A theorem on derivates *)

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Recently several papers have appeared dealing with a formulation of a W. H. YOUNG theorem [6] for approximate limits [3, 4, 5]. More precisely, it was proven in [4] that for an arbitrary real-valued function f the set of points at which the collection of upper right approximate limits of f differs from the collection of upper left approximate limits, is of the first category and of measure zero. A simple proof of this theorem was recently given in [3]. The purpose of the present paper is to show that this theorem is a special case of a theorem on derivates.

1. Let R be the set of real numbers and let $f\colon R\to R$ be a function. Denote by $f^+(x_0)$, $f_+(x_0)$ the upper right, lower right derivates of f at x_0, and denote by $f^-(x_0)$, $f_-(x_0)$ the corresponding left extreme derivates of f at x_0.

Theorem 1. *If $f\colon R\to R$ is continuous, then $E=\{x: f^-(x)\neq f^+(x)$ or $f_-(x)\neq\neq f_+(x)\}$ is a set of the first category.*

Proof. We will show that $A=\{x: f^-(x)<f^+(x)\}$ is of the first category. For r rational let $A_r=\{x: f^-(x)<r<f^+(x)\}$, and let

$$A_{rj}=\left\{x_0\colon x_0\in A_r \quad\text{and}\quad \frac{f(x)-f(x_0)}{x-x_0}<r,\ x_0-\frac{1}{j}<x<x_0\right\}.$$

We observe that $A=\bigcup_r\bigcup_{j\geqq 1}A_{rj}$, and thus it suffices to show that A_{rj} is nowhere dense. If we deny this, we have an interval (α,β) in which A_{rj} is dense. We may assume that $\beta-\alpha<\frac{1}{j}$.

Let $\alpha<x'<x''<\beta$, and let $\{x_n\}$ be a sequence in $A_{rj}\cap(\alpha,\beta)$ such that $\{x_n\}\to x''$ and $x'<x_n$ for each n. Since $x_n-\frac{1}{j}<x'<x_n$ and $x_n\in A_{rj}$, we have that $\frac{f(x')-f(x_n)}{x'-x_n}<r$, and in view of the continuity of f,

$$\text{(1)}\qquad \frac{f(x')-f(x'')}{x'-x''}\leqq r,\ \alpha<x',\ x''<\beta.$$

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For $x' \in A_{rj} \cap (\alpha, \beta)$ we have $f^+(x') > r$, and hence there is $x'' \in (x', \beta)$ such that $\frac{f(x') - f(x'')}{x' - x''} > r$, in contradiction with (1). Hence A_{rj} is nowhere dense, and the proof is complete.

Remark. The hypothesis of "continuity" in Theorem 1 cannot be omitted as the function

$$f(x) = \begin{cases} 1, & x \text{ rational} \\ 0, & x \text{ irrational} \end{cases}$$

shows.

Corollary. *If f: $R \to R$ is continuous and of bounded variation on every compact interval, then the set E of Theorem 1 is both of the first category and of measure zero.*

Remark. The example III in [2] shows that the hypothesis "bounded variation" cannot be omitted.

2. We will show that the theorem of M. KULBACKA [4] follows as a special case from the corollary to Theorem 1:

For a subset E of R and $x_0 \in R$, denote by $D^+(E; x_0)$, $D_+(E; x_0)$ the upper, lower right outer densities of E at x_0, and denote by $D^-(E; x_0)$, $D_-(E; x_0)$ the corresponding left extreme densities of E at x_0. Let H be a measurable cover of E. Then $D^+(H; x_0) = D^+(E; x_0)$, etc.

Lemma. *The set of points*

$$K = \{x: D^-(E; x) \neq D^+(E; x) \quad \text{or} \quad D_-(E; x) \neq D_+(E; x)\}$$

is both of the first category and of measure zero.

Proof. Let H be a measurable cover of E, and let $f(x) = \int_0^x \chi_H(t)\, dt$, where χ_H is the characteristic function of H. Then

$$K = \{x: f^-(x) \neq f^+(x) \quad \text{or} \quad f_-(x) \neq f_+(x)\},$$

and application of Theorem 1 completes the proof.

Let now f: $R \to R$. A real number y is an approximate right limit of f at x if and only if for every $\varepsilon > 0$, $D^+[f^{-1}((y-\varepsilon, y+\varepsilon)); x] > 0$; approximate left limit is defined similarly.

Theorem 2 (KULBACKA). *Let f: $R \to R$ and let $W^+(x)$, $W^-(x)$ be the set of approximate right, left limits of f at x. Then $E = \{x: W^+(x) \neq W^-(x)\}$ is both of the first category and of measure zero.*

Proof. Let $A = \{x: W^+(x) - W^-(x) \neq \emptyset\}$. For $r_1 < r_2$ rational numbers let

$$A_{r_1 r_2} = \{x: D^+[f^{-1}((r_1, r_2)); x] \neq D^-[f^{-1}((r_1, r_2)); x]\}.$$

Then $A \subseteqq \bigcup A_{r_1 r_2}$, and application of the lemma completes the proof.

The above proof admits of a sligthly more general theorem. For $f: R \to R$, let us call a real number y an *asymmetric approximate limit* of f at x if and only if there exists $\varepsilon > 0$ such that $y-\varepsilon < y' < y < y'' < y+\varepsilon$ implies

$$D^-[f^{-1}((y', y'')); x] \neq D^+[f^{-1}((y', y'')); x].$$

Theorem 3. *The set of points at which an arbitrary real-valued function possesses an asymmetric approximate limit is both of the first category and of measure zero.*

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