On the approximate limits of a real function*)

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The theorem of W. H. YOUNG, [1], on the symmetric structure of an arbitrary real function f asserts that the set of right limits of f is the same as the set of left limits of f at every point x, except for points belonging to a countable set.

It has recently been shown by L. BELOWSKA, [2], that the theorem of YOUNG no longer holds if ordinary limits are replaced by approximate limits. Belowska constructs a function whose right approximate limit superior is less than its left approximate limit superior on an uncountable set. On the other hand, M. KULBACKA, [3], has shown that the set of points for which the set of right approximate limits of f differs from the set of left approximate limits of f is both of the first category and of measure zero, for an arbitrary real function f.

The purpose of this note is to give short and simple proofs of these results.

Let f be an arbitrary real function on the real line. For any x, a number y is said to be a right approximate limit of f at x if for every $\varepsilon > 0$ the set $(x, \infty) \cap \cap f^{-1}((y-\varepsilon, y+\varepsilon))$ has positive upper exterior density at x; left approximate limit is defined similarly. Let $W^+(x)$ and $W^-(x)$ be the sets of right and left approximate limits at x, respectively. Let A be the set of points x for which $W^+(x)$ is not a subset of $W^-(x)$, and B the set of points x for which $W^-(x)$ is not a subset of $W^+(x)$. Then $A \cup B$ is the set for which $W^+(x) \neq W^-(x)$. It suffices to show that A is of the first category and of measure zero. It is evident that $A \subset \bigcup_{r_1 < r_2} A_{r_1 r_2}$ where $r_1 < r_2$ are rational numbers and $A_{r_1 r_2}$ is the set of points x such that $(x, \infty) \cap \cap f^{-1}((r_1, r_2))$ has positive upper exterior density at x and $(-\infty, x) \cap f^{-1}((r_1, r_2))$ has zero exterior density at x. Thus, in order to show that A is of the first category and of measure zero, it suffices to show that for every set S, the set E of points x such that $(x, \infty) \cap S$ has positive upper exterior density at x and $(-\infty, x) \cap S$ has zero exterior density at x is of the first category and of measure zero.

For every pair k, r of natural numbers, let

$$E_{kr} = \left(x\colon\ D_x^+(S) > \frac{1}{k} \quad \text{and} \quad \frac{m(S \cap (y, x))}{x-y} < \frac{1}{k} \quad \text{if} \quad 0 < x-y < \frac{1}{r}\right),$$

where $D_x^+(S)$ is the upper right exterior density of S at x. Then $E \subset \bigcup E_{kr}$. Suppose an E_{kr} is dense in an interval (a, b), where $b-a < \frac{1}{r}$. Then, for every $a \leqq y < x \leqq b$,

$$\frac{m(S \cap (y, x))}{x-y} \leqq \frac{1}{k},$$

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since $\frac{m(S \cap (y, x_n))}{x_n - y} < \frac{1}{k}$ for a sequence $\{x_n\}$ converging to x, and so $D_y^+(S) \leq \frac{1}{k}$ for every $y \in (a, b)$. It follows that $E_{kr} \cap (a, b)$ is empty, contradicting the assumption that it is dense in (a, b). Thus E_{kr} is of the first category, so that E itself is of the first category.

That E is of measure zero is merely a form of the Lebesgue density theorem. We thus have the

Theorem (KULBACKA). *For every real function f, the set for which $W^+(x) \neq W^-(x)$, is of the first category and of measure zero.*

We now prove the

Theorem (BELOWSKA). *There is a real function f such that the set of points for which $W^+(x) \neq W^-(x)$ is uncountable; indeed, the set for which the right approximate limit superior is less than the left approximate limit superior is uncountable.*

Proof. The intervals complementary to the Cantor ternary set are of the form

$$(\underbrace{.xx\ldots x}_{n}1,\ \underbrace{.xx\ldots x}_{n}2) \qquad (n = 1, 2, \ldots),$$

where $x = 0$ or 2. In each of these intervals, consider the subinterval

$$(\underbrace{.xx\ldots x}_{n}1,\ \underbrace{.xx\ldots x}_{n}1\ \underbrace{0\ldots 0}_{n}1).$$

Let S be the union of these subintervals. At every point of the Cantor set, the left metric density of S exists and is zero. However, the right metric density of S exists and is zero at some points of the Cantor set. We, accordingly, consider the subset E of the Cantor set whose points have ternary expansions of the form

$$.x0\,22\,x\,0\,222222\,x\,0\ldots$$

where $x = 0$ or 2 and after each pair $x0$ there are the same number of 2's as there are digits up to and including the pair $x0$. The set E has the power of the continuum.

Let $\xi \in E$ and let n be such that the n^{th} term in the expansion of E is the 0 of a pair $x0$. Then

$$\xi = \underbrace{.x\,0\,22\,x\,0\ldots x\,0}_{n}\ \underbrace{2\ldots 2}_{n}\,x\,0\ldots.$$

Let

$$I_n = (a_n, b_n) = (\underbrace{.x\,0\,22\,x\,0\ldots x}_{n}1,\ \underbrace{.x\,0\,22\,x\,0\ldots x}_{n}1\ \underbrace{0\ldots 0}_{n}1)$$

where the first $n-1$ digits in the expansions of ξ, a_n and b_n are the same. Then $I_n \subset S$. Now, since the expansion of a_n may be written $a_n = \underbrace{.x\,0\,22\,x\,0\ldots x0}_{n}\ 22\ldots$, we have $0 < a_n - \xi < .\underbrace{0\ldots01}_{2n}$. But $b_n - a_n = .\underbrace{0\ldots01}_{2n}$ so that $b_n - a_n > a_n - \xi$. Thus

$$\frac{b_n - a_n}{b_n - \xi} = \frac{b_n - a_n}{(b_n - a_n) + (a_n - \xi)} > \frac{b_n - a_n}{2(b_n - a_n)} = \frac{1}{2},$$

and so the upper right density of S at ξ is positive, since $\lim (b_n - \xi) = 0$.

To prove the theorem, we needed only consider the characteristic function of S.

References

[1] W. H. Young, La symétrie de structure des fonctions de variables réelles, *Bull. Sci. Math.*, (2) **52** (1928), 265–280.

[2] L. Belowska, Résolution d'un problème de M. Z. Zahorski sur les limites approximatives, *Fundamenta Math.*, **48** (1960), 277–286.

[3] M. Kulbacka, Sur l'ensemble des points de l'asymétrie approximative, *Acta Sci. Math.*, **21** (1960), 90–95.

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