A cosine functional equation in Banach algebras*

By SVETOZAR KUREPA in Zagreb (Yougoslavia)

§ 1. Introduction

Throughout this paper $B=\{a, b, \ldots\}$ denotes a real or complex Banach algebra[1]) with unit element e; $R=\{\alpha, \beta, \ldots, t, s, \ldots\}$ the set of all real numbers, and $X=\{x, y, \ldots\}$ a Banach space.

In this paper we study the functions f: $R\to B$ such that:

(1) $$f(t+s)+f(t-s) = 2f(t)f(s), \qquad f(0) = e,$$

for all $t, s\in R$, and functions F: $X\to B$ such that:

(2) $$F(x+y)+F(x-y) = 2F(x)\,F(y), \qquad F(0) = e,$$

for all $x, y\in X$.

The functional equation (1) was studied in our earlier papers. In [2] we have solved this equation under the assumption that the elements of B are square matrices of finite order. In [3] B was the algebra of all bounded normal operators defined on some Hilbert space. Assuming the weak continuity of f we have proved that

$$f(t)=\cos ta=\int \cos t\lambda\, de(\lambda),$$

where a is a normal operator which does not depend on t, $e(\lambda)$ is the spectral resolution of identity which corresponds to a and integration is over the complex plane. Furthermore in [4] B was the Banach algebra of all continuous and linear operators which are defined on some Banach space Y. Assuming that Y is reflexive and separable we have proved that weak measurability of f on one interval implies weak continuity of f on R. The functional equation (2) was also considered in [4]. Assuming that B is the set of complex numbers and F is continuous it was proved that there exists an additive and continuous functional A: $X\to B$ such that

$$F(x) = \cos A(x)$$

for all $x\in X$.

It is the object of this paper to treat the general case of functional equations (1) and (2) assuming in (1) that f is measurable and in (2) that F is measurable on every

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[1]) We follow the terminology of HILLE—PHILLIPS [1].

ray. We remark that the motivation for such considerations were the following problems A and B in Hille – Phillips [1], p. 278:

P r o b l e m A. Determine all measurable functions f on $(0, +\infty)$ to B such that for all t and s in $(0, +\infty)$

$$f(t+s) = f(t)\,f(s). \tag{3}$$

P r o b l e m B. Determine all functions F on a complex Banach space to a complex Banach algebra B which are measurable on rays and satisfy

$$F(x+y) = F(x)\,F(y) \tag{4}$$

for all x and y in a given cone.

It was proved ([1] pp. 280 – 291) that measurability of f which satisfies (3) implies continuity and that

$$f(t) = \sum_0^\infty a^n t^n/n! \qquad (a \in B)$$

for all $t \in (0, +\infty)$ provided that $f(t) \to e$ as $t \to 0$. If the function F satisfies (4), if it is measurable on every ray and has property that $\lim F(tx) = e$ $(t \to 0)$ uniformly with respect to x on some sphere, then

$$F(x) = \sum_0^\infty [P(x)]^n/n!,$$

where $P\colon X \to B$ is an additive and continuous function.

In § 2 we treat the problem A for the functional equation (1) and in § 3 we treat the problem B for the functional equation (2), i. e. we consider the following two problems:

P r o b l e m A′. Determine all measurable functions f from the set R of all real numbers in a real or complex Banach algebra B such that for all t and s

$$f(t+s)+f(t-s) = 2f(t)f(s), \quad f(0) = e.$$

P r o b l e m B′. Determine all functions F on a Banach space X to a Banach algebra B which are measurable on rays for all x, y and satisfy the functional equation

$$F(x+y)+F(x-y) = 2F(x)\,F(y), \quad F(0) = e.$$

We prove that in the first case there exists an element $a \in B$, independent of t, such that

$$f(t) = \sum_0^\infty a^n t^{2n}/(2n)! \tag{5}$$

for all $t \in R$, where the series converges uniformly and satisfies the functional equation (1). Furthermore it is always possible to imbed the Banach algebra B in another Banach algebra $\hat{B}$ which consists of all square matrices of order 2, the elements of which are elements of B, in such a way that

$$\hat{f}(t) = \begin{pmatrix} f(t) & 0 \\ 0 & f(t) \end{pmatrix} = \sum_0^\infty (\hat{a}t)^{2n}/(2n)!$$

where $\hat{a}$ is an element of $\hat{B}$. If the element a in (5) possesses a regular square root in B then the functional equation (1) can be reduced to the functional equation of type (3). If 1) B is a Banach algebra of bounded linear operators which are defined on some Hilbert space; 2) the element a in (5) possesses a regular square root in B and 3) $\sup_{t\in R}\|f(t)\| < +\infty$, then

$$f(t) = \sum_0^\infty (-1)^n (td_0)^{2n}/(2n)!,$$

where the operator d_0 is similar to a selfadjoint operator, i. e. there is a bounded and regular operator q such that $q\,d_0q^{-1}$ is a selfadjont operator.

Concerning the problem B' we prove in § 3 that

$$F(x) = \sum_0^\infty [A(x)]^n/(2n)!$$

for all $x\in X$, where $A\colon X\to B$. The function A is continuous if and only if $\lim F(tx) = e$ $(t\to 0)$ uniformly with respect to x in some sphere. If $\|e-A(x)\|<1$ for some x then $A(x)=[L(x)]^2$, where $L\colon X\to B$ is an additive function which is continuous if A is continuous.

§ 2. Problem A′ for a cosine functional equation

Theorem 1. *Let $R=\{\alpha, \beta, \ldots, t, s, \ldots\}$ be the set of all real numbers, $B=\{a, b, \ldots\}$ a real or complex Banach algebra with unit e and $f\colon R\to B$ a single-valued measurable function such that*

(6) $$f(t+s)+f(t-s) = 2f(t)f(s), \quad f(0) = e,$$

holds for all $t, s\in R$. Then there is one and only one element $a\in B$ such that

(7) $$f(t) = e+\frac{at^2}{2!}+\frac{a^2t^4}{4!}+\frac{a^3t^6}{6!}+\ldots = \sum_0^\infty \frac{a^nt^{2n}}{(2n)!},$$

the series (7) being absolutely convergent for every $t\in R$.

Proof. From (6) and $f(0)=e$ we see that $f(-t) = f(t)$ and $f(t)f(s)=f(s)f(t)$ for every pair $t, s\in R$. Since f is measurable the numerical function $\|f(t)\|$ is measurable in the Lebesgue sense. Hence there is a perfect set P of strictly positive and finite measure on which $\|f(t)\|$ is bounded. This in the same way as in [3] implies that $\|f(t)\|$ is bounded on every finite interval. Since f is measurable and locally bounded it is locally integrable in the Bochner sense ([1], theorem 3. 7. 4, p. 80).

Now set $f_0(t) = f(t) - e$. The function f_0 is measurable and locally bounded. Furthermore it satisfies the functional equation:

(8) $$f_0(t+s)+f_0(t-s) = 2f_0(t)+2f_0(s)+2f_0(t)f_0(s).$$

If in (8) we set $u = t+s$ and $v = t-s$, then we get:

$$f_0(u)+f_0(v) = 2f_0\left(\frac{u+v}{2}\right)+2f_0\left(\frac{u-v}{2}\right)+2f_0\left(\frac{u+v}{2}\right)f_0\left(\frac{u-v}{2}\right).$$

Integration from 0 to 1 leads to:

$$f_0(v) = \left[-\int_0^1 + 4\int_{\frac{v}{2}}^{\frac{1+v}{2}} + 4\int_{-\frac{v}{2}}^{\frac{1-v}{2}}\right] f_0(u)\,du + 4\int_{\frac{v}{2}}^{\frac{1+v}{2}} f_0(u) f_0(u-v)\,du.$$

We assert that f_0 is a continuous function. In order to prove this it is sufficient to consider the integral

$$\int_{\frac{v}{2}}^{\frac{1+v}{2}} f_0(u) f_0(u-v)\,du = \left[\int_{\frac{v}{2}}^{0} + \int_{\frac{1}{2}}^{\frac{1}{2}+\frac{v}{2}} + \int_0^{\frac{1}{2}}\right] f_0(u) f_0(u-v)\,du. \tag{9}$$

Suppose that $v \to v_0$. Then

$$\left\| \int_\alpha^{\alpha+\frac{v}{2}} f_0(u) f_0(u-v)\,du - \int_\alpha^{\alpha+\frac{v_0}{2}} f_0(u) f_0(u-v_0)\,du \right\| \leqq$$

$$\leqq 3M^2|(v-v_0)/2| + M\int_\alpha^{\alpha+\frac{v_0}{2}} \|f_0(u-v) - f_0(u-v_0)\|\,du,$$

where M is a suitably chosen constant. But the last integral tends to zero as $v \to v_0$ ([1], theorem 3. 8. 3, p. 86). Taking $\alpha = 0$ and $1/2$ we find that two of the integrals on the right hand side of (9) are continuous functions of v. For the third integral in (9) we have:

$$\left\| \int_0^{\frac{1}{2}} f_0(u) f_0(u-v)\,du - \int_0^{\frac{1}{2}} f_0(u) f_0(u-v_0)\,du \right\| \leqq M\int_0^{\frac{1}{2}} \|f_0(u-v) - f_0(u-v_0)\|\,du \to 0.$$

Thus f_0 is a continuous function on R and so is f too. This implies that

$$\lim_{\alpha\to 0} \frac{1}{\alpha}\int_0^\alpha f(t)\,dt = f(0) = e.$$

Hence there is a number γ such that

$$\left\| \frac{1}{\gamma}\int_0^\gamma f(t)\,dt - e \right\| < 1,$$

and consequently such that

$$c = \left[\int_0^\gamma f(t)\,dt\right]^{-1}$$

exists.

Now we integrate (6) from 0 to α with respect to s. We get

$$2f(t)\int_0^{\alpha} f(s)\,ds = \int_t^{\alpha+t} f(s)\,ds + \int_{-t}^{\alpha-t} f(s)\,ds. \tag{10}$$

From (10) we find:

$$2[f(t+u)-f(t)]\int_0^{\alpha} f(s)\,ds = \left|\int_{\alpha+t}^{\alpha+t+u} + \int_{\alpha-t}^{\alpha-t-u} - \int_t^{t+u} - \int_{-t}^{-t-u} f(s)\,ds\right|. \tag{11}$$

If in (11) we set $\alpha=\gamma$, multiply by c, divide by u and let $u\to 0$, we get:

$$2\lim_{u\to 0}[f(t+u)-f(t)]/u = [f(\gamma+t)-f(\gamma-t)]c. \tag{12}$$

Thus

$$g(t) = df/dt = \lim_{u\to 0}[f(u+t)-f(t)]/u \tag{13}$$

exists for every t, i. e. f is a differentiable (in fact strongly differentiable) function. Now we divide (11) by u and let $u\to 0$. We get

$$2g(t)\int_0^{\alpha} f(s)\,ds = f(\alpha+t)-f(\alpha-t). \tag{14}$$

From (14) and (6) we find:

$$f(t+s) = f(t)f(s)+g(t)\int_0^s f(u)\,du \tag{15}$$

which, because of the symmetry, leads to

$$g(t)\int_0^s f(u)\,du = g(s)\int_0^t f(u)\,du. \tag{16}$$

If we take $s=\gamma$ in (16) we get:

$$g(t)/t = cg(\gamma)(1/t)\int_0^t f(u)\,du.$$

Thus

$$\lim_{t\to 0} g(t)/t = cg(\gamma) = a \tag{17}$$

exists and it is an element of B. If we divide (16) by t and allow $t\to 0$ we find:

$$g(s) = a\int_0^s f(u)\,du, \tag{18}$$

i. e.

$$df/dt = a\int_0^t f(u)\,du. \tag{19}$$

Since $f(0)=e$ and $g(0)=0$ (in (18) set $s=0$), from (19) we find:

(20) $$f(t) = e+a\int_0^t ds\int_0^s f(u)\,du = e+a\int_0^t (t-s)f(s)\,ds.$$

The iteration method applied to the integral equation (20) leads to

$$f(t) = e+\frac{at^2}{2!}+\ldots+\frac{a^n t^{2n}}{(2n)!}+\frac{a^{n+1}}{(2n+1)!}\int_0^t (t-s)^{2n+1}f(s)\,ds$$

from which (6) follows. The uniqueness of the solution of (20) can be proved in the usual way. The uniqueness of a is obvious from (7) and direct calculation shows that (7) satisfies (2). Thus theorem 1 is proved.

If in B an element b exists such that $b^2=a$, then (7) can be written in the form:

(21) $$f(t) = \sum_0^\infty (tb)^{2n}/(2n)! = [\exp tb+\exp(-tb)]/2$$

which is natural to call the hyperbolic cosine. However, if b exists it is not unique. In that case as a rule there are infinitely many square roots of a and (7) is written in the form $f(t)=\mathrm{ch}\,tb$. However, f is in fact the function of $b^2=a$ which is unique.

Generally the square root of a does not exist in B and the solution (7) can not be written in the form (21). This also follows from theorem 3 of the paper [2]. We illustrate the situation by the example. Let B be the Banach algebra of 2×2 complex matrices. The function $f(t)=\begin{pmatrix}1 & 0\\ t^2 & 1\end{pmatrix}$ satisfies (6) and it is not of the form (21). In this case $a=\begin{pmatrix}0 & 0\\ 2 & 0\end{pmatrix}$ and there is no 2×2 matrix b with the property that $b^2=a$. On the other hand the matrix

$$\bar{a} = \left(\begin{array}{cc|cc} 1 & 0 & 1 & 0\\ \frac{1}{2} & 1 & -\frac{1}{2} & 1\\ \hline -1 & 0 & -1 & 0\\ \frac{1}{2} & -1 & -\frac{1}{2} & -1 \end{array}\right)$$

has the property that

$$\bar{a}^2 = \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0\\ 2 & 0 & 0 & 0\\ \hline 0 & 0 & 0 & 0\\ 0 & 0 & 2 & 0 \end{array}\right) = \begin{pmatrix} a & 0\\ 0 & a\end{pmatrix}.$$

In this case the 2×2 matrix function

$$f(t)=e+\frac{at^2}{2!}+\frac{a^2t^4}{4!}+\frac{a^3t^6}{6!}+\dots$$

can be written by use of the 4×4 matrix function in the form

$$\hat{f}(t)=\begin{pmatrix} f(t) & 0 \\ 0 & f(t)\end{pmatrix}=\hat{\imath}+\frac{\hat{a}^2t^2}{2!}+\frac{\hat{a}^4t^4}{4!}+\dots=\text{ch}\, t\hat{a}.$$

This example suggests the idea of imbedding the Banach algebra B in another Banach algebra $\hat{B}$ which has the property that any $a\in B$ as an element in $\hat{B}$ has a square root in $\hat{B}$; i. e., there is at least one element $\hat{b}\in\hat{B}$ such that $\hat{b}^2=\hat{a}$. The construction of such a Banach algebra $\hat{B}$ is very simple. It is sufficient to consider all 2×2 matrices $\hat{x}$, $\hat{y}$ the elements x_{ij}, y_{ij} $(i,j=1,2)$ of which are elements of B and to define the usual matrix operations between such matrices. Introducing the norm in $\hat{B}$ by the formula:

$$\|\hat{x}\|=\sum_{i,j=1}^{2}\|x_{ij}\|,$$

one easily verifies that $\hat{B}$ is a Banach algebra. In the Banach algebra $\hat{B}$ we imbed (isomorphically but not isometrically) the Banach algebra B by the correspondence:

$$a\to\hat{a}=\begin{pmatrix} a & 0 \\ 0 & a\end{pmatrix}\qquad (a\in B).$$

Now, simple calculation shows that

$$\begin{pmatrix} e+\frac{a}{4} & e-\frac{a}{4} \\ -\left(e-\frac{a}{4}\right) & -\left(e+\frac{a}{4}\right)\end{pmatrix}^2=\begin{pmatrix} a & 0 \\ 0 & a\end{pmatrix}$$

for every $a\in B$, i. e. in $\hat{B}$ every element $a\in B$ has a square root.

The function

$$\hat{f}(t)=\begin{pmatrix} f(t) & 0 \\ 0 & f(t)\end{pmatrix}$$

satisfies all conditions of theorem 1 and

$$\hat{f}(t)=\begin{pmatrix} f(t) & 0 \\ 0 & f(t)\end{pmatrix}=\text{ch}\, t\begin{pmatrix} e+\frac{a}{4} & e-\frac{a}{4} \\ -\left(e-\frac{a}{4}\right) & -\left(e+\frac{a}{4}\right)\end{pmatrix}$$

holds for every t, where the hyperbolic cosine is defined by the series. Thus we have:

T h e o r e m 2. *Let R, B and f be the same as in theorem 1 and let $\hat{B}$ be a Banach algebra of all 2×2 matrices the elements of which are elements of B and the norm of an $\hat{x}\in\hat{B}$ is defined by the formula:*

$$\|\hat{x}\| = \sum_{i,j=1}^{2} \|x_{ij}\|.$$

If we imbed B in $\hat{B}$ by the correspondence

$$a \to \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$$

then

$$\begin{pmatrix} f(t) & 0 \\ 0 & f(t) \end{pmatrix} = [\exp t\hat{a} + \exp(-t\hat{a})]/2$$

for every $t\in R$, where $\hat{a}$ can be taken as

$$\hat{a} = \begin{pmatrix} e+\frac{a}{4} & e-\frac{a}{4} \\ -\left(e-\frac{a}{4}\right) & -\left(e+\frac{a}{4}\right) \end{pmatrix}, \quad a\in B,$$

and it does not depend on t.

If in B there is a regular element b such that $b^2=a$, then the problem of solving (6) can be reduced to the problem of solving the functional equation $k(t+s) = k(t)\cdot k(s)$, $\lim k(t) = k(0) = e$ $(t\to 0)$ in B. In order to prove this we substitute $s+u$ for s in (15) and from the so obtained result we subtract (15). We get:

$$f(t+s+u)-f(t+s) = [f(s+u)-f(s)]f(t)+g(t)\int_s^{s+u} f(v)\,dv.$$

If we divide this by u and allow $u\to 0$ we get:

$$g(t+s) = g(s)\,f(t)+g(t)\,f(s). \tag{22}$$

Now write $a=b^2$ and define $h(t) = b^{-1}g(t)$. Then (15), (18) and (22) become:

$$f(t+s) = f(t)f(s)+h(t)h(s), \quad h(s) = b\int_0^s f(u)\,du, \quad h(t+s) = h(s)f(t)+h(t)f(s),$$

respectively. If we set

$$k(t) = f(t)+h(t) \tag{23}$$

then these equations imply:

$$k(t+s) = k(t)\,k(s), \quad k(0) = e.$$

But

$$dk/dt = df/dt+dh/dt = g(t)+b\,f(t) = b\,h(t)+b\,f(t) = b\,k(t).$$

Now $dk/dt = b\,k(t)$ and $k(0) = e$ lead to

$$k(t) = e + b\int_0^t k(s)\,ds$$

which by iteration gives

$$k(t) = \sum_0^{\infty} (tb)^n/n! = \exp tb$$

(cf. [1], p. 68). Thus

$$f(t) = [k(t)+k(-t)]/2 = [\exp tb + \exp(-tb)]/2.$$

Suppose that f satisfies all conditions of theorem 1 and that $M = \sup_{t\in R} \|f(t)\| < +\infty$. This and (14) for $\alpha=\gamma$ imply:

$$\|g(t)\| \leqq \|f(t+s)-f(t-s)\|\cdot\|c\|/2 \leqq M\|c\|;$$

i. e. $\sup_{t\in R}\|g(t)\| < +\infty$. In the case that a regular element b exists in B such that $b^2=a$ we derive:

$$\sup_{t\in R}\|k(t)\| = \sup_{t\in R}\|f(t)+h(t)\| < +\infty.$$

Thus in this case the one-parameter group $k(t)$ is uniformly bounded on R. If B is the Banach algebra of all bounded and linear operators on a Hilbert space endowed with the usual structure of a Banach space, then the well known result of BÉLA SZ.-NAGY [5] implies that the group $k(t)$ is similar to a one-parameter group of unitary operators; i. e., there is a nonsingular and bounded selfadjoint operator q such that

$$q^{-1}k(t)q$$

is a unitary operator for every $t\in R$. Since in our case $k(t) = \exp tb$ we find that $q^{-1}k(t)q = \exp tq^{-1}bq$ is a unitary group of operators. But this is possible if and only if $d = iq^{-1}bq$ is a selfadjoint operator. Thus

$$f(t) = q[\exp itd + \exp(-itd)]q^{-1}/2 = \cos td_0$$

where $d_0 = qdq^{-1}$. In such a way we have:

Theorem 3. *Let B be the Banach algebra of all bounded linear operators on some Hilbert space endowed with the usual structure of a Banach space, f the function which satisfies all conditions of theorem* 1 *where measurability is meant in the uniform operator topology.*

Then there exists a bounded operator $a\in B$ such that

$$f(t) = \sum_0^{\infty} a^n t^{2n}/(2n)! \tag{24}$$

for all $t\in R$.

If in addition: $\sup_{t\in R}\|f(t)\| < +\infty$ and the "infinitesimal operator" a which appears in (24) *possesses a regular square root in B, then $f(t)=\cos td_0$ where d_0 is similar to a selfadjoint operator, i. e. there is a regular element $q\in B$ such that $d=qd_0q^{-1}$ is selfadjoint and thus $f(t) = q^{-1}(\cos td)q$.*

§ 3. Problem B′ for the cosine functional equation

In this paragraph we prove the following theorem:

Theorem 4. *Let $R=\{\alpha, \beta, \ldots, t, s, \ldots\}$ be the set of all real numbers, $X=\{x, y, \ldots\}$ a Banach space, B a Banach algebra with unit element e, $F: X \to B$ the function which satisfies the functional equation*

$$F(x+y)+F(x-y) = 2F(x)F(y), \quad F(0) = e, \tag{25}$$

for all $x, y \in X$. Suppose the function F is measurable on every ray i. e. $F(tx)$ is measurable as a function of $t \in R$ for every $x \in X$. Then

(I) *there exists a function $A(x): X \to B$ such that*

$$F(x) = \sum_0^\infty [A(x)]^n/(2n)!, \tag{26}$$

$$A(tx)=t^2A(x) \qquad (x \in X, t \in R), \tag{27}$$

$$A(x)A(y)=A(y)A(x), \tag{28}$$

$$A(x+y)+A(x-y) = 2A(x)+2A(y), \tag{29}$$

$$A^2(x+y)+A^2(x-y) = 2A^2(x)+2A^2(y)+12A(x)A(y) \tag{30}$$

for all $x, y \in X$.

(II) *The function $A(x)$ is continuous if and only if*

$$\lim_{t \to 0} F(tx)=e \tag{31}$$

uniformly for x in some sphere.

(III) *If $\|e-A(x_0)\|<1$ for at least one $x_0 \in X$, then an additive function $L: X \to B$ exists such that $A(x) = [L(x)]^2$ for every $x \in X$ and therefore in this case*

$$F(x) = [\exp L(x)+\exp L(-x)]/2 = \text{ch } L(x).$$

If $A(x)$ is continuous so is $L(x)$.

Proof:

(I) For a given $x \in X$, the function $F_x(t) = F(tx)$ as a function of $t \in R$ satisfies all conditions of theorem 1. Hence an element $A(x) \in B$ exists such that

$$F(tx) = \sum_0^\infty [A(x)]^n t^{2n}/(2n)! \tag{32}$$

holds for all $t \in R$. Obviously (32) implies (27). Further $F(tx)\,F(ty) = F(ty)\,F(tx)$ for all $t \in R$ and $x, y \in X$ together with (32) leadr to (28). Replacing in (25) x by tx and y by ty we get

$$F_{x+y}(t)+F_{x-y}(t) = 2F_x(t)\,F_y(t)$$

which together with (32) implies (29) and (30).

(II) Next we observe that a symmetric function

$$M(x, y) = [A(x+y-A(x-y)]/4$$

is, because of (29), an additive function of x[1]). This and (27) imply that $M(x, y)$ is an additive and real-homogeneous function of each of its arguments. From (29) and the definition of M we find:

(33) $$A(x+y) = A(x)+A(y)+2M(x, y).$$

Now, suppose that $A(x)$ is continuous in the sphere S: $\|x-x_0\| < \varrho$, where we can without loss of generality take $x_0 \neq 0$. This assumption and (33) imply that $M(x_0, y)$ as a function of y is continuous in the sphere S_0: $\|y\| < \varrho$. Hence $A(y) = A(x_0+y) - A(x_0) - 2M(x_0, y)$ is continuous in the sphere S_0. The continuity of $A(y)$ in S_0 and (27) imply the continuity of $A(y)$ on every finite sphere. Thus $M(x, y)$ is continuous on every finite sphere. Since it is an additive function it is bounded and therefore the function $\|A(x)\| = \|M(x, x)\|$ is bounded on every finite sphere. Suppose that $\|A(x)\| < \alpha^2$, $\alpha > 0$, for all x from some finite sphere. This and (32) imply:

$$\|F(tx)-e\| \leqq \sum_1^\infty \|A(x)\|^n t^{2n}/(2n)! \leqq \frac{1}{2}(e^{\alpha t}+e^{-\alpha t})-1.$$

But this tends to zero as $t \to 0$, i. e. if $A(x)$ is continuous in some finite sphere then (31) holds uniformly for every x from any finite sphere.

Conversely, suppose that (31) holds uniformly in $x \in S$: $\|x-x_0\| < \varrho$. We then assert that $A(x)$ is a continuous function. First of all (31) implies the existence of a number $\gamma > 0$ such that

(34) $$\|F(tx)-e\| < 1/2$$

for all $|t| < \gamma$ and $x \in S$. This implies that $F(x)$ is bounded on the sphere S': $\|x-x_0\| < \gamma\varrho/2$. This and (25) lead to the boundedness of $F(x)$ on every finite sphere. Thus $F(tx)$ as a function of t is integrable on every finite interval for any $x \in S$. Now (34) leads to:

(35) $$\left\| e - \frac{1}{\gamma}\int_0^\gamma F(tx)\,dt \right\| \leqq 1/2$$

for every $x \in S$. From (35) we conclude that

(36) $$C_x = \left[\int_0^\gamma F(tx)\,dt\right]^{-1}$$

exists for every $x \in S$. Moreover we have:

$$C_x = \frac{1}{\gamma}\sum_0^\infty \left(e - \frac{1}{\gamma}\int_0^\gamma F(tx)\,dt\right)^n$$

from which we find:

(37) $$\|C_x\| \leqq 2/\gamma$$

[1]) Our attention on this fact was drawn by professor IVAN VIDAV at another occasion.

for every $x \in S$. In the same way as in the proof of theorem 1 we have the existence of

$$G_x(t) = \lim_{u \to 0} [F_x(t+u) - F_x(t)]/u$$

for every $x \in S$ and the function $G_x(t)$ has the property that:

$$2G_x(t) = [F_x(\gamma+t) - F_x(\gamma-t)]\, C_x. \tag{38}$$

Now (38), (37), and the fact that $F(x)$ is bounded on every finite sphere imply:

$$\sup_{x \in S} \|G_x(\gamma)\| < +\infty. \tag{39}$$

Furthermore we have

$$\lim_{t \to 0} G_x(t)/t = C_x G_x(\gamma) = A(x). \tag{40}$$

Now, (40), (37) and (39) imply $\sup_{x \in S} \|A(x)\| < +\infty$, i. e. the function $A(x)$ is bounded on one sphere. This and (27) imply that $A(x)$ is bounded on every finite sphere. Since the additive function $M(x, y)$ is bounded on every sphere it is continuous everywhere and therefore $A(x) = M(x, x)$ is also an everywhere continuous function.

(III) Suppose that $\|e - A(x_0)\| < 1$ for some $x_0 \in X$. Then

$$\sum_0^\infty \frac{(2n-1)!!}{(2n)!!} [e - A(x_0)]^n$$

converges to $[A(x_0)]^{-1/2}$, i. e. $[A(x_0)]^{-1/2}$ exists and it commutes with $A(x)$ and therefore with $M(x, y)$ for every pair $x, y \in X$.

Now we take the square of (29) and from this we substract (30). We get:

$$A(x+y)A(x-y) = [A(x) - A(y)]^2 \tag{41}$$

which together with (33) leads to:

$$[M(x, y)]^2 = A(x)A(y). \tag{42}$$

From (42) and the property of $[A(x_0)]^{-1/2}$ to commute with $M(x, y)$ we find

$$A(x) = [A(x_0)]^{-1}[M(x, x_0)]^2 = [L(x)]^2$$

where $L(x) = [A(x_0)]^{-1/2}\, M(x, x_0)$ is an additive function from X to B. If $A(x)$ is continuous then $M(x, y)$ is continuous and therefore $L(x)$ is also continuous. Thus theorem 4 is proved.

R e m a r k 1. Let X be a real Hilbert space and B the algebra of all 4×4 matrices over real numbers. The norm of a matrix $b = (b_{pq})$ will be defined as $\|b\| = \sum_{p,q=1}^{4} |b_{pq}|$. For an arbitrary bounded selfadjoint operator $A\colon X \to X$ set:

$$A(x) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 2(Ax, x) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2(Ax, x) & 0 \end{pmatrix}.$$

The functional $F(x)=e+A(x)/2$ satisfies the functional equation (2) and $A(x)$ is a continuous function of $x\in X$. We assert that there is no additive and continuous function L: $X\to B$ such that $A(x) = [L(x)]^2$. Indeed, if such a matrix $L(x) = = (l_{pq}(x))\,(p, q=1, 2, 3, 4)$ would exist, then each matrix element $l_{pq}(x)$ as a continuous and additive functional would have the form $l_{pq}(x)=(x, x_{pq})$, where x_{pq} are uniquely determined vectors. Thus the quadratic form $A(x)=2(Ax, x)$ would be determined by its values on 16 vectors. Since this is impossible the assertion is proved. However in this example the condition $\|e-A(x)\|<1$ is not satisfied for any x. On the other hand the existence of such an x is not necessary. Indeed, if in the above example we take $X=R$, $A(x) = A(t) = 2t^2$, then $\|e-A(x)\| = 4(1+t^2)$ and $A(t) = L^2(t)$ with $L(t) = 2^{1/2}\,t$.

Remark 2. Using the results obtained in this paper and in [4] one can generalise some results of the paper [4], e. g. one can solve the functional equation $f(t+s)+f(t-s) = 2f(t)\,g(s)$ where f, g: $R\to B$. If f is measurable, $g(0) = e$ and for some $s\neq 0$ $\|e-g(s)\|<1$ then $g(t) = \cos at$ and $f(t) = b\cos at+c\sin at$, where a, b and c are fixed elements of B.

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UNIVERSITY OF ZAGREB
AND UNIVERSITY OF MARYLAND

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