Bounds for the principal frequency of a membrane and the torsional rigidity of a beam

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1. We consider a simply connected or ring-shaped plane domain D of area A, its boundary C whose total length is L, its principal frequency Λ and its torsional rigidity P.

The quantities Λ and P^{-1} may be defined as the minima of the expressions

$$\text{(1a)} \quad \left(\frac{\iint (\operatorname{grad} u)^2\, d\sigma}{\iint u^2\, d\sigma}\right)^{\frac{1}{2}}, \qquad \text{(1b)} \quad \frac{\iint (\operatorname{grad} u)^2\, d\sigma}{4\left(\iint u\, d\sigma\right)^2},$$

respectively, where $d\sigma$ is the surface element of D, the integrations are extended over D; the function u is continuous in D, vanishes on C and has piecewise continuous first derivatives in D.[1])

We state that for a simply connected or ring-shaped domain

$$\text{(2a)} \quad \Lambda \leqq \sqrt{3}\,\frac{L}{A}, \qquad \text{(2b)} \quad P^{-1} \leqq \frac{L^2}{A^3}\ \text{[1a)]}$$

It is enough to show the validity of these inequalities for polygonal domains no two sides of which are parallel.[2]) The total statement follows hence by an argument of continuity.

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[1]) See e. g. G. PÓLYA—G. SZEGŐ, *Isoperimetric Inequalities in Mathematical Physics* (Princeton, 1951), pp. 87 and 102—103.

[1a]) *(Note added on February 25, 1959.)* The constants $\sqrt{3}$ and 1 on the right sides of (2a) and (2b), respectively, are not best possible. G. PÓLYA has shown that the precise upper bounds for $\Lambda A L^{-1}$ and $P^{-1} A^3 L^{-2}$ are $\frac{\pi}{2}$ and $\frac{3}{4}$, respectively; see his paper: Two more inequalities between physical and geometrical quantities (to be published in the *Journal of the Indian Math. Society*).

[2]) Cf. R. COURANT—D. HILBERT, *Methods of Mathematical Physics*. I (New York—London, 1955), pp. 419—423.

The inequalities (2) will be proved if one can find a particular function u for which the quantities (1a) and (1b) are less than (2a) resp. (2b). We shall see that such a function is the point function $d(P)$ which is defined as the distance of the point P from the boundary C. This function satisfies obviously the conditions imposed on the functions u.

Let the vertices of C be $A_1, A_2, \ldots, A_n$; the open line segment $A_i A_{i+1}$ $(A_{n+1} = A_1)$ will be denoted by a_i. We may now define subdomains D_i and D_i' of D in the following way. The interior of D_i resp. D_i' contains those points of D the nearest point of the boundary to which lies on a_i resp. it is the point A_i. The sum of the closures of the domains D_i and D_i' is D; D_i' is void if the inner angle at A_i is less than π.

The level lines of $d(P)$ are in D_i line segments parallel to a_i, in D_i' circular arcs, whose centre is A_i. In the interior of D_i or D_i' $|\text{grad}\, d(P)| = 1$, and so

$$\iint [\text{grad}\, d(P)]^2\, d\sigma = A. \tag{3}$$

Now the level line $d(P) = \xi$ is identical with the boundary of an inner parallel point set of the domain D. The length of this level line will be denoted by $l(\xi)$. We may transform the double integral $M_n = \iint [d(P)]^n d\sigma$ into a simple one by dividing D into narrow stripes the boundaries of which are the level lines $d(P) = \xi$ and the width of which is $d\xi$:

$$M_n = \iint [d(P)]^n\, d\sigma = \int_0^r \xi^n\, l(\xi)\, d\xi \tag{4}$$

where r is the radius of the greatest circle which can be inscribed in D.

If $n = 0$ we have from (4) that

$$\int_0^r l(\xi)\, d\xi = A.$$

Let now the quantity b be defined by $Lb = A$. As $0 \leqq l(\xi) \leqq L$ for $0 \leqq \xi \leqq r$,[3] it follows that $r \geqq b$, for $A \leqq \int_0^r L\, d\xi = Lr$. So we have for $n = 1, 2, \ldots$

$$\int_0^r \xi^n\, l(\xi)\, d\xi - \int_0^b \xi^n L\, d\xi = \int_b^r \xi^n\, l(\xi)\, d\xi - \int_0^b \xi^n \{L - l(\xi)\}\, d\xi \geqq$$

$$\geqq b^n \int_r^b l(\xi)\, d\xi - b^n \int_0^b \{L - l(\xi)\}\, d\xi = b^n \Big\{ \int_0^r l(\xi)\, d\xi - \int_0^b L\, d\xi \Big\} = 0$$

[3]) A proof may be found in the paper by B. Sz.-Nagy, Über Parallelmengen nichtkonvexer ebener Bereiche, *Acta Sci. Math.*, 20 (1959), 36—47.

by the definition of b. It follows that $M_n \geqq \frac{b^{n+1}L}{n+1}$, hence

$$M_1 \geqq \frac{A^2}{2L} \quad \text{and} \quad M_2 \geqq \frac{A^3}{3L^2}$$

and from these

$$\text{(2a)} \qquad \Lambda \leqq \left(\frac{\iint [\operatorname{grad} d(P)]^2\, d\sigma}{\iint [d(P)]^2\, d\sigma}\right)^{\frac{1}{2}} \leqq \left(\frac{A}{A^3/(3L^2)}\right)^{\frac{1}{2}} = \sqrt{3}\,\frac{L}{A},$$

resp.

$$\text{(2b)} \qquad P^{-1} \leqq \frac{\iint [\operatorname{grad} d(P)]^2\, d\sigma}{4\,[\iint d(P)\, d\sigma]^2} \leqq \frac{A}{4\,[A^2/2L]^2} = \frac{L^2}{A^3}.$$

2. There exists another upper estimate of Λ and P^{-1} for *star-shaped* domains, namely that of PÓLYA and SZEGŐ [4]). We consider the quantity $B_a = \int_C h^{-1}\, ds$ where a is a point inside D with respect to which C is star-shaped, h is the length of the perpendicular drawn from a to the tangent at a variable point of C where ds is the line element. If a varies and $B = \min B_a$, then $\Lambda \leqq j\sqrt{B/2A}$ with $j = 2.40\ldots$, and $P^{-1} \leqq BA^{-2}$.

It seems that for convex domains the estimate of PÓLYA and SZEGŐ gives better results than (2). Yet e. g. for the pentagonal domain whose consecutive vertices are $(1,0)$, $(1,1)$, $(0,\varepsilon)$, $(-1,1)$ $(-1,0)$, B tends to infinity as $\varepsilon \to 0$; on the other hand L and A remain bounded.

3. It may be noted that there does not exist a universal positive constant c such that for any simply connected domain $\Lambda \geqq cL/A$. (Contrary to the case when D is convex.) [5]) For let us consider the domains

$$D_1 (0 \leqq x \leqq 1, 0 \leqq y \leqq 1) \quad \text{and} \quad D_2 (1 \leqq x \leqq 1 + \varepsilon^{-1}, 0 \leqq y \leqq \varepsilon).$$

In the case of the domain $D = D_1 + D_2$ we have $L/A = 2 + \varepsilon^{-1}$ and Λ is bounded, for it is less than the principal frequency of the unit square.

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[4]) L. c. [1]) pp. 14—15 and 91—94.

[5]) E. MAKAI, On the principal frequency of a convex membrane and related problems, *Czechoslovak Math. Journal*, **9** (1959), 66—70.