On the functional calculus of an operator measure

By JACOB FELDMAN in Berkeley (California, USA)

1. Introduction

Let T be a set, $\mathbf{B}$ a σ-algebra of subsets of T, and F an operator measure on $\mathbf{B}$. That is,

(1) For each S in $\mathbf{B}$, $F(S)$ is a nonnegative bounded linear operator on the Hilbert space H,

(2) If S is the union of disjoint sets $S_1, S_2, \ldots$ in $\mathbf{B}$ then $F(S) = \sum_{n=1}^{\infty} F(S_n)$, where the sum converges in the strong topology,

(3) $F(T) = I$ (the identity operator on H).

NAĬMARK has shown ([1], p. 266, or [2]) that $F(\cdot)$ can be written in the form $PE(\cdot)|H$, where E is a projection-valued measure with values in some Hilbert space K containing H, and P is the orthogonal projection from K onto H.

Let $L_\infty(F)$ be the class of all bounded complex-valued Borel measurable functions on T, identified modulo functions f which are F-null in the sense that $F(\{t|f(t)\neq 0\})=0$. If H is separable, then by choosing a sequence $x_1, x_2, \ldots$ of unit vectors which span H, and setting $m(S) = \sum_{n=1}^{\infty} \frac{1}{2^n}(F(S)x_n, x_n)$, one sees that $L_\infty(F)$ is just $L_\infty(m)$. In any case, we can put the usual algebra, norm, and $*$ structure on $L_\infty(F)$.

There is defined a map φ from $L_\infty(F)$ to bounded operators on H by $\varphi(f) = P\int f(t)dE(t)|H$. The map φ is characterized by the property that $(\varphi(f)x, y) = \int f(t)d(F(t)x, y)$ for all x, y in H. Clearly φ is a linear, norm-nonincreasing, $*$-preserving, positivity-preserving map.

Sometimes we shall write $\int f(t)dF(t)$ for $\varphi(f)$. Indeed, this is a true equation if the operator integral is interpreted in the weak topology.

Of special interest is the case where T is the unit circle C in the complex plane, $\mathbf{B}$ consists of the Borel sets, and F is what is called a *strong* operator measure: that is, setting $A = \int tdF(t)$,

(4) $$\int t^n dF(t) = \begin{cases} A^n & \text{if } n>0, \\ (A^*)^{-n} & \text{if } n<0. \end{cases}$$

Clearly $\|A\| \leqq 1$. Furthermore, if we are given a *preassigned* A of norm $\leqq 1$, then, by a theorem of SZ.-NAGY [3], [4], there is precisely *one* strong operator measure F on the Borel sets of the unit circle related to A by $A = \int t dF(t)$. Without assumption (4), of course, there are many F for a given A. The corresponding operator $U = \int t dE(t)$ on the containing Hilbert space K is called the *unitary dilation* of A, and the operator $\varphi(f)$ is precisely $Pf(U)|H$. In this case, the function φ becomes multiplicative on polynomials in t, and hence also on their bounded F-a. e. limits. (Cf.[5], [6].) In the present paper, we do *not* make assumption (4). However, the results are new even for strong operator measures.

Let $C(r)$ be the circle of radius r in the complex plane, and $D(r)$ the closed disk of radius r. If $r=1$, we write simply C and D. For any bounded operator A on H, $\sigma(A)$ is defined as the spectrum of A, and $\alpha(A)$ its approximate point spectrum (which we interpret as including the point spectrum). Thus $\alpha(A) \subset \sigma(A) \subset D(\|A\|)$. In the following theorems, statements are made about $\alpha(A) \cap C(\|A\|)$. It should be noticed, however, that this is the same set as $\sigma(A) \cap C(\|A\|)$, since the boundary of $\sigma(A)$ is always contained in $\alpha(A)$ (this fact was pointed out to me by G. ORLAND).

In the following theorems F is a fixed operator measure on C, and we utilize the notation above.

Theorem 1. *For any $f \in L_\infty(F)$, $C(\|f\|) - \alpha(\varphi(f))$ is equal to the intersection of $C(\|f\|)$ with the union of all those open sets U of the plane for which $\|F(f^{-1}(U))\| < 1$.*

For $f \in L_\infty(F)$, let $\sigma(f)$ denote the spectrum of f as an element in the algebra $L_\infty(f)$. Thus z is in $\sigma(f)$ if and only if $f^{-1}(U)$ is F-nonnull for each neighborhood U of z. Furthermore $\sigma(f) \subset D(\|f\|)$, and $\sigma(f) \cap C(\|f\|)$ is nonempty.

Corollary. (a) *If φ is norm-preserving, then $\|F(S)\| = 1$ for all F-nonnull S in* **B**.

(b) *If $\|F(S)\| = 1$ for all F-nonnull S in* **B**, *then not only is φ norm-preserving, but in fact*

$$\alpha(\varphi(f)) \cap C(\|f\|) = \sigma(f) \cap C(\|f\|).$$

Theorem 2. *Let F be an operator measure on the Borel sets of the complex unit circle, and let F be absolutely continuous with respect to Lebesgue measure. Let φ be norm-preserving when restricted to those functions in $L_\infty(F)$ which have representatives in H_∞. Then φ is norm-preserving on all of $L_\infty(F)$.*

I would like to thank Professor M. SCHREIBER for a series of discussions on this subject, from which I have profited considerably.

2. Proofs of the Theorems

Proof of Theorem 1. First suppose that for each open neighborhood U of z_0 we have $\|F(f^{-1}(U))\| = 1$, and $|z_0| = \|f\|$. We wish to show that z_0 is in $\alpha(\varphi(f))$. There is clearly no loss of generality in assuming $\|f\| = 1$ and $z_0 = 1$, since the problem can be shifted to this by using $z_0^{-1}f$ instead of f. Choose $\varepsilon > 0$. Let U be an open disk about 1, of radius $2\varepsilon/3$. Write S for $f^{-1}(U)$, and choose x of norm 1 in H such that $(F(S)x, x) > 1 - \varepsilon/3$. Then $(F(C-S)x, x) < \varepsilon/3$. We write $(\varphi(f)x, x)$ as

$$\int_S d(F(t)x, x) + \int_S (f(t) - 1) d(F(t)x, x) + \int_{C-S} f(t) d(F(t)x, x).$$

Thus:

$$|(\psi(f)x, x) - 1| \leqq \left|\int_S d(F(t)x, x) - 1\right| + \int_S |f(t) - 1| d(F(t)x, x) + \int_{C-S} d(F(t)x, x) <$$

$$< |(F(S)x, x) - 1| + \varepsilon/3 + (F(C-S)x, x) < \varepsilon.$$

That is, $(\varphi(f)x, x)$ can be made arbitrarily close to 1 by appropriate choice of x. But since both $\varphi(f)x$ and x are vectors of length $\leqq 1$, this implies that $\varphi(f)x$ can be made arbitrarily close to x by appropriate choice of x, i. e. $1 \in \alpha(\varphi(f))$.

Conversely, suppose that z_0 is in $\alpha(\varphi(f)) \cap C(\|f\|)$. We wish to show that $\|F(f^{-1}(U))\| = 1$ for each open neighborhood U of z_0. Again there is no loss of generality in assuming that $\|f\|$ and z_0 are 1. Choose $\varepsilon > 0$, and x of norm 1 in H such that

$$|1 - \psi((f)x, x)| < \varepsilon^2,$$

that is

$$\left|1 - \int f(t) d(F(t)x, x)\right| < \varepsilon^2.$$

Then

$$\varepsilon^2 > \mathrm{Re}\left(1 - \int f(t) d(F(t)x, x)\right| = \int (1 - \mathrm{Re} f(t)) d(F(t)x, x),$$

where "Re" means "real part". Let $U_\varepsilon = \{z | \mathrm{Re}\, z \geqq 1 - \varepsilon\}$. Let $S_\varepsilon = f^{-1}(U_\varepsilon)$. Then we have $(F(C - S_\varepsilon)x, x) < \varepsilon$, so that $\|F(S_\varepsilon)\| > 1 - \varepsilon$. Since S_ε decreases as ε decreases, it follows that $\|F(S_\varepsilon)\| = 1$. If now U is *any* open neighborhood of 1 in the complex plane, then if ε is chosen sufficiently small we will have $U_\varepsilon \subset U$. Thus, $\|F(f^{-1}(U))\| = 1$ for any neighborhood U of 1.

P r o o f o f t h e C o r o l l a r y. (a) Suppose $0 < \|F(S)\| < 1$ for some S in $\mathbf{B}$. Let f be the characteristic function of S. Then $\|f\| = 1$, while $\|\varphi(f)\| = \|F(S)\| < 1$.

(b) This follows directly from Theorem 1.

P r o o f o f T h e o r e m 2. Let F be as described in our assumptions, and $0 < \|F(S)\| = c < 1$. Let $u(z)$ be the harmonic function which has the boundary value 0 on S and $\log(1 - c/2)$ on $C - S$. Let u^* be its harmonic conjugate. Then $e^{u + iu^*}$ is an H_∞ function whose values on C have absolute value 1 a. e. on S and $1 - c/2$ a. e. on $C - S$, with respect to Lebesgue measure. Let $f = e^{u + iu^*}|C$. Let g be the function on C which is equal to f on S and to 0 on $C - S$, while h is equal to 0 on S and to f on $C - S$. Thus $\varphi(f) = \varphi(g) + \varphi(h)$, so

$$\|\varphi(f)\| \leqq \|\varphi(g)\| + \|\varphi(h)\|.$$

Now, $\|\varphi(h)\| \leqq \|h\| = 1 - c/2$. We shall show that $\|\varphi(g)\| \leqq c$, which will show that $\|\varphi(f)\| < 1$, giving the desired contradiction.

For each Borel subset R of C, set

$$G(R) = c^{-1} F(R \cap S) + m(R - S) m(C - S)^{-1} (I - c^{-1} F(C - S)).$$

Then G is an operator measure on Borel sets of the unit circle, and is absolutely continuous with respect to Lebesgue measure. Let ψ be the map from $L_\infty(G)$ to operators obtained from G, i. e. $(\psi(k)x, y) = \int k(t)d(G(t)x, y)$. Then ψ is norm-nonincreasing. Applying this to the function g above, we get

$$|(\psi(g)x, y)| = \left|\int g(t)d(G(t)x, y)\right| = \left|c^{-1}\int g(t)d(F(t)x, y)\right|.$$

But $\|\psi(g)\| \leqq 1$, so $|(\varphi(g)x, y)| \leqq c\|x\|\,\|y\|$, and therefore $\|\varphi(g)\| \leqq c$.

3. Some remarks and a question

We have seen via the corollary to Theorem 1 that if the map φ arising from an operator measure F is norm-preserving, then $\alpha(\varphi(f)) \cap C(\|f\|)$ equals $\sigma(\|f\|) \cap \cap C(\|f\|)$. The opposite direction is obvious, of course. However, there are situations in which assumptions on the spectrum of $\varphi(f)$ for only a *single* function f lead to φ being an isometry.

Consider the case where F is an operator measure on the Borel sets of C. Let $A = \int t dF(t)$.

(1) M. SCHREIBER has shown in [7], and it also follows without difficulty from our Theorem 1, that if $\alpha(A)$ contains the support of F, then $\|\varphi(f)\| = \|f\|$ whenever f has a *continuous* representative.

(2) In the same paper, SCHREIBER, shows that if F is a strong operator measure, F being absolutely continuous with respect to Lebesgue measure, and $\alpha(A)$ contains some neighborhood of C in D, then φ is isometric on H_∞, and so by our Theorem 2 on all of L_∞ (where L_∞ refers to Lebesgue measure). SCHREIBER's theorem actually assumes that $\alpha(A) = D$, but his proof can be seen to give the stronger form we have stated.

So one question which naturally arises is this: let F be a strong operator measure on C. Does the condition $\alpha(A) \supset C$ suffice to make φ isometric on $L_\infty(F)$?

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UNIVERSITY OF CALIFORNIA
BERKELEY, CALIFORNIA

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