Some remarks on set theory. VII

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Professor L. Rédei on his 60th birthday

§ 1. Introduction

Let $\mathfrak{F}$ be a family of non empty sets and let $\mathfrak{r}$ be a cardinal number. The family $\mathfrak{F}$ is said to possess property $A(\mathfrak{r})$ if there exists a set X with $\overline{\overline{X}} < \mathfrak{r}$ which contains at least one element of each set of the family $\mathfrak{F}$.

Let $p(\mathfrak{F})$ denote the smallest cardinal number $\mathfrak{p}$, for which $\overline{\overline{F}} \leqq \mathfrak{p}$ for every $F \in \mathfrak{F}$.

Let now $\mathfrak{q}$ be a cardinal number. The family $\mathfrak{F}$ is said to possess property $A(\mathfrak{q}, \mathfrak{r})$ if each subfamily $\mathfrak{F}'$ of $\mathfrak{F}$ possesses property $A(\mathfrak{r})$ provided $\overline{\overline{\mathfrak{F}'}} < \mathfrak{q}$. We investigate the following problem: Suppose that the family $\mathfrak{F}$ possesses property $A(\mathfrak{q}, \mathfrak{r})$. Under what conditions on $\mathfrak{p}, \mathfrak{q}, \mathfrak{r}$ does the family $\mathfrak{F}$ possess property $A(\mathfrak{r})$?

More generally we introduce the symbol $[\mathfrak{p}, \mathfrak{q}, \mathfrak{r}] \rightarrow \mathfrak{s}$ to indicate the statement that every family $\mathfrak{F}$ (with $p(\mathfrak{F}) = \mathfrak{p}$) which possesses property $A(\mathfrak{q}, \mathfrak{r})$ possesses property $A(\mathfrak{s})$ too. ($[\mathfrak{p}, \mathfrak{q}, \mathfrak{r}] \not\rightarrow \mathfrak{s}$ indicates the negation of this statement.)

In Section 2 we are going to prove some results concerning this symbol which using the generalized continuum hypothesis enable us to give a complete discussion for the case $\mathfrak{p} \geqq \aleph_0$.

The problem for finite sets is posed and discussed in a paper of P. ERDŐS and T. GALLAI, and it is not yet entirely solved.[1]) That is why in what follows $\mathfrak{p}$ is supposed to be infinite.

Theorems in the proof of which the generalized continuum hypothesis will be used are marked with a star (*).

In Section 3 we investigate the question what results we can get by using weaker hypotheses or no hypothesis at all. The results in this Section are not quite complete. In Section 4 we investigate an analogous question to that treated in Section 2.

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1) See a forthcoming paper of P. ERDŐS and T. GALLAI.

§ 2.

First we make some obvious remarks.

(1) $$[\mathfrak{p}, \mathfrak{q}, \mathfrak{r}] \not\rightarrow \mathfrak{s} \quad \text{if} \quad \mathfrak{s} < \mathfrak{r}, \quad \text{for every} \quad \mathfrak{p}, \mathfrak{q}, \mathfrak{r};$$

(2) $$[\mathfrak{p}, \mathfrak{q}, \mathfrak{r}] \not\rightarrow \mathfrak{s} \quad \text{if} \quad \mathfrak{q} \leqq \mathfrak{r}, \quad \text{for every } \mathfrak{p} \text{ and } \mathfrak{s}.$$

The symbol has the following monotonity properties:

(3) $$\begin{aligned} &[\mathfrak{p}, \mathfrak{q}, \mathfrak{r}] \rightarrow \mathfrak{s} \quad \text{implies} \quad [\mathfrak{p}', \mathfrak{q}, \mathfrak{r}] \rightarrow \mathfrak{s} \quad \text{if} \quad \mathfrak{p} \geqq \mathfrak{p}', \\ &[\mathfrak{p}, \mathfrak{q}, \mathfrak{r}] \rightarrow \mathfrak{s} \quad \text{implies} \quad [\mathfrak{p}, \mathfrak{q}', \mathfrak{r}] \rightarrow \mathfrak{s} \quad \text{if} \quad \mathfrak{q} \leqq \mathfrak{q}', \\ &[\mathfrak{p}, \mathfrak{q}, \mathfrak{r}] \rightarrow \mathfrak{s} \quad \text{implies} \quad [\mathfrak{p}, \mathfrak{q}, \mathfrak{r}'] \rightarrow \mathfrak{s} \quad \text{if} \quad \mathfrak{r} \geqq \mathfrak{r}' \\ &[\mathfrak{p}, \mathfrak{q}, \mathfrak{r}] \rightarrow \mathfrak{s} \quad \text{implies} \quad [\mathfrak{p}, \mathfrak{q}, \mathfrak{r}] \rightarrow \mathfrak{s}' \quad \text{if} \quad \mathfrak{s} \leqq \mathfrak{s}'. \end{aligned}$$

We may omit the proofs and in what follows we shall often use these theorems without references.

First we are going to prove the negative results.

The following Lemma 1 gives a general method for the proof of negative theorems concerning the symbol.

For the sake of brevity we introduce the symbol $[\mathfrak{p}, \mathfrak{r}, \mathfrak{s}]^* \rightarrow \mathfrak{q}$ to indicate the following statement:

For every set S with $\overline{\overline{S}} = \mathfrak{p}$ there exists a set $\mathfrak{S}^*$ for which the following conditions hold:

a) every element X of $\mathfrak{S}^*$ is a subset of S of power less than $\mathfrak{s}$,

b) $\overline{\overline{\mathfrak{S}^*}} < \mathfrak{q}$, and

c) every subset Y of S with $\overline{\overline{Y}} < \mathfrak{r}$ is contained in an element of $\mathfrak{S}^*$.
($[\mathfrak{p}, \mathfrak{r}, \mathfrak{s}]^* \not\rightarrow \mathfrak{q}$ indicates the negation of this statement).

Lemma 1. *Suppose* $\mathfrak{p} \geqq \mathfrak{s} \geqq \mathfrak{r}$. *Then* $[\mathfrak{p}, \mathfrak{r}, \mathfrak{s}]^* \not\rightarrow \mathfrak{q}$ *implies* $[\mathfrak{p}, \mathfrak{q}, \mathfrak{r}] \not\rightarrow \mathfrak{s}$

Proof. Let S be a set of power $\mathfrak{p}$. Let $\mathfrak{F}$ be the family containing the complements of the elements of the set $[S]^{<\mathfrak{s}}$[2]. It is obvious that $p(\mathfrak{F}) = \mathfrak{p}$ and $\mathfrak{F}$ does not possess property $A(\mathfrak{s})$. We have to show that $\mathfrak{F}$ possesses property $A(\mathfrak{q}, \mathfrak{r})$. Let $\mathfrak{F}'$ be a subfamily of $\mathfrak{F}$, $\overline{\overline{\mathfrak{F}'}} < \mathfrak{q}$. Then by the assumption $[\mathfrak{p}, \mathfrak{r}, \mathfrak{s}]^* \not\rightarrow \mathfrak{q}$ there exists a subset X of S, $\overline{\overline{X}} < \mathfrak{r}$ which is not contained in the complement of any element of $\mathfrak{F}'$, hence $\mathfrak{F}'$ possesses property $A(\mathfrak{r})$.

Corollary 1. *Suppose* $\mathfrak{p} \geqq \mathfrak{q}$ *and* $\mathfrak{p}$ *is regular. Then* $[\mathfrak{p}, \mathfrak{q}, \mathfrak{r}] \not\rightarrow \mathfrak{s}$ *for* $\mathfrak{s} \leqq \mathfrak{p}$.

Proof. We may suppose $\mathfrak{q} > \mathfrak{r}$ and $\mathfrak{r} \leqq \mathfrak{s}$. But then obviously $[\mathfrak{p}, \mathfrak{r}, \mathfrak{s}]^* \not\rightarrow \mathfrak{q}$.

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[2] $[X]^{<\mathfrak{q}}$ denotes the set of all subsets of X of power less then $\mathfrak{q}$.

Corollary 2. *Suppose* $\mathfrak{p} \geqq \mathfrak{q}$ *and* $\mathfrak{p}$ *is singular. Then* $[\mathfrak{p}, \mathfrak{q}, \mathfrak{r}] \nrightarrow \mathfrak{s}$ *for* $\mathfrak{s} < \mathfrak{p}$.

Proof. Similar to that of Corollary 1.

Corollary 3. *Suppose* $\mathfrak{p}$ *is singular. Put* $\mathfrak{p} = \aleph_\alpha$ *where* α *is of the second kind* $\mathrm{cf}(\alpha) < \alpha$. *Then* $[\mathfrak{p}, \mathfrak{q}, \mathfrak{r}] \nrightarrow \mathfrak{s}$ *for* $\mathfrak{s} \leqq \mathfrak{p}$ *provided* $\mathfrak{q} \leqq \aleph_{\mathrm{cf}(\alpha)}$.

Proof. We may suppose $\mathfrak{q} > \mathfrak{r}$, $\mathfrak{r} \leqq \mathfrak{s}$. We have $[\mathfrak{p}, \mathfrak{r}, \mathfrak{s}]^* \nrightarrow \mathfrak{q}$ in this case too, since the sum of less than $\aleph_{\mathrm{cf}(\alpha)}$ sets each of which has power less than $\aleph_\alpha$, has power less than $\aleph_\alpha$.

Using the same idea we can prove the following negative

Theorem 1. *Suppose* $\mathfrak{p} = \aleph_\alpha$ *is singular,* $\mathfrak{r} > \aleph_{\mathrm{cf}(\alpha)}$ *and* $\mathfrak{q} \leqq \mathfrak{p}^+$. *Then* $[\mathfrak{p}, \mathfrak{q}, \mathfrak{r}] \nrightarrow \mathfrak{s}$ *for* $\mathfrak{s} \leqq \mathfrak{p}$.

Proof. It is enough to prove $[\mathfrak{p}, \mathfrak{p}^+, \aleph_{\mathrm{cf}(\alpha)+1}] \nrightarrow \mathfrak{p}$. By Lemma 1 it is enough to prove $[\aleph_\alpha, \aleph_{\mathrm{cf}(\alpha)+1}, \aleph_\alpha]^* \nrightarrow \aleph_{\alpha+1}$. Let S be a set, $\overline{\overline{S}} = \aleph_\alpha$ and $\mathfrak{S}^*$ a set of subsets of S for which $\overline{\overline{\mathfrak{S}^*}} < \aleph_{\alpha+1}$, and the elements of which are subsets of S of power less than $\aleph_\alpha$. We have to construct a set $X_0 \subseteq S$ such that $\overline{\overline{X_0}} \leqq \aleph_{\mathrm{cf}(\alpha)}$ and $X_0 \not\subseteq X$ for any element X of $\mathfrak{S}^*$. Let $\{\alpha_\nu\}_{\nu < \omega_{\mathrm{cf}(\alpha)}}$ be a monotone increasing sequence of type $\omega_{\mathrm{cf}(\alpha)}$ of ordinal numbers less than α cofinal with α.

Put $\mathfrak{S}^*_\nu = \{X : X \in \mathfrak{S}^* \text{ and } \overline{\overline{X}} \leqq \aleph_{\alpha_\nu}\}$. We have

$$\mathfrak{S}^* = \bigcup_{\nu < \omega_{\mathrm{cf}(\alpha)}} \mathfrak{S}^*_\nu .$$

Since $\overline{\overline{\mathfrak{S}^*_\nu}} \leqq \aleph_\alpha$ for every ν, we may split S^*_ν into the sum of subsets $S^*_{\nu,\mu}$ for $\mu < \omega_{\mathrm{cf}(\alpha)}$ in such a way that

$$\mathfrak{S}^*_\nu = \bigcup_{\mu < \omega_{\mathrm{cf}(\alpha)}} \mathfrak{S}^*_{\nu,\mu} \quad \text{and} \quad \overline{\overline{\mathfrak{S}^*_{\nu,\mu}}} \leqq \aleph_{\alpha_\mu}$$

for every $\mu < \omega_{\mathrm{cf}(\alpha)}$.

Put $\mathfrak{S}^{**}_\lambda = \bigcup_{\nu \leqq \lambda} \bigcup_{\mu \leqq \lambda} \mathfrak{S}^*_{\nu,\mu}$ for every $\lambda < \omega_{\mathrm{cf}(\alpha)}$. It is obvious that

$$\mathfrak{S}^* = \bigcup_{\lambda < \omega_{\mathrm{cf}(\alpha)}} \mathfrak{S}^{**}_\lambda .$$

Let now $(\mathfrak{F})$ denote the set $\bigcup_{X \in \mathfrak{F}} X$ for an arbitrary family $\mathfrak{F}$ of sets.

We have by the construction that

$$\overline{\overline{(\mathfrak{S}^{**}_\lambda)}} \leqq \aleph_{\alpha_\lambda} \cdot \aleph_{\alpha_\lambda} \cdot \aleph_{\mathrm{cf}(\alpha)} < \aleph_\alpha$$

for every $\lambda < \omega_{\mathrm{cf}(\alpha)}$. Therefore we can define by transfinite induction a sequence $\{x_\lambda\}_{\lambda < \omega_{\mathrm{cf}(\alpha)}}$ of type $\omega_{\mathrm{cf}(\alpha)}$ of the elements of S in such a way that $x_\lambda \notin \bigcup_{\lambda' < \lambda} (\mathfrak{S}^{**}_{\lambda'})$. Put $X_0 = \{x_\lambda\}_{\lambda < \omega_{\mathrm{cf}(\alpha)}}$. It is obvious that X_0 satisfies our requirements.

To obtain positive results we need the following lemmas. Let $\mathfrak{F}$ be an arbitrary family with $(\mathfrak{F})=S$.

Let $\mathfrak{F}|S'$ denote the family $\{F\cap S'\}_{F\in\mathfrak{F}}$ for an arbitrary subset S' of S.

Lemma 2. *Let $\mathfrak{F}$ be a family $((\mathfrak{F})=S, p(\mathfrak{F})=\mathfrak{p})$ which possesses property $A(\mathfrak{q},\mathfrak{r})$ for certain $\mathfrak{q}$ and $\mathfrak{r}$, where $\mathfrak{q}>\mathfrak{r}$.*

α) Suppose $\mathfrak{q}\leqq\mathfrak{p}^+$, $\mathfrak{q}$ is regular. Then there exists a subset S' of S such that $\overline{\overline{S'}}\leqq p$ and $\mathfrak{F}|S'$ possesses property $A(\mathfrak{q},\mathfrak{r})$ too.

β) Suppose $\mathfrak{q}>\mathfrak{p}^+$. Then for every $\mathfrak{t}$ with $\mathfrak{r}\cdot\mathfrak{p}\leqq\mathfrak{t}<\mathfrak{q}$ there exists a subset S' of S such that $\overline{\overline{S'}}\leqq\mathfrak{t}$ and $\mathfrak{F}|S'$ possesses property $A(\mathfrak{t}^+,\mathfrak{r})$.

Proof. We are going to prove α). The proof of β) is quite similar and will be omitted. We have formulated β) only to make clear Problems 1 and 2 which will be formulated in Section 3.

If $\mathfrak{r}$ is singular then the family $\mathfrak{F}$ possesses property $A(\mathfrak{q},\mathfrak{r}')$ for an $\mathfrak{r}'<\mathfrak{r}$, since if corresponding to every $\mathfrak{r}'<\mathfrak{r}$, there exists a subfamily $\mathfrak{F}_{\mathfrak{r}'}$ of $\mathfrak{F}$ such that $\overline{\overline{\mathfrak{F}_{\mathfrak{r}'}}}=\mathfrak{r}'$ and $\overline{\overline{\mathfrak{F}_{\mathfrak{r}'}}}$ does not possess property $A(\mathfrak{r}')$, then the family $\mathfrak{F}'=\bigcup_{\mathfrak{r}'<\mathfrak{r}}\mathfrak{F}_{\mathfrak{r}'}$ has the power $\mathfrak{r}<\mathfrak{q}$ and does not possess property $A(\mathfrak{r})$.

Thus we may suppose $\mathfrak{r}$ to be regular.

Let φ denote the initial number of $\mathfrak{r}$. We are going to define a sequence S_α of subsets of S and a sequence $\mathfrak{F}_\alpha$ of subfamilies of $\mathfrak{F}$ for every $\alpha<\varphi$ by transfinite induction on α as follows.

Let S_0 be an arbitrary subset of S of power $\leqq\mathfrak{p}$.

Suppose now that $\alpha<\varphi$, and the sets S_β as well as the families F_β are already defined for $\beta<\alpha$. Put $S^*_\alpha=\bigcup_{\beta<\alpha}S_\beta$.

Now we distinguish two cases:

(i) $\mathfrak{F}|S^*_\alpha$ does not possess property $A(\mathfrak{q},\mathfrak{r})$,

(ii) $\mathfrak{F}|S^*_\alpha$ possesses property $A(\mathfrak{q},\mathfrak{r})$.

Let $\mathfrak{F}_\alpha$ be a subfamily of $\mathfrak{F}$ of power less than $\mathfrak{q}$ such that $\mathfrak{F}_\alpha|S^*_\alpha$ does not possess property $A(\mathfrak{r})$, if (i) holds and put $\mathfrak{F}_\alpha=0$ if (ii) holds. Put further $S_\alpha=(\mathfrak{F}_\alpha)$. Thus the sets $S_\alpha (0\leqq\alpha<\varphi)$ and the families F_α $(1\leqq\alpha<\varphi)$ are defined. Put

$$S_\varphi=\bigcup_{\alpha<\varphi}S^*_\alpha \quad\text{and}\quad \mathfrak{F}_\varphi=\bigcup_{\alpha<\varphi}\mathfrak{F}_\alpha.$$

Now we have $\overline{\overline{S_\alpha}}\leqq\mathfrak{p}$ for every $\alpha<\varphi$ since $p(\mathfrak{F})=\mathfrak{p}$ and $\overline{\overline{\mathfrak{F}_\alpha}}<\mathfrak{q}$ for every α, hence $\overline{\overline{\mathfrak{F}_\alpha}}\leqq p$ by the assumption $\mathfrak{q}\leqq\mathfrak{p}^+$.

Taking into consideration that $\mathfrak{p}^+\geqq\mathfrak{q}>\mathfrak{r}$ implies $\mathfrak{r}\leqq\mathfrak{p}$, it follows that $\overline{\overline{S^*_\alpha}}\leqq\mathfrak{p}\cdot\mathfrak{r}=\mathfrak{p}$.

Thus if for an $\alpha<\varphi$ (ii) holds then Lemma 2 is proved.

We have to show that the assumption: for every $\alpha < \varphi$ (i) holds leads to a contradiction. In fact we have $\overline{\overline{\mathfrak{F}}}_\varphi < \mathfrak{q}$, since $\overline{\varphi} = \mathfrak{r} < \mathfrak{q}$ and $\mathfrak{q}$ is supposed to be regular. Thus by our assumption it follows that $\mathfrak{F}_\varphi$ possesses property $A(\mathfrak{r})$.

It is obvious that $(\mathfrak{F}_\varphi) = S_\varphi$. Therefore there exists a set X_0, $X_0 \subseteq S_\varphi$ such that $\overline{\overline{X}}_0 < \mathfrak{r}$ and X_0 intersects every element of $\mathfrak{F}_\varphi$. But since $\overline{\varphi} = \mathfrak{r}$ is regular there exists an $\alpha_0 < \varphi$ such that $X_0 \subseteq S^*_{\alpha_0}$. Therefore $\mathfrak{F}_\varphi | S^*_{\alpha_0}$ possesses property $A(\mathfrak{r})$, and $\mathfrak{F}_{\alpha_0} \subseteq \mathfrak{F}_\varphi$ implies that $\mathfrak{F}_{\alpha_0} | S^*_{\alpha_0}$ possesses property $A(\mathfrak{r})$ in contradiction with the construction of $\mathfrak{F}_{\alpha_0}$.

Lemma 3. *Let $\mathfrak{F}$ be a family which possesses property $A(\mathfrak{q}, \mathfrak{r})$. Suppose $(\overline{\overline{\mathfrak{F}}}) = \mathfrak{t}$. The family $\mathfrak{F}$ possesses property $A(\mathfrak{s})$, provided* $[\mathfrak{t}, \mathfrak{r}, \mathfrak{s}]^* \to \mathfrak{q}$.

Proof. Let $\mathfrak{S}^*$ be a set of subsets of S satisfying conditions a), b), c) (with $\mathfrak{p} = \mathfrak{t}$). Then one of the elements of $\mathfrak{S}^*$ has to intersect every element of $\mathfrak{F}$, for if not, we can single out corresponding to every element X of $\mathfrak{S}^*$ an element $f(X)$ of $\mathfrak{F}$ in such a way that $f(X) \cap X = 0$.

Put $\mathfrak{F}' = \{f(X)\}_{X \in \mathfrak{S}^*}$. Then $\overline{\overline{\mathfrak{F}'}} = \overline{\overline{\mathfrak{S}^*}} < \mathfrak{q}$ by b) and therefore it possesses property $A(\mathfrak{r})$ in contradiction with c).

Theorem 2. *Suppose $\mathfrak{p} = \aleph_\alpha$ is singular, $\mathfrak{q} > \aleph_{\mathrm{cf}(\alpha)}$, $\mathfrak{r} \leqq \aleph_{\mathrm{cf}(\alpha)}$. Then* $[\mathfrak{p}, \mathfrak{q}, \mathfrak{r}] \to \mathfrak{p}$.

Proof. It is enough to prove $[\aleph_\alpha, \aleph_{\mathrm{cf}(\alpha)+1}, \aleph_{\mathrm{cf}(\alpha)}] \to \aleph_\alpha$. Let $\mathfrak{F}$ be a family (with $p(\mathfrak{F}) = \aleph_\alpha$) which possesses property $A(\aleph_{\mathrm{cf}(\alpha)+1}, \aleph_{\mathrm{cf}(\alpha)})$. Since the conditions of Lemma 2 hold, we may suppose that $(\overline{\overline{\mathfrak{F}}}) = \overline{\overline{S}} = \aleph_\alpha$. Therefore by Lemma 3 it is enough to see that

$$[\aleph_\alpha, \aleph_{\mathrm{cf}(\alpha)}, \aleph_\alpha]^* \to \aleph_{\mathrm{cf}(\alpha)+1}.$$

This may be seen as follows: Let $\{x_\varrho\}_{\varrho < \omega_\alpha}$ be a well ordering of S, and let $\{\alpha_\nu\}_{\nu < \omega_{\mathrm{cf}(\alpha)}}$ be a sequence of type $\omega_{\mathrm{cf}(\alpha)}$ of ordinal numbers less than α cofinal with α.

Put $S_\nu = \{x_\varrho : \varrho < \omega_{\alpha_\nu}\}$ and $\mathfrak{S}^* = \{S_\nu\}_{\nu < \omega_{\mathrm{cf}(\alpha)}}$. It is well known that

a) $S_\nu \subseteq S$, $\overline{\overline{S}}_\nu < \aleph_\alpha$ for every $\nu < \omega_{\mathrm{cf}(\alpha)}$,

b) $\overline{\overline{\mathfrak{S}^*}} = \aleph_{\mathrm{cf}(\alpha)} < \aleph_{\mathrm{cf}(\alpha)+1}$,

c) if $Y \subseteq S$, $\overline{\overline{Y}} < \aleph_{\mathrm{cf}(\alpha)}$ then Y can not be cofinal with S, and therefore it is contained in one element of $\mathfrak{S}^*$.

Corollary 4. *If $\mathfrak{q} > \mathfrak{r}$, then* $[\mathfrak{p}, \mathfrak{q}, \mathfrak{r}] \to \mathfrak{p}^+$.

Proof. It is enough to show that $[\mathfrak{p}, \mathfrak{r}^+, \mathfrak{r}] \to \mathfrak{p}^+$. Using Lemma 2 and 3 we have to show that $[\mathfrak{p}, \mathfrak{r}, \mathfrak{p}^+]^* \to \mathfrak{r}^+$. But we have trivially $[\mathfrak{p}, \mathfrak{r}, \mathfrak{p}^+]^* \to 2$.

Corollaries 1—4 and Theorems 1, 2 give a complete discussion of the symbol $[\mathfrak{p}, \mathfrak{q}, \mathfrak{r}] \to \mathfrak{s}$, for the cases $\mathfrak{p} \geqq \mathfrak{q}$.

In what follows we may suppose $\mathfrak{p}<\mathfrak{q}$ and $\mathfrak{q}>\mathfrak{r}$. Theorem 1 shows that these assumptions do not assure $[\mathfrak{p}, \mathfrak{q}, \mathfrak{r}]\rightarrow\mathfrak{r}$. Using the hypothesis we are going to prove that the only exception is that given by Theorem 1. First we prove a theorem which without using the hypothesis can not be proved to be best possible, but using the hypothesis we can obtain from it all results.

Theorem 3. *Suppose* $\mathfrak{q}>\mathfrak{r}$ *with* $\sum_{\mathfrak{r}'<\mathfrak{r}}\mathfrak{p}^{\mathfrak{r}'}<\mathfrak{q}$, *then* $[\mathfrak{p}, \mathfrak{q}, \mathfrak{r}]\rightarrow\mathfrak{r}$.

Proof. Put $\mathfrak{t}=\sum_{\mathfrak{r}'<\mathfrak{r}}\mathfrak{p}^{\mathfrak{r}'}$. Then we have

(1) $$\sum_{\mathfrak{r}'<\mathfrak{r}}\mathfrak{t}^{\mathfrak{r}'}=\mathfrak{t}<\mathfrak{t}^{+}$$

and it is enough to prove that $[\mathfrak{t}, \mathfrak{t}^{+}, \mathfrak{r}]\rightarrow\mathfrak{r}$.

Let $\mathfrak{F}$ be a family with $p(\mathfrak{F})=\mathfrak{t}$ which possesses property $A(\mathfrak{t}, \mathfrak{r})$.

Then by Lemma 2 we may suppose $\overline{\overline{(\mathfrak{F})}}=\mathfrak{t}$ and by Lemma 3 we have to prove only $[\mathfrak{t}, \mathfrak{r}, \mathfrak{r}]^{*}\rightarrow\mathfrak{t}^{+}$.

Put $\mathfrak{S}^{*}=[S]^{\mathfrak{r}}$ then S^{*} clearly satisfies conditions a), c) and by (1) it satisfies condition b) too.

(*) Theorem 4. *Suppose* $\mathfrak{p}<\mathfrak{q}$ *with* $\mathfrak{q}>\mathfrak{r}$, *then*

$$[\mathfrak{p}, \mathfrak{q}, \mathfrak{r}]\rightarrow\mathfrak{r}$$

except if $\mathfrak{p}=\aleph_{\alpha}$ *is singular,* $\mathfrak{q}=\aleph_{\alpha+1}$ *and* $\mathfrak{r}>\aleph_{\mathrm{cf}(\alpha)}$.

Proof. We have $\sum_{\mathfrak{r}'<\mathfrak{r}}\mathfrak{p}^{\mathfrak{r}'}\leqq\mathfrak{p}\mathfrak{r}<\mathfrak{q}$ if $\mathfrak{p}$ is regular, or if $\mathfrak{p}$ is singular, but $\mathfrak{r}\leqq\aleph_{\mathrm{cf}(\alpha)}$, and we have $\sum_{\mathfrak{r}'<\mathfrak{r}}\mathfrak{p}^{\mathfrak{r}'}<\mathfrak{p}^{+}\mathfrak{r}<\mathfrak{q}$ if $\mathfrak{q}>\mathfrak{p}^{+}$. The statement of Theorem 4 follows then from Theorem 3 in both cases.

Theorem 4 with Theorem 1 completes the discussion of the symbol $[\mathfrak{p}, \mathfrak{q}, \mathfrak{r}]\rightarrow\mathfrak{s}$ for the case $\mathfrak{p}<\mathfrak{r}, \mathfrak{q}>\mathfrak{r}$.

§ 3.

Lemma 4. $[\aleph_{\alpha+n}, \aleph_{\alpha+1}, \aleph_{\alpha+1}]^{*}\rightarrow\aleph_{\alpha+n+1}$ *where n is finite and α is arbitrary.*

The proof of Lemma 4 is a slight modification of the proof of BERNSTEIN's well known equality

$$\aleph_{\alpha+n}^{\aleph_{\alpha}}=\aleph_{\alpha+n}\aleph_{\alpha}^{\aleph_{\alpha}}.$$ [3])

[3]) See e. g. A. TARSKI, Quelques théorèmes sur les alephs, *Fundamenta Math*, 7 (1925), 1—14.

Thus we may omit the proof. In the same way one can prove the more general statement

$$[\aleph_{\alpha+n}, \aleph_\alpha, \aleph_\alpha]^* \to \aleph_{\alpha+n+1}$$

if n is finite and $\aleph_\alpha$ is regular.

As a corollary of Lemmas 3, 4 and 5 we obtain the following theorem.

Theorem 5. $[\aleph_{\alpha+n}, \aleph_{\alpha+n+1}, \aleph_\alpha] \to \aleph_\alpha$ *if* $\aleph_\alpha$ *is regular.*

It results that we can obtain all the results concerning the symbol $[\mathfrak{p}, \mathfrak{q}, \mathfrak{r}] \to \mathfrak{s}$ without the hypothesys $(*)$ prowided $\mathfrak{p} < \aleph_\omega$.

We have $[\aleph_\omega, \aleph_{\omega+1}, \aleph_1] \not\to \aleph_\omega$ by Theorem 1 and

$$[\aleph_\omega, \aleph_1, \aleph_0] \begin{smallmatrix} \to \aleph_\omega \\ \not\to \aleph_n \end{smallmatrix}$$

by Theorem 2 and by Corollary 2 respectively. Thus the symbol is completely discussed without the hypotheses for $\mathfrak{p} = \aleph_\omega$.

$\mathfrak{p} = \aleph_{\omega+1}$ is the first cardinal number for which there remains unsolved problem if we do not assume the hypothesis. We can not prove $[\aleph_{\omega+1}, \aleph_{\omega+2}, \aleph_1] \to \aleph_1$ or at least $[\aleph_{\omega+1}, \aleph_{\omega+2}, \aleph_1] \to \aleph_\omega$. ($[\aleph_{\omega+1}, \aleph_{\omega+2}, \aleph_1] \to \aleph_{\omega+1}$ follows from Theorem 5 and $[\mathfrak{p}, \mathfrak{p}^+, \aleph_0] \to \aleph_0$ follows from Theorem 3, since $\sum_{\mathfrak{r}' < \aleph_0} \mathfrak{p}^{\mathfrak{r}'} = \mathfrak{p}$ for every cardinal number $\mathfrak{p}$.)

Lemma 1 shows that a proof of $[\aleph_{\omega+1}, \aleph_{\omega+2}, \aleph_1] \to \aleph_\omega$ proves $[\aleph_{\omega+1}, \aleph_1, \aleph_\omega]^* \to \aleph_{\omega+2}$, i. e. such a proof would furnish a proof of the inequality

$$\aleph_{\omega+1}^{\aleph_0} \leqq \aleph_{\omega+1} \cdot \left(\sum_{n=1}^{\infty} \aleph_n^{\aleph_0} \right) = 2^{\aleph_0} \cdot \aleph_{\omega+1}.$$

It is well known that this is one of the hopeless unsolved problems of set theory.

But we can not decide the truth of the above statement even if we assume this inequality.

Problem 1. Is it true that $\aleph_{\omega+1}^{\aleph_0} \leqq 2^{\aleph_0} \cdot \aleph_{\omega+1}$ implies $[\aleph_{\omega+1}, \aleph_{\omega+2}, \aleph_1] \to \aleph_\omega$?

Lemma 1 shows that $[\mathfrak{p}, \mathfrak{r}, \mathfrak{s}]^* \to \mathfrak{q}$ is a necessary condition of $[\mathfrak{p}, \mathfrak{q}, \mathfrak{r}] \to \mathfrak{s}$ at least in the case $\mathfrak{p} \geqq \mathfrak{s} \geqq \mathfrak{r}$. The problem whether this condition is sufficient or not remains open if we do not assume the generalized continuum hypothesis. Lemma 2 shows only that the condition is sufficient for $\mathfrak{q} \leqq \mathfrak{p}^+$.

Thus it is not quite obvious that $[\aleph_{\omega+1}, \aleph_1, \aleph_1]^* \to \aleph_{\omega+3}$ implies $[\aleph_{\omega+1}, \aleph_{\omega+3}, \aleph_1] \to \aleph_1$.

The part β) of Lemma 2 shows only that $[\aleph_{\omega+2}, \aleph_1, \aleph_1]^* \to \aleph_{\omega+3}$ implies $[\aleph_{\omega+1}, \aleph_{\omega+3}, \aleph_1] \to \aleph_1$. But using the same idea as one uses for the proof of Lemma 4 it is easy to see that the following theorem is valid:

$[\aleph_{\omega+1}, \aleph_1, \aleph_1]^* \rightarrow \aleph_{\omega+n+1}$ implies that $[\aleph_{\omega+n}, \aleph_1, \aleph_1]^* \rightarrow \aleph_{\omega+n+1}$ and therefore $[\aleph_{\omega+1} \aleph_1, \aleph_1]^* \rightarrow \aleph_{\omega+n+1}$ assures the validity of

$$[\aleph_{\omega+1}, \aleph_{\omega+n+1}, \aleph_1] \rightarrow \aleph_1.$$

Moreover it is easy to see that $[\aleph_{\omega+1}, \aleph_1, \aleph_1]^* \rightarrow \aleph_{\omega+\omega+1}$ implies that $[\aleph_{\omega+1}, \aleph_1, \aleph_1]^* \rightarrow \aleph_{\omega+n+1}$ for a finite n and therefore $[\aleph_{\omega+1}, \aleph_1, \aleph_1]^* \rightarrow \aleph_{\omega+\omega+1}$ is a sufficient condition for the validity of $[\aleph_{\omega+1}, \aleph_{\omega+\omega+1}, \aleph_1] \rightarrow \aleph_1$ too.

The simplest unsolved problem here is

Problem 2. Is the condition $[\aleph_{\omega+1}, \aleph_1, \aleph_1]^* \rightarrow \aleph_{\omega+\omega+2}$ sufficient for $[\aleph_{\omega+1}, \aleph_{\omega+\omega+2}, \aleph_1] \rightarrow \aleph_1$?

§ 4.

Let $\mathfrak{F}$ be a family of non empty sets and $\mathfrak{t}$ a cardinal number. The family $\mathfrak{F}$ is said to possess property $B(\mathfrak{t})$ if, for every $\mathfrak{F}' \subseteq \mathfrak{F}$ with $\overline{\overline{\mathfrak{F}'}} = \mathfrak{t}$, $\mathfrak{F}'$ has a subfamily $\mathfrak{F}''$ with $\overline{\overline{\mathfrak{F}''}} = \mathfrak{t}$ such that the set $\bigcap_{F \in \mathfrak{F}''} \mathfrak{F}$ is not empty.

We are going to prove the following

Theorem 6. *If the family $\mathfrak{F}$ with $p(\mathfrak{F}) = \mathfrak{p}$ possesses property $B(\mathfrak{p})$, it possesses property $A(\mathfrak{p})$ too.*

Proof. If a family F possesses property $B(\mathfrak{p})$ then the same holds for every subfamily of it. It is easy to see that our theorem holds if $\overline{\overline{\mathfrak{F}}} \leqq \mathfrak{p}$. It follows that a family $\mathfrak{F}$ satisfying the requirements of Theorem 6 possesses property $A(\mathfrak{p}^+, \mathfrak{p})$ hence it has the property $A(\mathfrak{p})$ by Theorem 5, provided $\mathfrak{p}$ is regular.

Therefore we may suppose that $\mathfrak{p}$ is singular $\mathfrak{p} = \aleph_\alpha$ where α is of the second kind, and $\operatorname{cf}(\alpha) < \alpha$. Let $\{\alpha_\nu\}_{\nu < \omega_{\operatorname{cf}(\alpha)}}$ be a sequence of type $\omega_{\operatorname{cf}(\alpha)}$ of ordinal numbers less than α cofinal with α. Put $S = (\mathfrak{F})$. We may suppose $\overline{\overline{S}} \geqq \aleph_\alpha$.

Now we define a double sequence $\{S_{\nu,\mu}\}_{\nu < \omega_{\operatorname{cf}(\alpha)}, \mu < \omega_{\operatorname{cf}(\alpha)}}$ of subsets of S and a sequence $\{\mathfrak{F}_\nu\}_{1 \leqq \nu < \omega_{\operatorname{cf}(\alpha)}}$ of subfamilies of $\mathfrak{F}$, by transfinite induction on ν as follows:

Let S_0 be an arbitrary subset of S of power $\aleph_\alpha$. Let $S_0 = \{x_\varrho^0\}_{\varrho < \omega_\alpha}$ be a well ordering of type ω_α of the set S_0 and put $S_{0,\mu} = \{x_\varrho^0 : \varrho < \omega_{\alpha_\mu}\}$ for every $\mu < \omega_{\operatorname{cf}(\alpha)}$.

It is obvious that

(0) $\quad S_0 = \bigcup_{\mu < \omega_{\operatorname{cf}(\alpha)}} S_{0,\mu}$ and $\overline{\overline{S_{0,\mu}}} \leqq \aleph_{\alpha_\mu}$ for every $\mu < \omega_{\operatorname{cf}(\alpha)}$.

Suppose that the sets $S_{\nu'}$, $S_{\nu',\mu}$ are already defined for every $\nu'<\nu<\omega_{\mathrm{cf}(\alpha)}$ and for every $\mu<\omega_{\mathrm{cf}(\alpha)}$ in such a way that

(00) $\overline{\overline{S}}_{\nu'}=\aleph_\alpha$, $S_{\nu'}=\bigcup_{\mu<\omega_{\mathrm{cf}(\alpha)}} S_{\nu',\mu}$ for every $\nu'<\nu$, and $\overline{\overline{S}}_{\nu'\mu}\leqq\aleph_{\alpha_\mu}$ for every $\nu'<\nu$ and $\mu<\omega_{\mathrm{cf}(\alpha)}$.

Put

$$S_\nu^*=\bigcup_{\nu'<\nu}\bigcup_{\mu<\nu} S_{\nu',\mu}.$$

Then we have by (0) and (00) $\overline{\overline{S_\nu^*}}\leqq\aleph_{\alpha_\nu}\cdot\aleph_{\mathrm{cf}(\alpha)}<\aleph_\alpha$.

If there exists only less than $\aleph_{\alpha_\nu}$ elements of $\mathfrak{F}$ disjoint to S_ν^* then $\mathfrak{F}$ possesses property $A(\aleph_\alpha)$, hence we may assume:

(000) there is an $\mathfrak{F}'\subseteqq\mathfrak{F}$ with $\overline{\overline{\mathfrak{F}'}}=\aleph_{\alpha_\nu}$ such that $(\mathfrak{F}')\cap S_\nu^*=0$.

Let $\mathfrak{F}_\nu$ be such a subfamily of $\mathfrak{F}$ and put

$$S_\nu=(\mathfrak{F}_\nu).$$

We have $\overline{\overline{S}}_\nu\leqq\aleph_\alpha$ and we may suppose $\overline{\overline{S}}_\nu=\aleph_\alpha$. Put $S_\nu=\{x_\varrho^\nu\}_{\nu<\omega_{\mathrm{cf}(\alpha)}}$ and $S_{\nu,\mu}=\{x_\varrho^\nu:\varrho<\omega_{\alpha_\mu}\}$ for every $\mu<\omega_{\mathrm{cf}(\alpha)}$. We have:

(0000) $S_\nu=\bigcup_{\mu<\omega_{\mathrm{cf}(\alpha)}} S_{\nu,\mu}$ and $\overline{\overline{S}}_{\nu,\mu}\leqq\aleph_{\alpha_\mu}$ for every $\mu<\omega_{\mathrm{cf}(\alpha)}$.

Thus S_ν, $S_{\nu,\mu}$ and $\mathfrak{F}_\nu$ are defined and it is proved that (0000) holds for every ν and $\overline{\overline{\mathfrak{F}}}_\nu=\aleph_{\alpha_\nu}$.

Put $S_{\omega_{\mathrm{cf}(\alpha)}}=\bigcup_{\nu<\omega_{\mathrm{cf}(\alpha)}} S_\nu$ and $\mathfrak{F}_{\omega_{\mathrm{cf}(\alpha)}}=\bigcup_{\nu<\omega_{\mathrm{cf}(\alpha)}}\mathfrak{F}_\nu$. We have

$$S_{\omega_{\mathrm{cf}(\alpha)}}=\bigcup_{\nu<\omega_{\mathrm{cf}(\alpha)}}\bigcup_{\mu<\omega_{\mathrm{cf}(\alpha)}} S_{\nu,\mu}=\bigcup_{\mu<\omega_{\mathrm{cf}(\alpha)}}\Big(\bigcup_{\nu'<\nu}\bigcup_{\mu<\nu} S_{\nu'\mu}\Big)=\bigcup_{\nu<\omega_{\mathrm{cf}(\alpha)}} S_\nu^*$$

and therefore

$$(\mathfrak{F}_{\omega_{\mathrm{cf}(\alpha)}})=S_{\omega_{\mathrm{cf}(\alpha)}}=\bigcup_{\nu<\omega_{\mathrm{cf}(\alpha)}} S_\nu^*.$$

On the other hand we have

$$\overline{\overline{\mathfrak{F}}}_{\omega_{\mathrm{cf}(\alpha)}}=\aleph_\alpha.$$

It follows by the assumption that there exists an $\mathfrak{F}'\subseteqq\mathfrak{F}_{\omega_{\mathrm{cf}(\alpha)}}$, $\overline{\overline{\mathfrak{F}'}}=\aleph_\alpha$ such that the set $P=\bigcap_{F\in\mathfrak{F}'} F$ is non-empty.

Suppose $x_0\in P$, then $x_0\in(\mathfrak{F}_{\omega_{\mathrm{cf}(\alpha)}})$, hence $x_0\in S_{\nu_0}^*$ for a $\nu_0<\omega_{\mathrm{cf}(\alpha)}$. $\mathfrak{F}'=\bigcup_{\nu<\omega_{\mathrm{cf}(\alpha)}}\mathfrak{F}'\cap\mathfrak{F}_\nu$ since $\mathfrak{F}'\subseteqq\mathfrak{F}_{\omega_{\mathrm{cf}(\alpha)}}$ and $\overline{\overline{\mathfrak{F}'}}$ being $\aleph_\alpha$ there exists a $\nu_1\geqq\nu_0$ such that $\mathfrak{F}'\cap\mathfrak{F}_{\nu_1}\neq 0$.

But $S_{\nu_0}^*\subseteqq S_{\nu_1}^*$ and therefore $x_0\in(\mathfrak{F}_{\nu_1})\cap S_{\nu_1}^*$ in contradiction with the construction of $\mathfrak{F}_{\nu_1}$ based upon the indirect hypothesis (000).

It follows that the family $\mathfrak{F}$ possesses property $A(\aleph_\alpha)$, q. e. d.

Theorem 7. *Suppose* $\mathfrak{p}=\aleph_\alpha$ *is singular and the family F (with* $p(\mathfrak{F})=\aleph_\alpha$*) possesses property* $B(\aleph_{\mathrm{cf}(\alpha)})$, *then it possesses property* $A(\aleph_\alpha)$ *too.*

The proof is an easy modification of the proof of Theorem 6 taking for $\mathfrak{F}_\nu$ a subfamily of $\mathfrak{F}'$ of power 1. We may omit the details.

Remarks. 1. The property $B(\mathfrak{t})$ is not "monotonic" in any direction. The fact that $\mathfrak{F}$ possesses property $B(\mathfrak{t})$ implies the same neither for $\mathfrak{t}'<\mathfrak{t}$ nor for $\mathfrak{t}'>\mathfrak{t}$.

2. It is easy to see that a family $\mathfrak{F}$ with $p(\mathfrak{F})=\aleph_\alpha$ may possess property $B(\mathfrak{t})$ for every $\mathfrak{t}\leqq\aleph_\alpha$ different from $\aleph_\alpha$ and $\aleph_{\mathrm{cf}(\alpha)}$ without possessing property $A(\aleph_\alpha)$ as shows the example of the family $\mathfrak{F}$ which consists of the complements of the elements of $[S]^{<\aleph_\alpha}$ where S is a set of power $\aleph_\alpha$.

3. A family $\mathfrak{F}$ with $p(\mathfrak{F})=\aleph_\alpha$ may possess property $B(\mathfrak{t})$ for every $\mathfrak{t}\leqq\aleph_\alpha$ without possessing property $A(\mathfrak{r})$ for any fixed $\mathfrak{r}<\aleph_\alpha$.

In fact let S be a set of power $\aleph_\alpha$. Let $\mathfrak{F}$ be the family of the complements of the elements of $[S]^{<\mathfrak{r}}$.

It is obvious that $\mathfrak{F}$ possesses property $B(\mathfrak{t})$ for every $\mathfrak{t}<\aleph_\alpha$ and it does not possess property $A(\mathfrak{r})$.

The fact that it possesses property $B(\aleph_\alpha)$ is a corollary of a theorem of P. ERDŐS.[4])

(Received October 27, 1959)

[4]) See P. ERDŐS, Some remarks on set theory. III, *Michigan Math. Journal*, **2** (1953), 55.