\infty} \frac{1}{t} f_a(p, t),$$

$$R_a(p^0) - R_a(p^s) = \sum_{i=1}^{s} \sum_{t=i}^{\infty} \frac{1}{t} f_a(p, t),$$

$$R_a(p^s) = 1 - \sum_{i=1}^{s} \sum_{t=i}^{\infty} \frac{1}{t} f_a(p, t).$$

It can be shown that if p, q are relatively prime then $1 - R_a(pq) = (1 - R_a(p)) + (1 - R_a(q))$.

Thus in any von Neumann ring the function $R_a(q)$ for all q in P is determined by the values of $f_a(ep, s)$ for all p in P with ep pure irreducible in $e\mathfrak{R}$ and e a non-zero central idempotent and all $s \geqq 1$. We shall say for each non-zero central idempotent e and p in P with ep pure irreducible in $e\mathfrak{R}$, and $f_a(ep, s) > 0$ that *$q = p^s$ is an elementary divisor of a in $e\mathfrak{R}$ occurring with normalized frequency $f_a(ep, s)$.* This definition agrees with the usual one for the classical case (there, the only possibility for e is 1) except that the normalized frequency is the usual frequency multiplied by the factor $\dfrac{s \cdot (\text{degree of } p)}{n}$. It can be shown that in every irreducible von Neumann ring

$$1 - \sum_{p, s} f_a(p, s) = R(ae_0) \geqq 0$$

where ae_0 is the transcendental part of a [4] (thus „$=$" holds if a is almost-algebraic, in particular for all a in the classical case).

We have noted that each p in P can be expressed „locally" as a product of pure irreducible factors. We shall call a subset P_1 of P *fully factorizable* if for each p in P_1 there are central idempotents $\{e\}$ such that $\bigcup (e)_r = \mathfrak{R}$ and such that each ep is a product $ep_1 \ldots p_m$ with each p_i in P_1 and ep_i pure irreducible in $e\mathfrak{R}$.

Clearly P itself is fully factorizable. If $\mathfrak{R}$ is irreducible then P_1 is fully factorizable if it contains all irreducible p in P.

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[10]) If $x \geqq y$ then $[x - y]$ denotes any (fixed) w such that $y \dot{\cup} w = x$ (the dot in $\dot{\cup}$ indicates independence of the addends); such w exist in every complemented modular lattice.

1.3. Statement of main theorem. The main object of this paper is to prove the following theorem, a generalization of (1.2.1) to any von Neumann ring:

Theorem 1.1. *Let a and b be arbitrary elements of a von Neumann ring.*
(i) *For a and b to be similar it is necessary that*

(1.3.1) $R_a(q) = R_b(q)$ *for all q in P.*

(ii) *For a and b to be similar it is sufficient that for some fully factorizable P_1:*

(1.3.2) $R_a(p^s) = R_b(p^s)$ *for all p in P_1 and $s \geqq 1$,*

(1.3.3) *a and b are P_1-almost-algebraic.*[11]

(1.3.4) *Whenever $\bar{e}$ is central idempotent such that $\bar{e}\mathfrak{R}$ contains minimal elements then $\bar{e}_1\mathfrak{R}$ is a finite dimensional matrix ring over $\bar{e}_1 Z$ for some non-zero central idempotent $\bar{e}_1$ such that $\bar{e}_1\bar{e} = \bar{e}_1$.*

(1.3.5) *$\mathfrak{R}_2$ is a von Neumann ring.*[12]

Remark. The definition of $\mathfrak{R}_2$ is given in footnote [6]. It is shown in the Corollary to Lemma 3.2 below that $\mathfrak{R}_2$ is a von Neumann ring whenever $\mathfrak{R}$ is an irreducible von Neumann ring (equivalently, if $\bar{R}_{\mathfrak{R}}$ is a discrete or continuous geometry), more generally whenever $\mathfrak{R}$ is a direct sum of irreducible von Neumann rings.

Also, it follows from Lemma 3.1 and Lemma 3.2 below that every von Neumann ring can be expressed as a direct sum $\overline{\mathfrak{R}} \oplus \mathfrak{R}'$ in such a way that $(\mathfrak{R}')_2$ is a von Neumann ring and $\overline{\mathfrak{R}}$ is a von Neumann ring in which every idempotent is central (equivalently, $\overline{\mathfrak{R}}_{\bar{R}}$ is a Boolean algebra).

Let E be the central idempotent for which $\mathfrak{R}' = \mathfrak{R}E$; then clearly, a and b are similar in $\mathfrak{R}$ if and only if Ea, Eb are similar in $\mathfrak{R}'$ and $(1-E)a$, $(1-E)b$ are similar in $\overline{\mathfrak{R}}$. If a, b satisfy (1.3.2), (1.3.3) and (1.3.4), then at least Ea, Eb are similar in $\mathfrak{R}'$ (hence in $\mathfrak{R}$) since $\mathfrak{R}'$ satisfies (1.3.5). Thus a and b will be similar in $\mathfrak{R}$ if and only if $(1-E)a$, $(1-E)b$ are similar in the ring $\overline{\mathfrak{R}}$ ($(1-E)a$ and $(1-E)b$ satisfy (1.3.2), (1.3.3), (1.3.4) in $\overline{\mathfrak{R}}$).

In such a ring $\overline{\mathfrak{R}}$ condition (1.3.2) can be expressed in the simpler equivalent form:

(1.3.2)′ $(p(a))_r = (p(b))_r$ *for all p in P_1.*

We shall postpone to another occasion further discussion of the case of a ring $\overline{\mathfrak{R}}$, noting here only that it is easy to see that Theorem 1.1 (ii) holds without (1.3.5), if $\overline{\mathfrak{R}}$ is the example given by Kaplansky (and described in footnote [7]).

Corollary to Theorem 1.1. *Suppose $\mathfrak{R}$ is a von Neumann ring which is irreducible, or more generally, is a direct sum of irreducible von Neumann rings, or more generally, has the property: $\mathfrak{R}_2$ is a von Neumann ring and that (1.3.4) holds.*

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[11]) In the presence of (1.3.1) the condition (1.3.3) for a will imply (1.3.3) for b.

[12]) For the classical case $\mathfrak{R} = \mathfrak{D}_n$ ($\mathfrak{D}$ commutative) our proof specializes of course, to a proof of the known result (1.2.1).

If a and b in $\mathfrak{R}$ are almost algebraic then they are similar if and only if they have the same elementary divisors.

However, we shall not use rank (or dimension) functions. In (1. 3. 1) and (1. 3. 2) we shall replace equality of rank by perspectivity of corresponding principal right ideals.

1. 4. Plan of the proof of Theorem 1. 1. Corollary 1 to Lemma 2. 1 below will show that $(dp(a))_r \sim (p(a))_r$ if d is invertible. From this follows (i) of Theorem 1. 1.

To prove (ii) of Theorem 1. 1 we prove first the special case:

(1. 4. 1) *a and b are similar in an $\aleph_0$-ring $\mathfrak{R}$ if $\mathfrak{R}_2$ is an $\aleph_0$-ring, and $(a^s)_r \sim (b^s)_r$ for all $s \geqq 1$ and* $\cap\, ((a^s)_r | s \geqq 1) = 0$ (see Theorem 4. 1 below),

and then the case:

(1. 4. 2) *a and b are similar in a von Neumann ring $\mathfrak{R}$ if* (1. 3. 4), (1. 3. 5) *hold and for some pure irreducible p in P, $(p^s(a))_r \sim (p^s(b))_r$ for all $s \geqq 1$ and* $\cap\, ((p^s(a))_r | s \geqq 1) = 0$ (see Theorem 4. 2 below).

Then in the general case we show that the unit in $\mathfrak{R}$ can be decomposed into orthogonal idempotents e (not necessarily in the centre) with $\cup\, (e)_r = \mathfrak{R}$ and (using Theorem 3. 1 below) such that, for some $\bar{b} = dbd^{-1}$: for each e, $ae = ea$ and $\bar{b}e = e\bar{b}$ and ae, $\bar{b}e$ satisfy the hypotheses of (1. 4. 2) in $e\mathfrak{R}e$.

This will yield: ae and $\bar{b}e$ are similar in $e\mathfrak{R}e$. Then, using a theorem which permits „combining" such local similarities in the case that $\mathfrak{R}_2$ is a von Neumann ring (Theorem 3. 2 and its Corollary 1 below) we deduce that a and $\bar{b}$, and hence also a and b are similar.

2. Proof of (1. 2. 2) and Theorem 1. 1 (i)

If d is in $\mathfrak{R}$ we shall write d^r to denote $\{b | db = 0\}$. If $x \subset \mathfrak{R}$ we write x^r to denote $\{b | xb = 0\}$. Similarly for d^l and x^l.

Lemma 2. 1. *Suppose $d \in \mathfrak{R}$ and $x \in \bar{R}_{\mathfrak{R}}$. Let $x_0 = d^r \cap x$. If $x \cap dx = 0$ or if $\mathfrak{R}$ is an $\aleph_0$-ring, then $[x - x_0] \sim dx$.*

Corollary 1. *If also $x_0 = 0$ (in particular, if d is invertible), then $x \sim dx$ so* (i) *of Theorem* 1. 1 *holds.*

Corollary 2. *If $\mathfrak{R}$ is a von Neumann ring, then $D(x) = D(dx) + D(x_0)$.*

Proof of Lemma 2. 1. Let e, f be idempotents such that $x = (e)_r$, $d\mathfrak{R} = (f)_r$, and let $x_1 = [x - x_0]$. Then $a \in x$ implies $a = a_0 + a_1$ with $a_i \in x_i$. Thus $dx = dx_1$.

Let T denote the mapping of $0 \leqq y \leqq x_1$ onto $0 \leqq w \leqq dx_1$ defined by: $T(a)_r = (da)_r$. Then T has the properties:

(i) T is order-preserving: indeed, $(db)_r \leqq (dc)_r$ is equivalent in turn to each of: for some a in $\mathfrak{R}$, $db = dca$, $d(b - ca) = 0$, $b - ca = 0$, $(b)_r \leqq (c)_r$.

(ii) $T(a)_r \sim (a)_r$ if $T(a)_r \cap (a)_r = 0$: indeed, $(a+da)_r$ is an axis of perspectivity since

$$T(a)_r \dot{\cup} (a+da)_r = (da)_r \dot{\cup} (a+da)_r = (a)_r \dot{\cup} (a+da)_r.$$

From [4, Lemma 6. 1] it follows that $x_1 \sim dx_1 = dx$.

Proof of Corollary 2. $D(x) = D(x_0) + D(x_1)$ and $D(dx) = D(x_1)$ since $dx \sim x_1$.

Lemma 2. 2. (1. 2. 2) *holds in an* $\aleph_0$*-ring.*

Proof. Let $x^s = (b^s)_r \cap b^r$. By Lemma 2. 1, $[(b^s)_r - (b^{s+1})_r] \dot{\cup} \bar{x}^s = (b^s)_r$ for some $\bar{x}^s \sim x^s$. Since $x^{s+1} \leqq x^s$, $\bar{x}^{s+1} \precsim x^s$ (perspectivity is transitive in an $\aleph_0$-geometry [2]). Now $(b^{s+1})_r \leqq (b^s)_r$; so from [2, Lemma 6. 5] follows (1. 2. 2).

3. Lattice sums of ring elements

3. 1. Preliminary Lemmas.

Lemma 3. 1. *Suppose $\bar{R}_{\mathfrak{R}}$ has a basis*[13] x_1, x_2, x_3 *with* $x_2 \sim x_1$, $x_3 \precsim x_1$. *Then if $\mathfrak{R}$ is an $\aleph$-ring (respectively von Neumann ring) so is $\mathfrak{R}_2$.*

Proof. This coincides with [5, Corollary 2 to Theorem 3. 1].

Lemma 3. 2. *Every von Neumann ring $\mathfrak{R}$ is a direct sum $\bar{\mathfrak{R}} \oplus \mathfrak{R}'$ with $\mathfrak{R}'$ satisfying the hypotheses of Lemma* 3. 1 *and $\bar{\mathfrak{R}}$ a von Neumann ring in which every idempotent is central.*[14])

Proof. If L is a von Neumann geometry then $L = \sum_{i=0}^{\infty} \oplus L_i$ where L_i has a homogeneous basis consisting of i minimal[15]) elements if $i \geqq 1$, and L_0 has the property: $0 \neq x \in L_0$ implies $0 \neq y_1 \sim y_2$ for some $y_1 \dot{\cup} y_2 \leqq x$ [9, Part III, Theorem 3.2].

There are elements $x_1^{(0)}, x_2^{(0)}$ which form a homogeneous basis for L_0: indeed take a maximal class of pairs $\{y_1^\alpha, y_2^\alpha\}$ with $\{y_1^\alpha, y_2^\alpha \mid \text{all } \alpha\} \perp$[16]) and $y_1^\alpha \sim y_2^\alpha$ for each α and set $x_1^{(0)} = \cup_\alpha y_1^\alpha$, $x_2^{(0)} = \cup_\alpha y_2^\alpha$.

For $i > 1$, L_i has a basis $x_1^{(i)}, x_2^{(i)}, x_3^{(i)}$ with $x_2^{(i)} \sim x_1^{(i)}$, $x_3^{(i)} \precsim x_1^{(i)}$: indeed, if $y_1, \ldots, y_i$ is a homogeneous basis for L_i then according as $i = 2m$ or $i = 2m+1$, take $x_1^{(i)} = y_1 \cup \ldots \cup y_m$, $x_2^{(i)} = y_{m+1} \cup \ldots \cup y_{2m}$, $x_3^{(i)} = 0$ or y_i respectively.

Let $\bar{L} = L_1$, $L' = L_0 \oplus \sum_{i=2}^{\infty} \oplus L_i$. Then $L = \bar{L} \oplus L'$ and $\bar{L}$ is a Boolean algebra whereas L' has a basis $x_j = \cup(x_j^{(i)} \mid i \neq 1)$, $j = 1, 2, 3$, with $x_2 \sim x_1$, $x_3 \precsim x_1$.

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[13]) $x_1, \ldots x_m$ are said to be a *basis* for a lattice L if $\dot{\cup}_i x_i =$ the unit of L; the basis is called *homogeneous* if $x_i \sim x_j$ for all i, j [9].

[14]) If $\mathfrak{R}$ is a regular ring with unit, all idempotents in $\mathfrak{R}$ are central if and only if $\bar{R}_{\mathfrak{R}}$ is a Boolean algebra [9, Part II, Theorem 2.5 (Note) and Theorem 2.10].

[15]) A non-zero element x in a lattice L is called *minimal* or *locally-atomic* if $y_1 \leqq x$, $y_2 \leqq x$, $y_1 \sim y_2$ together imply $y_1 = y_2$ (for another definition, see [9, Part III, Definition 3.1]).

[16]) $\perp$ indicates „independence".

Every direct decomposition of $L=\overline{R}_{\mathfrak{R}}^{\bullet}$ is determined by a corresponding direct decomposition of $\mathfrak{R}$ and from this follows Lemma 3. 2.

Corollary. *Suppose $\mathfrak{R}$ is a von Neumann ring. Then $\mathfrak{R}$ has the property: $\mathfrak{R}_2$ is a von Neumann ring whenever $\mathfrak{R}$ is irreducible, more generally whenever $\mathfrak{R}$ is a direct sum of irreducible von Neumann rings, more generally whenever $\mathfrak{R}$ is a direct sum of von Neumann rings $\mathfrak{R}^\alpha$ each of which has the property: $(\mathfrak{R}^\alpha)_2$ is a von Neumann ring.*

Proof. Since $(\sum \oplus \mathfrak{R}^\alpha)_2 = \sum \oplus (\mathfrak{R}^\alpha)_2$ we need only show that $\mathfrak{R}_2$ is a von Neumann ring whenever $\mathfrak{R}$ is an irreducible von Neumann ring. But if $\mathfrak{R}$ is irreducible, then with the decomposition $\mathfrak{R} = \overline{\mathfrak{R}} \oplus \mathfrak{R}'$ of Lemma 3. 2, we must have $\mathfrak{R}=\overline{\mathfrak{R}}$ or $\mathfrak{R}=\mathfrak{R}'$. Since $(\mathfrak{R}')_2$ is a von Neumann ring (according to Lemma 3. 2), we need only prove: $\mathfrak{R}_2$ is a von Neumann ring whenever $\mathfrak{R}$ is an irreducible von Neumann ring in which every idempotent is central, equivalently, $\mathfrak{R}$ is a division ring. But in this case $\mathfrak{R}_2$ is (trivially) a discrete von Neumann ring.

Lemma 3. 3. *If $(a)_r=\bigcup_\alpha(a_\alpha)_r$ in $\overline{R}_{\mathfrak{R}}$ then $ba_\alpha=0$ for all α if and only if $ba=0$.*

Proof. [9, Part II, Corollary 2 to Lemma 2. 2.]

Lemma. 3. 4. *If $\mathfrak{R}$ is an $\aleph$-ring and $\{x_\alpha|\alpha\in I\}\perp$ in $\overline{R}_{\mathfrak{R}}$ with cardinal of $I\leqq\aleph$, there exist orthogonal idempotents $\{e_\alpha\}$ with $(e_\alpha)_r=x_\alpha$ for all α.*

Proof. Let $y=[\mathfrak{R}-\bigcup_\alpha x_\alpha]$ and choose e_α so that $(e_\alpha)_r=x_\alpha$ and $(1-e_\alpha)_r= =(\bigcup_{\beta\neq\alpha}x_\beta)\cup y$.

Lemma 3. 5. *Suppose $\{e_\alpha|\alpha\in I\}$ are orthogonal idempotents with cardinal of $I\leqq\aleph$ in an $\aleph$-ring and let e be an idempotent with $(e)_r=\bigcup_\alpha(e_\alpha)_r$. Then e is the unique idempotent with also $(1-e)_r=\bigcap_\alpha(1-e_\alpha)_r$*[17]) *if and only if $(e)_l=\bigcup_\alpha(e_\alpha)_l$; then $de_\gamma=e_\gamma$, $de_\beta=0$ for $\beta\neq\gamma$ imply $de=e_\gamma$; $e_\gamma\overline{d}=e_\gamma$, $e_\beta\overline{d}=0$ for $\beta\neq\gamma$ imply $e\overline{d}=e_\gamma$.*

Proof. By [9, Part II, Lemma 2. 2, Corollary 2] $(1-e)_r=\bigcap_\alpha(1-e_\alpha)_r$ is equivalent to $(e)_l=(1-e)^l=\bigcup_\alpha(1-e_\alpha)^l=\bigcup_\alpha(e_\alpha)_l$.

Next, $(de-e_\gamma)e_\alpha=de_\alpha-e_\gamma e_\alpha=0$ for *all* α. Hence, by Lemma 3. 3, $(de-e_\gamma)e=0$, $de=e_\gamma$.

Lemma 3. 6. *Suppose $\bigcup_\alpha x_\alpha$ exists in $\overline{L}_{\mathfrak{R}}$. Then for any d in $\mathfrak{R}$, $\bigcup_\alpha(x_\alpha d)$ exists and is equal to $(\bigcup_\alpha x_\alpha)d$.*

Proof. Let $x=\bigcup_\alpha x_\alpha$. Then for each α, $xd\geqq x_\alpha d$ since $x\geqq x_\alpha$. To prove Lemma 3. 6 we need show: if $y\geqq x_\alpha d$ for all α then $y\geqq xd$.

Suppose $y\geqq x_\alpha d$ for all α. Then $y\cap(d)_l\geqq x_\alpha d$ for all α and therefore it clearly suffices to prove: $(d)_l\geqq y\geqq x_\alpha d$ for all α implies $y\geqq xd$. Now for some a in $\mathfrak{R}$, $y=(ad)_l=(a)_l d$. Let $u=(a)_l\cup d^l$. Then $y=ud$. Hence it is sufficient to prove that $u\geqq x_\alpha$ for all α; this would yield $u\geqq x$ and hence $y=ud\geqq xd$ as required.

To prove $u\geqq x_\alpha$ suppose $c\in x_\alpha$; then $cd\in ud$ since $x_\alpha d\leqq ud$, hence $cd=c_1d$ for some $c_1\in u$. Then $(c-c_1)d=0$, so $c-c_1\in u$. Now u is a left ideal so $(c_1+c-c_1)\in u$, $c\in u$. Thus $x_\alpha\leqq u$, as required. This proves Lemma 3. 6.

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[17]) This e exists because $\bigcup_\alpha(e_\alpha)_r$ and $\bigcap_\alpha(1-e_\alpha)_r$ are complements ($\mathfrak{R}$ is an $\aleph$-ring).

3. 2. Lattice sums of ring elements. In this section, $\mathfrak{R}$ will be an $\aleph$-ring for some $\aleph$, I a set of indices α with cardinal $\leqq \aleph$.

Definition 3. 1. A set of orthogonal idempotents $\sigma = \{e_\alpha\}$ will be called a *separating system* (s. s.); then $e = e_\sigma$ will denote the unique idempotent with $(e)_r = \bigcup_\alpha (e_\alpha)_r$, $(e)_l = \bigcup_\alpha (e_\alpha)_l$ (existing by Lemma 3. 5).

Definition 3. 2. An s. s. σ will be called a *right separating system* (r. s. s.) for $\{d_\alpha\}$ if $e_\alpha d_\alpha = d_\alpha$ for all α.

Definition 3. 3. If σ is a r. s. s. for $\{d_\alpha\}$ then $\sum_\alpha^{\sigma, r} \oplus d_\alpha$ will denote an element d such that $d \in \bigcup_\alpha (d_\alpha)_r$ and $e_\alpha d = d_\alpha$ for each α; such an element d (if existing) will be called a *σ-right lattice sum of the d_α*. Similarly for *σ-left lattice sum*.

Lemma 3. 7. *A r. s. s. σ exists for $\{d_\alpha\}$ if and only if $\{(d_\alpha)_r\} \perp$ (by Lemma 3. 4). If for some r. s. s. σ, a σ-right lattice sum of the d_α exists then its value d is unique, $d = e_\sigma d$, $(d)_l = \bigcup_\alpha (d_\alpha)_l$, and for any element b, $d_\gamma b = d$ for some γ and $d_\alpha b = 0$ for $\alpha \neq \gamma$ imply $db = d_\gamma$.*

Proof. If $e_\alpha(d - \bar{d}) = 0$ for all α, then by the right-left dual of Lemma 3. 3, $e_\sigma(d - \bar{d}) = 0$; $e_\sigma d = e_\sigma \bar{d}$. This means: the σ-right lattice sum (if existing) is unique.

Next, $(d)_l = (e_\sigma d)_l = (e_\sigma)_l d$, so by Lemma 3. 6, $(d)_l = \bigcup_\alpha (e_\alpha d)_l = \bigcup_\alpha (d_\alpha)_l$.

Finally, $e_\alpha(db - d_\gamma) = 0$ for *all* α, so by the right-left dual of Lemma 3. 3, $e_\sigma(db - d_\gamma) = 0$. Hence $db = d_\gamma$.

Definition 3. 4. If $\{(d_\alpha)_r\} \perp$, we denote by $\sum_\alpha^{r} \oplus d_\alpha$ an element d such that d is a σ-right lattice sum of the d_α for every r. s. s. σ for $\{d_\alpha\}$. This d (unique, if it exists, by Lemma 3. 7) will be called the *right lattice sum of the d_α*.

Definition 3. 5. If $\{(d_\alpha)_r\} \perp$ and $\{(d_\alpha)_l\} \perp$ we denote by $\sum_\alpha \oplus d_\alpha$ an element d such that d is a right lattice sum and a left lattice sum of the d_α. This d (unique, if it exists, by Lemma 3. 7) will be called the *lattice sum of the d_α*.

Lemma 3. 8. *If $\{(d_\alpha)_r\} \perp$ and $\{(d_\alpha)_l\} \perp$ and d is a σ-right-lattice sum of the d_α (for some r. s. s. σ) then d is a lattice sum of the d_α.*

Proof. Let $\tau = \{f_\alpha\}$ be any left separating system for $\{d_\alpha\}$ and let $f = f_\tau$. Then $e_\beta(d_\alpha - df_\alpha) = 0$ for all α, β so $e_\sigma(d_\alpha - df_\alpha) = 0$, $d_\alpha = df_\alpha$. This shows that d is a τ-left lattice sum of the d_α (by Lemma 3. 7, $(d)_l = \bigcup_\alpha (d_\alpha)_l \leqq \bigcup_\alpha (f_\alpha)_l$).

Now by right-left duality, d is a $\bar{\sigma}$-right lattice sum of the d_α for every r. s. s. $\bar{\sigma}$ for $\{d_\alpha\}$.

Lemma 3. 9. *If $\{(d_\alpha)_r | \alpha \in I\} \perp$ and I is finite then $\sum_\alpha^{r} \oplus d_\alpha$ exists and coincides with the ordinary (ring) sum $\sum_\alpha d_\alpha$.*

Proof. Obvious.

Lemma 3. 10. *If $\{e_\alpha\}$ are orthogonal idempotents then $\sum_\alpha \oplus e_\alpha$ exists and coincides with the unique idempotent e with properties: $(e)_r = \bigcup_\alpha (e_\alpha)_r$, $(e)_l = \bigcup_\alpha (e_\alpha)_l$.*

Proof. By Definition 3. 3 and Lemma 3. 5.

Corollary. *Suppose* $\{e_\alpha\}$ *are orthogonal idempotents and* $e=\sum_\alpha \oplus e_\alpha$. *If for some* a *in* $\mathfrak{R}$, $e_\alpha a = a e_\alpha$ *for each* α, *then* $\sum_\alpha \oplus (e_\alpha a)$ *exists and equals* $ae=ea$.

Proof. First we show $ae=ea$. We have $(e)_r=\bigcup_\alpha (e_\alpha)_r$ by Lemma 3. 10, and $(a-ea)e_\alpha = ae_\alpha - eae_\alpha = ae_\alpha - ee_\alpha a = ae_\alpha - e_\alpha a = 0$ for all α. By Lemma 3. 3, $(a-ea)e=0$ so $ae=eae$. By a left-right dual argument, $ea=eae$. So $ae=ea$.

Next, $\sigma=\{e_\alpha\}$ is a r. s. s. for $\{ae_\alpha\}$ and $\{(ae_\alpha)_l\}$ are independent since $(ae_\alpha)_l \leqq (e_\alpha)_l$. So by Lemma 3. 8, $ae=\sum_\alpha \oplus (ae_\alpha)$ if only ae is a σ-right lattice sum of the ae_α.

So from Definition 3. 3 we need only show (i): $e_\alpha ae = ae_\alpha$ for each α and (ii): $ae \in \bigcup_\alpha (ae_\alpha)_r$. But (i) holds since $e_\alpha ae = e_\alpha ea = e_\alpha a = ae_\alpha$. As for (ii), $(ae)_r \geqq (aee_\alpha)_r = (ae_\alpha)_r$ so $(ae)_r \geqq \bigcup_\alpha (ae_\alpha)_r$. If $(ae)_r \neq \bigcup_\alpha (ae_\alpha)_r$, then there exists a non-zero idempotent $g\in(ae)_r$ such that $(1-g)_r \geqq (ae_\alpha)_r$ for all α. Then $gae_\alpha = g(1-g)ae_\alpha = 0$, $(ga)e_\alpha=0$ for all α, so by the left-right dual of Lemma 3. 3, $(ga)e=0$. But $g=aed$ for some d, so $g=gg=gaed=0$, a contradiction. Thus $(ae)_r=\bigcup_\alpha (ae_\alpha)_r$ so (ii) holds and the Corollary is established.

Lemma 3. 11. *It* σ *is a r. s. s. for* $\{d_\alpha\}$, *then a* σ*-right lattice sum of the* d_α *does exist if* $\mathfrak{R}_2$ *is an* $\aleph$*-ring.*[18])

Proof. Let $e=e_\sigma$ and form the matrices:

$$D_\alpha = \begin{Vmatrix} 0 & 0 \\ d_\alpha & e_\alpha \end{Vmatrix}, \quad M = \begin{Vmatrix} 0 & 0 \\ 0 & e \end{Vmatrix}.$$

$\{D_\alpha\}$ are orthogonal idempotents in $\mathfrak{R}_2$ so by Lemma 3. 5 an idempotent E in $\mathfrak{R}_2$ exists such that $(E)_r=\bigcup_\alpha (D_\alpha)_r$ and $(E)_l=\bigcup_\alpha (D_\alpha)_l$ in $\overline{R}_{\mathfrak{S}}$ with $\mathfrak{S}=\mathfrak{R}_2$. Now $MD_\alpha = D_\alpha$ for all α so $(M)_r \supset \bigcup_\alpha (D_\alpha)_r = (E)_r$, $ME=E$. Thus E must have the form

$$E = \begin{Vmatrix} 0 & 0 \\ d & g \end{Vmatrix}$$

with $ed=d$. Since $D_\alpha E = D_\alpha$ for all α it also follows that $e_\alpha d = d_\alpha$ for all α. Thus this element d is a σ-right lattice sum of the d_α.

Lemma 3. 12. *Suppose* $d=\sum_\alpha \oplus d_\alpha$, $c=\sum_\alpha \oplus c_\alpha$, *and some* $\sigma=\{e_\alpha\}$ *is a r. s. s. for* $\{c_\alpha\}$ *and a l. s. s. for* $\{d_\alpha\}$. *Then* $\sum_\alpha \oplus (d_\alpha c_\alpha)$ *exists and is equal to* dc.

Proof. Since $(d_\alpha c_\alpha)_r \leqq (d_\alpha)_r$, $\{(d_\alpha c_\alpha)_r\}\perp$. Similarly, $\{(d_\alpha c_\alpha)_l\}\perp$. If $\tau=\{g_\alpha\}$ is a r. s. s. for $\{d_\alpha\}$, then $g_\alpha dc = d_\alpha c = d_\alpha e_\alpha c = d_\alpha c_\alpha$.

Theorem 3. 1. *Suppose* $e=\sum_\alpha \oplus e_\alpha$ *and* $f=\sum_\alpha \oplus f_\alpha$ *for idempotents* e_α, f_α *in a von Neumann ring* $\mathfrak{R}$. *Suppose* $(e_\alpha)_r \sim (f_\alpha)_r$ *for each* α. *Then there exist* d, $\bar{d}$ *in* $\mathfrak{R}$ *such that* $d=edf$, $\bar{d}=f\bar{d}e$, $d\bar{d}=e$, $\bar{d}d=f$, $df_\alpha\bar{d}=e_\alpha$ *and* $\bar{d}e_\alpha d=f_\alpha$ *for each* α. *Moreover, if* $\bigcup_\alpha (e_\alpha)_r = \mathfrak{R}$ *then* $e=1=f$ *and* d *is invertible with* $\bar{d}$ *as its reciprocal.*

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[18]) If $\aleph$ is an infinite cardinal, $\mathfrak{R}_2$ may fail to be an $\aleph$-ring and such d may not exist; this happens in KAPLANSKY's ring (see footnote [7])) if $e_m=(0,\ldots,0,1,0,\ldots)$, $d_m=(0,\ldots,0,\sqrt{-1},0,\ldots)$ $(m\geqq 1)$ where the non-zero components are in the m-th place.

Proof. The last statement would follow from the additivity of perspectivity in a von Neumann geometry [9, Part III], [3].

We recall VON NEUMANN's proof of Theorem 3. 1 for the case I has a single index [9, Part II, Theorem 15.3 (a)]; suppose $(e)_r$ and $(f)_r$ are perspective, hence have a common complement. Then there exist idempotents e', f' such that:

$$(e)_r = (e')_r; \quad (1-e')_r = (1-f')_r; \quad (f')_r = (f)_r.$$

Define $d(e,f)=e'f$, $\bar{d}(e,f)=f'e$. Then it follows that $e'=e'f', f'=f'e', e=e'e$, $e'=ee'$, $f=f'f$, $f'=ff'$. Therefore:

$$ed(e,f)f=d(e,f); \; f\bar{d}(e,f)e=\bar{d}(e,f);$$

$$d(e,f)f\bar{d}(e,f)=d(e,f)\,\bar{d}(e,f)=e;$$

$$\bar{d}(e,f)fd(e,f)=\bar{d}(e,f)d(e,f)=f.$$

Next, if $\mathfrak{R}_2$ is also a von Neumann ring, we need only define $d=\sum_\alpha \oplus d(e_\alpha,f_\alpha)$, $\bar{d}=\sum_\alpha \oplus \bar{d}(e_\alpha,f_\alpha)$, using Lemma 3. 11. and Lemma 3. 12.

Finally, every von Neumann ring $\mathfrak{R}$ has a direct decomposition $\bar{\mathfrak{R}}\oplus\mathfrak{R}'$ as in Lemma 3. 2 and we let E be the central idempotent for which $\mathfrak{R}'=\mathfrak{R}E$.

Then $Ea=aE$ for all a in $\mathfrak{R}$. Let a' denote aE, $\bar{a}$ denote $a(1-E)$. Then in $\bar{\mathfrak{R}}$, $(\bar{e}_\alpha)_r \sim (\bar{f}_\alpha)_r$ and since $\bar{R}_{\bar{\mathfrak{R}}}$ is a Boolean algebra, necessarily $\bar{e}_\alpha=\bar{f}_\alpha$.

In $\mathfrak{R}'$, $(e'_\alpha)_r \sim (f'_\alpha)_r$ and we can apply the argument of the preceding paragraph since $\bar{R}_{\mathfrak{S}}$ is a von Neumann ring when $\mathfrak{S}=\mathfrak{R}'_2$. Now

$$d = \left(\textstyle\sum_\alpha \oplus \bar{e}_\alpha\right) + \textstyle\sum_\alpha \oplus d(e'_\alpha, f'_\alpha),$$

$$\bar{d} = \left(\textstyle\sum_\alpha \oplus \bar{f}_\alpha\right) + \textstyle\sum_\alpha \oplus \bar{d}(e'_\alpha, f'_\alpha)$$

satisfy the requirements of Theorem 3. 1.

Theorem 3. 2. *Suppose $\{e_\alpha\}$ are orthogonal idempotents in a von Neumann ring $\mathfrak{R}$. Suppose for each α, $d_\alpha=e_\alpha d_\alpha=d_\alpha e_\alpha$. If $\mathfrak{R}_2$ is also a von Neumann ring, then $\sum_\alpha \oplus d_\alpha=d$ exists.*

Proof. $\sigma=\{e_\alpha\}$ is a r. s. s. for $\{d_\alpha\}$, so by Lemma 3. 11, $\{d_\alpha\}$ possesses a σ-right lattice sum d. But $\{(d_\alpha)_l\}\perp$ since $(d_\alpha)_l \leqq (e_\alpha)_l$, so by Lemma 3. 8, d is a lattice sum $\sum_\alpha \oplus d_\alpha$.

Corollary 1. *If in Theorem 3.2, $\{\bar{d}_\alpha\}$ satisfy $\bar{d}_\alpha=\bar{d}_\alpha e_\alpha=e_\alpha\bar{d}_\alpha$, $d_\alpha\bar{d}_\alpha=\bar{d}_\alpha d_\alpha=e_\alpha$ for each α, then $\bar{d}=\sum_\alpha\oplus\bar{d}_\alpha$ satisfies $d\bar{d}=\bar{d}d=\sum_\alpha\oplus e_\alpha$.*

Proof. Since $\sigma=\{e_\alpha\}$ is a r. s. s. for $\{\bar{d}_\alpha\}$ and l. s. s. for $\{d_\alpha\}$, it follows from Lemma 3. 12 that $d\bar{d}=\sum_\alpha\oplus(d_\alpha\bar{d}_\alpha)=\sum_\alpha\oplus e_\alpha$. Similarly, $\bar{d}d=\sum_\alpha\oplus e_\alpha$.

Corollary 2. *Suppose $\{e_\alpha\}$ are orthogonal idempotents and $\{f_\alpha\}$ are orthogonal idempotents in a von Neumann ring $\mathfrak{R}$ such that $\mathfrak{R}_2$ is a von Neumann ring. Suppose for each α, $d_\alpha=e_\alpha d_\alpha=d_\alpha f_\alpha$. Then $d=\sum_\alpha\oplus d_\alpha$ exists. Moreover if $\{\bar{d}_\alpha\}$ exists such that for each α: $\bar{d}_\alpha=f_\alpha\bar{d}_\alpha=\bar{d}_\alpha e_\alpha$, $d_\alpha\bar{d}_\alpha=e_\alpha$, $\bar{d}_\alpha d_\alpha=f_\alpha$, then $\bar{d}=\sum_\alpha\oplus\bar{d}_\alpha$ exists and $d\bar{d}=\sum_\alpha\oplus e_\alpha$, $\bar{d}d=\sum_\alpha\oplus f_\alpha$.*

Proof. The argument for Theorem 3.2 and its Corollary 1 is valid in the present case.

4. Proof of the special cases (1. 4. 1), (1. 4. 2)

Lemma 4. 1. *Suppose that c is in an $\aleph_0$-ring $\mathfrak{R}$ and $\cap((c^s)_r|s\geqq 1)=0$. Then $\mathfrak{R}$ can be espressed as the union of independent principal right ideals:*

$$\mathfrak{R} = \dot{\cup}(x_{i,j}|1\leqq i<\infty;\ 1\leqq j\leqq i) \tag{4. 1. 1}$$

such that $cx_{i,j}=x_{i,j+1}$ and $c^r\cap x_{i,j}=0$ for $1\leqq j<i<\infty$, and $cx_{i,i}=0$ for $1\leqq i<\infty$. Then[19] *necessarily $x_{i,j}\sim x_{i,j+1}$ for $1\leqq j<i$, $c\mathfrak{R}\dot{\cup}(\cup(x_{i,1}|i\geqq 1))=\mathfrak{R}$, and for each $s\geqq 1$, $(c^s)^r\geqq x_{i,j}$ if $i-s<j\leqq i$ and $(c^s)^r\cap x_{i,j}=0$ if $j\leqq i-s$, so (by* (1. 1. 1) *and the modular law) $(c^s)^r=\cup(x_{i,j}|i\geqq 1;\ i-s<j\leqq i)$.*

Moreover any value of $[(c^i)^r-((c^i)^r\cap((c^{i-1})^r\dot{\cup}c\mathfrak{R}))]=[(c^i)^r-((c^{i-1})^r\cup((c^i)^r\cap\cap c\mathfrak{R}))]$ may be used as $x_{i,1}$;[20] *on the other hand, any value of $[(c^r\cap c^{i-1}\mathfrak{R})-(c^r\cap c^i\mathfrak{R})]$ may be used as $x_{i,i}$.*

Proof. Suppose $x_{i,1}$ given as described and define $x_{i,j}=c^{j-1}x_{i,1}$ for $1\leqq j\leqq i$. Then for $1\leqq j<i$, $cx_{i,j}=x_{i,j+1}$. If $d\in x_{i,1}$ and $c^jd=0$ with $1\leqq j<i$ then $c^{i-1}d=0$, $d\in((c^{i-1})^r\cup((c^i)^r\cap c\mathfrak{R}))$, hence (see the definition of $x_{i,1}$) $d=0$. Thus $c^r\cap x_{i,j}=0$ for $1\leqq j<i$. Clearly $cx_{i,i}=c^ix_{i,1}=0$.

Next we show that for each $j\geqq 1$

$$\{c^j\mathfrak{R},\ x_{i,j}|i\geqq j\}\perp.$$

For suppose $c^jv=c^{j-1}v_j+\ldots+c^{j-1}v_s$ with all $v_i\in x_{i,1}$. Then we must have $c^{j-1}v_s=0$. Otherwise, $j-1<s$ and left multiplication by c^{s-j} yields: $c^sv=c^{s-1}v_s$. Then $v_s=(v_s-cv)+cv$ and $(v_s-cv)\in(c^{s-1})^r$, $cv\in(c^s)^r\cap c\mathfrak{R}$; this implies that $v_s=0$ since $v_s\in x_{s,1}$ and hence $c^{j-1}v_s=0$, after all. Repetition of this argument shows that $c^{j-1}v_i=0$ for all $i=s, s-1, \ldots, j$ and hence $c^jv=0$. This proves the assertion.

From this it follows that for each fixed $j\geqq 1$: $\{x_{i,j}|i\geqq j\}\perp$. Also $\{\dot{\cup}(x_{i,j}|i\geqq j)\,|\,j\geqq 1\}\perp$ since $\dot{\cup}(x_{i,j}|i\geqq j)\cap\dot{\cup}(x_{i,s}|i\geqq s>j)\leqq\dot{\cup}(x_{i,j}|i\geqq j)\cap c^j\mathfrak{R}=0$. This implies that $\{x_{i,j}|i\geqq 1;\ 1\leqq j\leqq i\}\perp$.

Next, by [4, Lemma 6. 2], $\cap((c^j)_l|j\geqq 1)=0$ since by assumption $\cap((c^j)_r|j\geqq 1)=0$. Hence $\mathfrak{R}=(\cap((c^j)_l|j\geqq 1))^r=\cup((c^j)^r|j\geqq 1)$ (by [9, Part II, Lemma 2. 2, Corollary 2]). Since $(\cup(x_{i,1}|i\geqq 1))\cup(c\mathfrak{R})\geqq(c^j)^r$ for all $j\geqq 1$ it follows that $\cup(x_{i,j}|1\leqq j\leqq i<\infty)\cup\cup c\mathfrak{R}\geqq\mathfrak{R}$. Successive left multiplication by c now gives: $\cup(x_{i,j}|1\leqq j\leqq i<\infty)\cup\cup c^m\mathfrak{R}\geqq\mathfrak{R}$ for all $m\geqq 1$, and since $\cap(c^m\mathfrak{R}|m\geqq 1)=\cap((c^m)_r|m\geqq 1)=0$, therefore $\dot{\cup}(x_{i,j}|1\leqq j\leqq i<\infty)=\mathfrak{R}$.

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[19]) $x_{i,j}\sim x_{i,j+1}$ follows from $cx_{i,j}=x_{i,j+1}$ and $c^r\cap x_{i,j}=0$ because of Corollary 1 of Lemma 2. 1. Further, from $cx_{i,j}=x_{i,j+1}$ follows, because of formula (4. 1. 1), $c\mathfrak{R}=\dot{\cup}(x_{i,j}|1<\,<j\leqq i)$.

[20]) When specialized to the classical case, this result yields: let T be a linear transformation of a finite dimensional vector space V into itself (V shall be finite dimensional over a division ring D but D need not be commutative) and suppose $T^p=0$ but $T^{p-1}\neq 0$ for some $p\geqq 1$. Let $N(T)=\{v|Tv=0\}$. Let $\xi_{i,1},\ldots,\xi_{i,s_i}$ be a basis for the difference-space $[N(T^i)-N(T^i)\cap\cap(N(T^{i-1})\cup T(V))]$. Then $\{T^j\xi_{i,k}|i=1,\ldots,p;\ k=1,\ldots,s_i;\ j=0, 1, \ldots, i-1\}$ are a basis for V.

On the other hand, if the $x_{i,i}$ are pre-assigned as some given $[(c^r \cap c^{i-1}\mathfrak{R}) - (c^r \cap c^i\mathfrak{R})]$, set $x_{i,1} = [\{d | c^{i-1}d \in x_{i,i}\} - (c^{i-1})^r]$. We shall show that these values for $x_{i,1}$ satisfy the conditions given in the first part of Lemma 4. 1 and that $c^{i-1}x_{i,1}$ will coincide with the given $x_{i,i}$.

First, if $d \in x_{i,1}$, then $c^{i-1}d \in x_{i,i}$ and $x_{i,i} \leqq c^r$. Hence $c^i d = 0$. This proves: $x_{i,1} \leqq (c^i)^r$.

Next, $x_{i,1}$ is a relative complement of $(c^i)^r \cap ((c^{i-1})^r \cup (c\mathfrak{R}))$ with respect to $(c^i)^r$; to show this we must prove: (i) $x_{i,1} \cap ((c^{i-1})^r \cup c\mathfrak{R}) = 0$, (ii) $x_{i,1} \cup ((c^{i-1})^r \cup c\mathfrak{R}) \geqq (c^i)^r$.

To prove (i), suppose $d \in x_{i,1}$ and $d = u + cv$ with $c^{i-1}u = 0$. Then $c^{i-1}d = c^i v \in x_{i,i}$ and $c(c^i v) = 0$. Hence $c^i v \in (c^r \cap c^i\mathfrak{R})$ so, from the definition of $x_{i,i}$ it follows that $c^i v = 0$. Thus $c^{i-1}d = 0$. Now we have $d \in (c^{i-1})^r$, so from the definition of $x_{i,1}$ it follows that $d = 0$. This proves (i).

To prove (ii), we remark that from the definition of $x_{i,1}: x_{i,1} \cup (c^{i-1})^r = \{d | c^{i-1}d \in x_{i,i}\}$. Hence

$$x_{i,1} \cup ((c^{i-1})^r \cup c\mathfrak{R}) = \{d | c^{i-1}d \in x_{i,i}\} \cup c\mathfrak{R}.$$

Now suppose $u \in (c^i)^r$. Then $c^{i-1}u \in c^r$ so $c^{i-1}u \in (c^r \cap c^{i-1}\mathfrak{R})$. Then from the definition of $x_{i,i}: c^{i-1}u = v + w$ for some $v \in x_{i,i}$ and some $w \in (c^r \cap c^i\mathfrak{R})$. Now $w = c^i q$ for some q. Therefore $u = d + cq$ where $d = u - cq$ has the property: $c^{i-1}d = c^{i-1}u - c^i q = v \in x_{i,i}$. Hence $u \in (x_{i,1} \cup ((c^{i-1})^r \cup c\mathfrak{R}))$, which implies (ii).

Finally, if $d \in x_{i,1}$, then $c^{i-1}d \in x_{i,i}$, so $c^{i-1}x_{i,1} \leqq x_{i,i}$; on the other hand, if $u \in x_{i,i}$, then $u = c^{i-1}w$ for some w, so $w = d + v$ for some $d \in x_{i,1}$ and some $v \in (c^{i-1})^r$. So $u = c^{i-1}d \in c^{i-1}x_{i,1}$. Thus $x_{i,i} \leqq c^{i-1}x_{i,1}$. Hence $x_{i,i} = c^{i-1}x_{i,1}$ as stated.

Now all parts of Lemma 4. 1 are established.

Remark. If c is an element in an arbitrary regular ring $\mathfrak{R}$ with unit and $c^h = 0$ for some integer h, then the proof of Lemma 4. 1 is valid; moreover the range of i may be restricted to $1 \leqq i \leqq h$ (the appeal to [4, Lemma 6. 2] and [9, Part II, Lemma 2. 2, Corollary 2]) is unnecessary here since $\mathfrak{R} = \cup((c^j)^r | j \geqq 1)$ is an immediate consequence of $c^h = 0$, $(c^h)^r = \mathfrak{R}$.

Lemma 4. 2. *Suppose the hypotheses of Lemma 4. 1 hold and that* $c = p(a)$ *for some element a and some pure irreducible p in P,* $p(t) = t^m + z_{m-1}t^{m-1} + \ldots + z_0$.

If z_0 *is invertible, in particular if* $m > 1$, *then the element a is invertible. In every case if* $\mathfrak{R}$ *is a von Neumann ring*[21]) *and* (1. 3. 4) *holds* $\mathfrak{R}$ *has a decomposition as described in Lemma 4. 1 with the additional properties: For each* $i \geqq 1$,

(i) $x_{i,i} = \dot{\cup}(a^j x_i | 0 \leqq j < m)$ *for some* x_i;

(ii) $x_{i,1} = \cup(a^j y_i | 0 \leqq j < m)$ *for some* y_i *with* $c^{i-1}y_i = x_i$;

(iii) $(a^j)^r \cap y_i = 0$ *so* $a^j y_i \sim y_i$ *for* $0 \leqq j < mi$.

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[21]) Lemma 4.1 (and Lemma 4.2 for the case $m = 1$) hold if $\mathfrak{R}$ is any $\aleph_0$-ring. But if $m > 1$ our proof of Lemma 4.2 uses transfinite induction (or ZORN's Lemma) and requires $\mathfrak{R}$ to be a von Neumann ring in which (1. 3. 4) holds.

Then (necessarily implied by (i), (ii), (iii) *in any* $\aleph_0$*-ring)*

(iv) $\bigcup(a^j y_i \mid 0 \leqq j < ms) = \dot{\bigcup}(x_{i,j} \mid 1 \leqq j \leqq s)$ *for* $1 \leqq s \leqq i$ *and* $\{a^j y_i \mid 0 \leqq j < mi\} \perp$;

(v) $\dot{\bigcup}(a^j y_i \mid i \geqq 1, 0 \leqq j < mi) = \mathfrak{R}$.

Proof. Suppose that $d \in a^r$. Then

$$0 = \bigcap((p^s(a))_r \mid s \geqq 1) \geqq \bigcap((p^s(a)d)_r \mid s \geqq 1) = \bigcap((z_0^s d)_r \mid s \geqq 1) = (d)_r$$

if z_0 is invertible. Then $a^r = 0$, $(a)_l = \mathfrak{R}$. Thus if z_0 is invertible then (in any $\aleph_0$-ring by [4, Lemma 6. 2]) a is invertible.

Suppose $m > 1$. Suppose e is a non-zero central idempotent. If $ez_0 = 0$, then $ep(t) = et(t^{m-1} + \ldots + z_1)$ which is impossible since p is pure irreducible. Hence $ez_0 \neq 0$ for every non-zero central idempotent e. But $z_0\mathfrak{R} = e_0\mathfrak{R}$ for some central idempotent e_0 [9, Part II, Theorem 2. 5], and $(1-e_0)z_0 \in (1-e_0)e_0\mathfrak{R} = 0$. This forces $1-e_0$ to be 0, so $e_0 = 1$, $z_0\mathfrak{R} = \mathfrak{R}$. This shows that z_0 is invertible, and therefore, by the preceding paragraph, a also is invertible.

Next, suppose (i), (ii) and (iii) hold. Then, by (ii), $c^{i-1}(a^j y_i) = a^j c^{i-1} y_i = a^j x_i$ so $a^j y_i \sim a^j x_i$ if $0 \leqq j < m$ by Corollary 1 to Lemma 2. 1 (since $(c^{i-1})^r \cap x_{i,1} = 0$). But $x_{i,1} \sim x_{i,i}$ so, by (i), $\bigcup(a^j y_i \mid 0 \leqq j < m) \sim \dot{\bigcup}(a^j x_i \mid 0 \leqq j < m)$. This forces: $\{a^j y_i \mid 0 \leqq j < m\} \perp$ by [2, Lemmas 6. 15, 4. 4], in any $\aleph_0$-ring. The same argument applies to inclusion relation

$$\bigcup(a^j y_i \mid 0 \leqq j < ms) \geqq \dot{\bigcup}(c^k a^j y_i \mid 0 \leqq j < m; 0 \leqq k \leqq s-1) = \dot{\bigcup}(x_{i,j} \mid 1 \leqq j \leqq s)$$

and forces the addends on the left to be independent and the inclusion to be equality. Thus (i), (ii), (iii) imply (iv) and hence (v).

We need now only show that (i), (ii) and (iii) can be satisfied.

If $m = 1$, choose $x_{i,i}$ and $x_{i,1}$ as in Lemma 4. 1. Let $x_i = x_{i,i}$, $y_i = x_{i,1}$. Then (i) and (ii) hold obviously ($m = 1$). Suppose for some j with $0 \leqq j < i$ and some $d \in y_i$ that $a^j d = 0$; then $(c - z_0)^j d = 0$ so $c^j d \in (x_{i,1} \cup \ldots \cup x_{i,j}) \cap x_{i,j+1} = 0$, hence $d = 0$. Thus (iii) holds by Corollary 1 to Lemma 2. 1.

We may therefore suppose $m > 1$. Let $A_i = c^r \cap c^{i-1}\mathfrak{R}$. Then $aA_i \leqq A_i$ so $aA_i = A_i$ (since a is invertible).

Now since p is pure irreducible and (1. 3. 4) holds, an argument of von Neumann [4, Lemma 5. 1] applies here[21]) and shows, by transfinite induction that for some x_i: $A_{i+1} \dot{\cup} (\dot{\bigcup}(a^j x_i \mid 0 \leqq j < m)) = A_i$. Hence we may use $\bigcup(a^j x_i \mid 0 \leqq j < m)$ as the pre-assigned $x_{i,i}$ in Lemma 4. 1 and (i) will hold.

Let $B_i = \{d \mid c^{i-1} d \in x_i\} \leqq (c^i)^r$ and define $y_i = [B^i - (c^{i-1})^r]$. Then $c^{i-1} y_i = x_i$. Also

$$(\dot{\bigcup}(a^j y_i \mid 0 \leqq j < m)) \dot{\cup} (c^{i-1})^r = \{d \mid c^{i-1} d \in x_{i;i}\}$$

so we may also (in the proof of Lemma 4. 1) choose $\bigcup(a^j y_i \mid 0 \leqq j < m)$ as $x_{i,1}$. Then (ii) holds.

As for (iii), since $m > 1$, a is invertible and $(a^j)^r = 0$; thus (iii) does hold.

This completes the proof of Lemma 4. 2.

Lemma 4. 3. *Suppose a and b are elements in a regular ring $\mathfrak{R}$ with unit and suppose m is an integer $\geqq 1$. Suppose $x_1, \ldots, x_m$ is a basis for $\bar{R}_{\mathfrak{R}}$ such that $ax_i = x_{i+1} = bx_i$ for $1 \leqq i < m$ and $a^r = b^r = x_m$. Then a and b are similar.*

Proof. We may suppose $m \geqq 2$ (if $m=1$ then $a=b=0$ and so $b=dad^{-1}$ with $d=1$).

Since $x_1, \ldots, x_m$ is a basis for $\overline{R}_{\mathfrak{R}}: \dot{\cup}(x_i|1 \leqq i \leqq m)=\mathfrak{R}$, in particular $x_i \cap x_j=0$ if $i \neq j$. But if $1 \leqq i < m$, then $ax_i=x_{i+1}$ and $a^r \cap x_i=x_m \cap x_i=0$, so by Corollary 1 to Lemma 2. 1, $x_i \sim x_{i+1}$. Hence $x_1, \ldots, x_m$ is a homogeneous basis for $\overline{R}_{\mathfrak{R}}$. Then by [9, Part II, Lemma 3. 6] there exist matrix units s_{ij} $(i, j=1, \ldots, m)$ with $(s_{ii})_r=x_i$ for all i. Finally, the proof of [9, Part II, Theorem 3. 3] (note especially [9, page 99, lines 13, 14]) shows that $\mathfrak{R}=\mathfrak{S}_m$ with $\mathfrak{S}_1=s_{11}\mathfrak{R}s_{11}$.

We shall call $c=(c_{ij})$ *off-diagonal* if (i) $c_{ij}=0$ except when $i=j+1$ and (ii) $c_{j+1,j}$ is invertible (in $s_{11}\mathfrak{R}s_{11}$) for $1 \leqq j < m$. Let c_0 be the off-diagonal element with non-zero entries all $1(=s_{11})$.

Now the hypotheses of Lemma 4. 3 force a and b to be off-diagonal; so it is sufficient to prove a and c_0 are similar. Thus we need only find an invertible $d=(d_{ij})$ such that $ad=dc_0$. For this purpose choose $d_{ij}=0$ for $i \neq j$, and $d_{11}=1$, $d_{ii}=a_{i,i-1}a_{i-1,i-2} \ldots a_{21}$ for $i > 1$; then $ad=dc_0$.

This completes the proof of Lemma 4. 3.

Theorem 4. 1. *Suppose that a, b are in an $\aleph_0$-ring, $\mathfrak{R}$ such that $\mathfrak{R}_2$ is an $\aleph_0$-ring, and $(a^s)_r \sim (b^s)_r$ for $s \geqq 1$ and $\cap((a^s)_r|s \geqq 1)=0$. Then a and b are similar.*

Proof. Since $\cap((b^s)_r|s \geqq 1) \precsim (a^m)_r$ for all $m \geqq 1$ it follows by [2, Lemma 6. 11] that $\cap((b^s)_r|s \geqq 1)=0$.

Let $x^a_{i,j}$ and $x^b_{i,j}$ be determined for a, b respectively as in Lemma 4. 1. First we shall show that $x^a_{i,j} \sim x^b_{i,j}$. We have: $(a^s)_r \sim (b^s)_r$ for $s \geqq 1$, hence $[(a^{s-1})_r-(a^s)_r] \sim [(b^{s-1})_r-(b^s)_r]$ for $s \geqq 1$. Then by Lemma 4. 1, $\dot{\cup}(x^a_{i,s}|i \geqq s) \sim$ $\sim \dot{\cup}(x^b_{i,s}|i \geqq s)$.

Since $x^a_{i,s} \sim x^a_{i,1}$ for each $i \geqq s$, $\dot{\cup}(x^a_{i,s}|i \geqq s) \sim \dot{\cup}(x^a_{i,1}|i \geqq s)$. Hence $\dot{\cup}(x^a_{i,1}|i \geqq s) \sim$ $\sim \dot{\cup}(x^b_{i,1}|i \geqq s)$, and so by subtraction, $x^a_{i,1} \sim x^b_{i,1}$ for all $i \geqq 1$. Then $x^a_{i,j} \sim x^a_{i,1} \sim$ $\sim x^b_{i,1} \sim x^b_{i,j}$ so $x^a_{i,j} \sim x^b_{i,j}$ for all $1 \leqq j \leqq i < \infty$, as stated.

Now let $\{e_{i,j}\}$, $\{f_{i,j}\}$ be families of orthogonal idempotents such that $(e_{i,j})_r=x^a_{i,j}$ and $(f_{i,j})_r=x^b_{i,j}$. Then by Theorem 3. 1, $df_{i,j}d^{-1}=e_{i,j}$ for some invertible d.

The element $c=dbd^{-1}$ has the property: $(c^s)_r=(db^s)_r \sim (b^s)_r$ for $s \geqq 1$ (use Corollary 1 to Lemma 2. 1), so $(c^s)_r \sim (a^s)_r$. Hence $\cap((c^s)_r|s \geqq 1)=0$ (the argument used above for b applies to c also). Finally $(df_{i,j}d^{-1})_r$ may be used as $x^c_{i,j}$ since the mapping: $(u)_r \to (dud^{-1})_r=(du)_r$ is a lattice automorphism of $\overline{R}_{\mathfrak{R}}$.

So we may suppose $x^c_{i,j}=(e_{i,j})_r=x^a_{i,j}$ and clearly, we need only prove a and c are similar. In other words, we may assume $x^a_{i,j}=x^b_{i,j}=x_{i,j}$ (say).

Let $\{e_i\}$ be orthogonal idempotents with $(e_i)_r=\dot{\cup}(x_{i,j}|1 \leqq j \leqq i)$, $\sum_i \oplus e_i=1$. Then $ae_i=e_ia$, $be_i=e_ib$ for all i and the hypotheses of Lemma 4. 3 are satisfied in the ring $e_i\mathfrak{R}e_i$ by ae_i, be_i and $x_{i,j}e_i$ $(1 \leqq j \leqq i)$.

Thus for some d_i, $\overline{d}_i$ in $e_i\mathfrak{R}e_i$, $d_i\overline{d}_i=e_i=\overline{d}_id_i$ and $e_ib=d_iae_i\overline{d}_i$.

Now $d=\sum_i \oplus d_i$, $\overline{d}=\sum_i \oplus \overline{d}_i$ exist by Lemma 3. 11; and by Lemma 3. 12, $d\overline{d}=\overline{d}d=1$, $b=da\overline{d}$. Thus Theorem 4. 1 is established.[22])

[22]) Theorem 4.1 together with Lemma 4.3, yields a „canonical" representation for any a in $\mathfrak{R}$ for which $\cap((a^s)_r|s \geqq 1)=0$.

Lemma 4. 4. *Suppose a and b are invertible elements in a regular ring $\mathfrak{R}$ with unit. Suppose $m \geqq 1$ and $p(t) = t^m + z_{m-1}t^{m-1} + \ldots + z_0$ is in P and $p(a) = p(b) = 0$. Suppose $x, ax, \ldots, a^{m-1}x$ is a basis for $\overline{R}_{\mathfrak{R}}$ and $a^i x = b^i x$ for $1 \leqq i < m$. Then a and b are similar.*

Proof. We may suppose $m \geqq 2$ (if $m = 1$ then $a = -z_0 = b$ and $b = dad^{-1}$ with $d = 1$).

Then a is invertible and Corollary 1 to Lemma 2. 1 shows that $x, ax, \ldots, a^{m-1}x$ is a homogeneous basis for $\overline{R}_{\mathfrak{R}}$. Hence $\mathfrak{R}$ possesses matrix units s_{ij} $(i, j = 1, \ldots, m)$ with $(s_{i1})_r = a^{i-1}x$ for $1 \leqq i \leqq m$.

Call $c = (c_{ij})$ *p-off-diagonal* if:

(i) $c_{i+1,i}$ is invertible (in $s_{11}\mathfrak{R}s_{11}$) for $1 \leqq i < m$,

(ii) $c_{i,m}c_{m,m-1}c_{m-1,m-2}\ldots c_{i+1,i} = -z_{i-1}$ for $1 \leqq i \leqq m$, and

(iii) $c_{ij} = 0$ for all other i, j.

Let c_0 be the p-off-diagonal element with $c_{i+1,i} = 1$ for $1 \leqq i < m$.

The hypotheses of Lemma 4. 4 force a and b to be p-off-diagonal. Hence, we need only show that $ad = dc_0$ for some invertible d. For this purpose take $d_{11} = 1$, $d_{ii} = a_{i,i-1}\ldots a_{21}$ for $1 < i \leqq m$ and $d_{ij} = 0$ for $i \neq j$. This completes the proof of Lemma 4. 4.

Theorem 4. 2. *Suppose that a and b are elements in a von Neumann ring $\mathfrak{R}$ and that* (1. 3. 4), (1. 3. 5) *hold, and $m \geqq 1$ and $p(t) = t^m + z_{m-1}t^{m-1} + \ldots + z_0$ is in P and pure irreducible. Suppose $(p^s(a))_r \sim (p^s)b))_r$ for all $s \geqq 1$ and $\bigcap((p^s(a))_r | s \geqq 1) = 0$. Then a and b are similar.*

Proof. Theorem 4. 1 applies to $p(a)$ and $p(b)$ and shows that $p(a) = dp(b)d^{-1} = p(dbd^{-1})$ for some invertible d. If $m = 1$, then $b + z_0 = d(a + z_0)d^{-1}$ so $b = dad^{-1}$ for some invertible d, as required. Thus we may assume $m \geqq 2$. Since we need only show that a and dbd^{-1} are similar, we may now assume that $p(b) = p(a)$.

Now Lemma 4. 2 can be applied to yield elements y_i^a, y_i^b for a, b respectively, as described in Lemma 4. 2. The corresponding values of $x_{i,1}$ (as described in Lemma 4. 1) $x_{i,1}^a, x_{i,1}^b$ may not be the same but they are of the form $[x - \bar{x}]$ for the same $x, \bar{x}$; hence they are perspective. So, by (ii) of Lemma 4. 2: $\bigcup(a^j y_i^a | 0 \leqq j < m) \sim \bigcup(b^j y_i^b | 0 \leqq j < m)$. Moreover, by (v) of Lemma 4. 2, the elements in each of these unions form an independent family, and by (iii) of Lemma 4. 2, the elements in the same family are mutually perspective.

Now $y_i^a \sim y_i^b$ follows from the theorem that $u_1 \sim v_1$ in a von Neumann geometry whenever $\dot{\bigcup}(u_i | 1 \leqq i \leqq m) \sim \dot{\bigcup}(v_i | 1 \leqq i \leqq m)$ with $\{u_i\}$ mutually perspective and $\{v_i\}$ mutually perspective (in the terminology of [9, Part III, page 272]: if $mA = mB$ with $m \geqq 1$ then $A = B$). To prove this theorem assume if possible that $u_1 \sim v_1$ is false. Then for some w in the centre of the geometry:

$w \cap u_1 \sim v_1^0$ where $v_1^0 \leqq v_1$ but $v_1^0 \neq v_1$ (here we use [9, Part III, Theorem 2. 7], and interchange u_1, v_1 if necessary). Then there exist elements v_i^0 such that

$$w \cap \bigcup_{i=1}^{m} u_i = \bigcup_{i=1}^{m} (w \cap u_i) \sim \bigcup_{i=1}^{m} (w \cap v_i^0) = w \cap \bigcup_{i=1}^{m} v_i^0 \leqq$$

$$\leqq w \cap \bigcup_{i=1}^{m} v_i \quad \text{but with} \quad w \cap \bigcup_{i=1}^{m} v_i^0 \neq w \cap \bigcup_{i=1}^{m} v_i^0.$$

On the other hand, $\left(w \cap \bigcup_{i=1}^{m} u_i\right) \sim \left(w \cap \bigcup_{i=1}^{m} v_i\right)$ by [9, Part III, Theorem 1. 4, (*a*), with $w = \bar{a}$, $\bigcup_{i=1}^{m} u_i = a$, $\bigcup_{i=1}^{m} v_i = b$]. So by the transitivity of perspectivity the lattice element $c = w \cap \bigcup_{i=1}^{m} v_i$ satisfies: $c \sim c_1$ with $c_1 \leqq c$, $c_1 \neq c$. But this is impossible. Hence $u_1 \sim v_1$ must hold, and so $y_i^a \sim y_i^b$.

Then $a^j y_i^a \sim b^j y_i^b$ for all j. Now by Theorem 3. 1 there exists a similarity mapping which maps $b^j y_i^b$ onto $a^j y_i^a$ for all $0 \leqq j < mi$. Hence, in proving Theorem 4.2, we may suppose $b^j y_i^b = a^j y_i^a$ for all $0 \leqq j < mi$.

Now set $Y_i = \bigcup (a^j y_i | 0 \leqq j < mi)$. Then $\dot{\bigcup}_i Y_i = \mathfrak{R}$ and $aY_i = bY_i$ for all $i \geqq 1$. By Lemma 3. 4 there exist orthogonal idempotents F_i with $(F_i)_r = Y_i$, $aF_i = F_i a$, $bF_i = F_i b$.

The hypotheses of Lemma 4. 4 are satisfied in the ring $F_i \mathfrak{R} F_i$ by aF_i and bF_i and $\{a^j y_i F_i | 0 \leqq j < mi\}$. Hence aF_i and bF_i are similar in the ring $F_i \mathfrak{R} F_i$ and, as in the proof of Theorem 4. 1, Lemmas 3. 11 and 3. 12 can be used to derive: a and b are similar in $\mathfrak{R}$.[23])

5. Proof of the Main Theorem

We suppose $\mathfrak{R}$ is a von Neumann ring satisfying (1.3.4), (1.3.5) and need only prove Theorem 1.1 (ii). It will be sufficient to prove the following „augmentation" lemma.

Lemma 5. 1. *Suppose P_1, a, b satisfy the hypotheses* (1. 3. 2) *and* (1. 3. 3). *Suppose $S^a = \{e_\alpha^a, p_\alpha | \alpha \in I\}$ and $S^b = \{e_\alpha^b, p_\alpha | \alpha \in I\}$ have the properties:*[24])

(5. 1. 1) *e_α^a, e_α^b are non-zero idempotents with $\bar{e}_\alpha^a = \bar{e}_\alpha^b = \bar{e}_\alpha$* (say) *for each $\alpha \in I$, $p_\alpha \in P_1$ and $\bar{e}_\alpha p_\alpha$ is pure irreducible in $\bar{e}_\alpha \mathfrak{R}$;*

(5. 1. 2) $\bar{e}_\alpha\left(\bigcap ((p_\alpha^s(a))_r | s \geqq 1)\right) = (\bar{e}_\alpha - e_\alpha^a)_r$,
$\bar{e}_\alpha\left(\bigcap ((p_\alpha^s(a))_l | s \geqq 1)\right) = (\bar{e}_\alpha - e_\alpha^a)_l$; *similarly for b in place of a.*

(5. 1. 3) $\bar{e}_\alpha \bar{e}_\beta p_\alpha \neq \bar{e}_\alpha \bar{e}_\beta p_\beta$ if $\bar{e}_\alpha \bar{e}_\beta \neq 0$.

Then:

(5. 1. 4) $(e_\alpha^a)_r \sim (e_\alpha^b)_r$ *for each $\alpha \in I$, $(\Sigma_\alpha \oplus e_\alpha^a)_r \sim (\Sigma_\alpha \oplus e_\alpha^b)_r$;*

(5. 1. 5) *$\{e_\alpha^a | \alpha \in I\}$, $\{e_\alpha^b | \alpha \in I\}$ are sets of orthogonal idempotents,*
$a e_\alpha = e_\alpha a$, $e_\alpha b = b e_\alpha$;

(5. 1. 6) *If $\sum_\alpha \oplus e_\alpha \neq 1$ it is possible to augment S^a, S^b by pairs (e^a, p), (e^b, p) preserving* (5. 1. 1), (5. 1. 2) *and* (5. 1. 3).

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[23]) Theorem 4.2 together with Lemma 4.4 yields a „canonical" representation for any a in $\mathfrak{R}$ for which $\bigcap ((p^s(a))_r | s \geqq 1) = 0$ for some *pure irreducible* p in P.

[24]) For any idempotent e in a von Neumann ring $\mathfrak{R}$, we write $\bar{e}$ to denote the central cover of e, that is, the central idempotent $\bar{e}$ with the properties: $\bar{e}e = e$ and for any central idempotent f, $fe = e$ implies $f\bar{e} = \bar{e}$.

Remark. Theorem 1. 1 (ii) can be deduced from Lemma 5. 1. To see this, note that by transfinite induction (or ZORN's Lemma) it is possible to choose S^a, S^b to be maximal with the properties (5. 1. 1), (5. 1. 2), (5. 1. 3). Then from (5. 1. 6) it will follow that $\sum_\alpha \oplus e_\alpha^a = 1$ and hence $\sum_\alpha \oplus e_\alpha^b = 1$.

Then by Theorem 3. 2 there will exist d, d' in $\mathfrak{R}$ such that $dd' = d'd = 1$ (so $d' = d^{-1}$) and $de_\alpha^b d' = e_\alpha^a$, $d' e_\alpha^a d = e_\alpha^b$ for each α. The mapping $u \to dud^{-1}$ is a ring isomorphism of $e_\alpha^b \mathfrak{R} e_\alpha^b$ onto $e_\alpha^a \mathfrak{R} e_\alpha^a$.

Let $c = dbd^{-1}$. Then $de_\alpha^b bd^{-1} = dbe_\alpha^b d^{-1} = e_\alpha^a c = ce_\alpha^a$ and (5. 1. 1), (5. 1. 2), (5. 1. 3) hold if b is replaced by c and each e_α^b is replaced by $e_\alpha^a = e_\alpha$ (say).

In each ring $e_\alpha \mathfrak{R} e_\alpha$, the elements $e_\alpha c$, $e_\alpha a$ satisfy the hypothesis of Theorem 4. 2. Hence g_α, g'_α exist in $e_\alpha \mathfrak{R} e_\alpha$ such that $g_\alpha g'_\alpha = g'_\alpha g_\alpha = e_\alpha$ and $g_\alpha e_\alpha c g'_\alpha = e_\alpha a$.

If now $\mathfrak{R}_2$ is also a von Neumann ring then, by Corollary 1 to Theorem 3. 2, the elements $g = \sum \oplus g_\alpha$, $g' = \sum_\alpha \oplus g'_\alpha$ exist and satisfy: $gg' = g'g = 1$ (so $g' = g^{-1}$). Then by the Corollary to Lemma 3. 10, $c = \sum_\alpha \oplus e_\alpha c$ and by Lemma 3. 12, $gcg^{-1} = \sum_\alpha \oplus (g_\alpha(e_\alpha c) g_\alpha^{-1}) = \sum_\alpha \oplus (e_\alpha a) = a$.

Thus *if* $\mathfrak{R}_2$ *is a von Neumann ring*, c and a are similar, hence b and a are similar, which establishes Theorem 1. 1 (ii).

Thus we need only prove Lemma 5. 1 to complete the proof of Theorem 1. 1.

Proof of Lemma 5. 1. The hypotheses of Theorem 1. 1 (ii) imply that (recall the definition of $\bar{e}_\alpha$ given in footnote [24]):

$$\bar{e}_\alpha\big(\cap (p_\alpha^s(a))_r \,|\, s \geqq 1\big) \sim \bar{e}_\alpha\big(\cap (p_\alpha^s(b))_r \,|\, s \geqq 1\big).$$

Hence (5. 1. 2) implies $(e_\alpha^a)_r \sim (e_\alpha^b)_r$. Now (5. 1. 5) follows from [4, § 7. 1]. Then $\dot{\bigcup} (e_\alpha^a)_r \sim \dot{\bigcup} (e_\alpha^b)_r$, by the additivity of perspectivity in von Neumann geometries [3]. So (5. 1. 5) and (5. 1. 4) both hold.

Finally, we establish (5. 1. 6). Suppose $E = 1 - \sum_\alpha \oplus e_\alpha^a \neq 0$. Then $Ee_\alpha^a = e_\alpha^a E = 0$ and the Corollary to Lemma 3. 10 shows that $aE = Ea$ so $p(a)E = Ep(a)$ for all $p \in P$.

Now a is assumed to be P_1-almost algebraic, so $\cap_p (p(a))_r = 0$ when p varies over all products of factors from P_1. Hence $\cap_p (Ep(a))_r = \cap_p (p(a)E)_r = 0$. Thus for some such p, $(Ep(a))_r \neq (E)_r$.

Since P_1 is fully factorizable there is a set of orthogonal non-zero central idempotents $\{\bar{e}\}$ such that $\bigcup (\bar{e})_r = \mathfrak{R}$ and each $\bar{e}p$ is a product $\bar{e}p_1 \dots p_m$ with all p_i in P_1 and $\bar{e}p_i$ pure irreducible in $\bar{e}\mathfrak{R}$.

Now for at least one of these $\bar{e}$ we have $(\bar{e}E)_r \neq (\bar{e}Ep(a))_r$ since for every c in $\mathfrak{R}$: $(c)_r = \bigcup (\bar{e}c)_r$ (use the Corollary to Lemma 3. 10). Hence with this $\bar{e}$: $(\bar{e}Ep(a))_r = (\bar{e}Ep_1(a) \dots p_m(a))_r \neq (\bar{e}E)_r$ where the p_i are all in P_1 and each $\bar{e}p_i$ is pure irreducible in $\bar{e}\mathfrak{R}$. If $\bar{e}Ep_i(a) b_i = \bar{e}E$ were to hold for some b_i for $i = 1, \dots, m$ we would have

$$\begin{aligned} \bar{e}Ep_1(a) \dots p_m(a) b_m \dots b_1 &= \bar{e}Ep_1(a) \dots p_{m-1}(a) \bar{e}Eb_{m-1} \dots b_1 = \\ &= \bar{e}E\bar{e}Ep_1(a) \dots p_{m-1}(a) b_{m-1} \dots b_1 = \dots = \bar{e}E\bar{e}E \dots \bar{e}E = \bar{e}E, \end{aligned}$$

a contradiction. Thus, if p is replaced by a suitable p_i, we can assert: p is in P_1, $\bar{e}p$ is pure irreducible in $\bar{e}\mathfrak{R}$ and $(\bar{e}Ep(a))_r \neq (\bar{e}E)_r$. For the rest of this proof we keep p fixed with this value.

Now we apply the well known method of „exhaustion". Let $\{\bar{f}\}$ be a set of orthogonal non-zero central idempotents maximal with the property: $\bar{f}\bar{e} = \bar{f}$ and

$(\bar{f}Ep(a))_r = (\bar{f}E)_r$. Let $\bar{f}_0 = \sum \oplus (\bar{f})$. Then $\bar{f}_0 \bar{e} = \bar{f}_0$ and, using the Corollary to Lemma 3. 10, we deduce $(\bar{f}_0 Ep(a))_r = (\bar{f}_0 E)_r$.

Thus, if $\bar{e}$ is replaced by $\bar{e} - \bar{f}_0$ we can assert: $(\bar{g}Ep(a))_r \neq (\bar{g}E)_r$ whenever $\bar{g}$ is a non-zero central idempotent with $\bar{g} = \bar{e}\bar{g}$. For the rest of this proof we keep $\bar{e}$ fixed with this value (clearly, $\bar{e} \neq 0$).

Applying [4, § 7. 1] to the ring $\bar{e}\mathfrak{R}$, we choose e^a to be the unique idempotent with $e^a = \bar{e}e^a$ and

$$(\bar{e} - e^a)_r = \bigcap \left((\bar{e}p^s(a))_r \mid s \geqq 1 \right),$$

and
$$(\bar{e} - e^a)_l = \bigcap \left((\bar{e}p^s(a))_l \mid s \geqq 1 \right);$$

similarly, with $p^s(b)$ in place of $p^s(a)$, we choose e^b.

Since we assume (1. 3. 2) and (1. 3. 3) it follows that for each $s \geqq 1$, $(\bar{e}p^s(a))_r \sim \sim (\bar{e}p^s(b))_r$, hence

$$\bigcap \left((\bar{e}p^s(a))_r | s \geqq 1 \right) \sim \bigcap \left((\bar{e}p^s(b))_r | s \geqq 1 \right)$$

(use [2] or [3]). Hence, by subtraction: $(e^a)_r \sim (e^b)_r$, and so $\overline{e^a} = \overline{e^b}$ (use [9, Part III, Theorem 1. 4 (*d*)]).

We now prove that if S^a, S^b are augmented by the pairs (e^a, p), (e^b, p) then (5. 1. 1), (5. 1. 2), (5. 1. 3) are preserved.

First, we shall show that $\overline{e^a} = \bar{e}$. If this were false then, since $e^a \bar{e} = e^a$ it follows that $\overline{\bar{g}e^a} = 0$ (and hence $\bar{g}e^a = 0$) for some $\bar{g} = \bar{g}\bar{e} \neq 0$. But our choice of e^a implies, by [4, § 7. 1], that $((\bar{e} - e^a)p(a))_r = (\bar{e} - e^a)_r$ so $(\bar{g})_r = (\bar{g}(\bar{e} - e^a))_r = (\bar{g}(\bar{e} - e^a)p(a))_r = = (\bar{g}p(a))_r$; $(\bar{g})_r = (\bar{g}p(a))_r$. But also, by our choice of $\bar{e}$: $(\bar{g}E)_r \neq (\bar{g}Ep(a))_r$. This is a contradiction, for if $\bar{g}p(a)c = \bar{g}$, then $\bar{g}Ep(a)c = E\bar{g}p(a)c = E\bar{g} = \bar{g}E$. This contradiction shows that $\overline{e^a} = \bar{e}$. Since $\bar{e} \neq 0$, it follows that $e^a \neq 0$ and so (5. 1. 1) and (5. 1. 2) continue to hold.

Next we show that (5. 1. 3) also continues to hold. We suppose for some α that $\bar{g} = \bar{e}_\alpha \bar{e} \neq 0$ and we need only show that $\bar{g}p_\alpha \neq \bar{g}p$. It is sufficient to show that $\bar{g}p_\alpha(a) \neq \bar{g}p(a)$.

Since $e^a_\alpha E = 0$ it follows from (5. 1. 2) that

$$(\bar{e}_\alpha E)_r = ((\bar{e}_\alpha - e^a_\alpha)E)_r \leqq \bigcap \left((p^s_\alpha(a))_r \mid s \geqq 1 \right) \geqq (p_\alpha(a))_r,$$

so $E\bar{e}_\alpha = p_\alpha(a)c$ for some c in $\mathfrak{R}$. Then $\bar{g}Ep_\alpha(a)c = \bar{g}E\bar{e}_\alpha = \bar{g}E$ so $(\bar{g}Ep_\alpha(a))_r = (\bar{g}E)_r$. But by our choice of $\bar{e}$, since $\bar{g} \neq 0$ and $\bar{g} = \bar{e}\bar{g}$: $(\bar{g}Ep(a))_r \neq (\bar{g}E)_r$. Hence $\bar{g}p_\alpha(a) \neq \neq \bar{g}p(a)$, as required to show that (5. 1. 3) continues to hold.

This completes the proof of Lemma 5. 1 and so Theorem 1. 1 is established.

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