Separation of two linear subspaces.

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A pair of non-trivial linear subspaces of Euclidean 3-space, whose dimensionalities are known, forms a geometrical figure which is determined up to Euclidean congruence by the non-obtuse angle between them — single number between 0 and $\pi/2$.

The situation is not so simple in Euclidean n-space, $n > 3$, but the proper generalization has been known since early in the history of study of these spaces [14, § 48]. For example, given two 2-dimensional subspaces $\mathfrak{P}$ and $\mathfrak{Q}$ of 4-space, intersecting in a single point 0; then there exist perpendicular 2-subspaces $\mathfrak{S}_1$ and $\mathfrak{S}_2$, intersecting in 0, each intersecting $\mathfrak{P}$ and $\mathfrak{Q}$ perpendicularly in a line; the angles θ_i $(0 < \theta_i \leqq \pi/2)$ between $\mathfrak{S}_i \cap \mathfrak{P}$ and $\mathfrak{S}_i \cap \mathfrak{Q}$ may have any values independently; these two numbers are determined uniquely by the figure $\mathfrak{P}$, $\mathfrak{Q}$ and determine it up to congruence.

This behavior does generalize directly, not only to complex finite-dimensional Hilbert spaces, but also, with the natural changes, to infinite-dimensional real and complex Hilbert spaces. Because of the decomposition (essentially) into orthogonal 2-spaces which do not interfere with each other, much of the geometry of a pair of subspaces is like that of a pair of lines.

Why then should there be an article written about it? For three reasons: (1) To establish the decomposition described in full generality (§ 5). This result is due to DIXMIER [4, Chap. I], [3]. (2) In infinite-dimensional spaces the decomposition is not quite into 2-dimensional subspaces $\mathfrak{S}_i$ as above (because C below may have continuous spectrum). Therefore many general facts which might be provable as corollaries of the decomposition, together with facts about angles in 2-space, must instead be proved as *generalizations* of the trigonometric facts. Of this sort are most of §§ 3—6. Some of the main results are due to DIXMIER[2]) and others, and some simple

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[2]) There is enough overlap with [3] in particular that I give a partial glossary of terms: my C, S, and $\mathfrak{Y}$) are DIXMIER's A^2, B^2, and V_0. The development is rather different, both in methods and in subjects treated. (Cf. e. g. my Thm. 4.1 with his Thm. 1.)

facts have been known for years to me and doubtless to others. I have not collected all known theorems which would fit in, but I hope I have simplified the subject by basing it on the "closeness" and "separation" operators, generalizing trigonometric functions (§ 2). (3) Some of the results, since they concern distinctions which cannot be made in 2-space, cannot be proved even in n-space by reducing them to 2-space problems *via* the decomposition. This applies in particular to § 7, and to the intended sequel [2] on several numerical measures for the separation of two subspaces.

The geometry is simple as long as only two subspaces — or, equivalently, two hermitian projections, or two symmetries — are involved. If there are more than two, it is inevitably complicated. Here are two indications: Every unitary operator on an infinite-dimensional Hilbert space is a product of four symmetries [8]; the algebra of all bounded operators on a finite- or denumerable-dimensional Hilbert space can be generated by three projections [1].

1. Notations. The Hilbert space $\mathfrak{H}$ is in much of what follows of arbitrary dimensionality, and either real or complex. Specializations will be mentioned when they are made. P, Q are hermitian projections, and other bounded operators also are denoted by capital letters. Subspaces are denoted by the same letters as the projections onto them, but in gothic type: $P\mathfrak{H}=\mathfrak{P}$. Always $\tilde{P}=1-P$, accordingly $\tilde{\mathfrak{P}}=\tilde{P}\mathfrak{H}=\mathfrak{P}^{\perp}$; similarly $\tilde{Q}$, etc. $P\cap Q$ and $P\cup Q$ are the meet and join respectively of P and Q in the lattice of projections; accordingly, $\mathfrak{P}\cup\mathfrak{Q}$ is not the set-union of $\mathfrak{P}$ and $\mathfrak{Q}$. The symbol "$\leftrightarrow$" means "commutes with". $\mathfrak{N}(A)$ is the nullspace of A.

2. Closeness and separation operators [1]. Given any two hermitian projections P, Q, consider the *closeness operator*

$$C=C(P,Q)=PQP+\tilde{P}\tilde{Q}\tilde{P} \tag{2.11}$$

and the *separation operator*

$$S=S(P,Q)=P\tilde{Q}P+\tilde{P}Q\tilde{P} \tag{2.12}$$

associated with P and Q. For 1-subspaces of 2-space, C is constant $=\cos^2\theta$, where θ is (either) angle between them; similarly $S=\sin^2\theta$. This, with the introduction above, should explain the idea behind the definitions and the following properties, which may be verified without trouble in the order given.

(2.21) $\quad 0\leqq C\leqq 1.$

(2.22) $\quad C(P,Q)=1-P-Q+PQ+QP.$

(2.23) $\quad C(P,Q)=C(\tilde{P},\tilde{Q})=C(Q,P)=C(\tilde{Q},\tilde{P}).$

(2.24) $\quad C(P,Q)\leftrightarrow P, Q.$

(2. 25) $$C(P, Q)P = PQP.$$

(2. 31) $$S + C = 1.$$

(2. 32) $$S(P, \tilde{Q}) = C(P, Q).$$

(2. 4) $$S(P, Q) = (P - Q)^2 = (\tilde{P} - \tilde{Q})^2.$$

(2. 5) $$\mathfrak{N}(C(P, Q)) = \mathfrak{N}(P - \tilde{Q}) = \mathfrak{N}(\tilde{P} - Q) = (\mathfrak{P} \cap \tilde{\mathfrak{Q}}) \cup (\tilde{\mathfrak{P}} \cap \mathfrak{Q}).$$

One may permute S and C by (2. 31), (2. 32), and one may permute $P, Q, \tilde{P}, \tilde{Q}$ in various ways by (2. 23). The many resulting corollaries to the formulas above are taken for granted.

3. Unitary applications of one subspace onto another. The opening remark is again trivial:

(3. 11) $$(PQ + \tilde{P}\tilde{Q})(QP + \tilde{Q}\tilde{P}) = (QP + \tilde{Q}\tilde{P})(PQ + \tilde{P}\tilde{Q}) = C(P, Q).$$

Therefore if by definition

(3. 12) $$U = U(P, Q) = C^{-1/2}(QP + \tilde{Q}\tilde{P})$$

then U looks formally like a unitary operator, because $U(P, Q)^* = U(Q, P)$, so (3. 11) gives (by (2. 24)) $U^*U = UU^* = 1$. Also $\tilde{Q}UP = 0$ so $U\mathfrak{P} \subseteq \mathfrak{Q}$; $\tilde{P}U^*Q = 0$, so $U^*\mathfrak{Q} \subseteq \mathfrak{P}$; thus $U\mathfrak{P} = \mathfrak{Q}$, and similarly $U\tilde{\mathfrak{P}} = \tilde{\mathfrak{Q}}$. It seems U is unitary taking $\mathfrak{P}$ onto $\mathfrak{Q}$ and $\tilde{\mathfrak{P}}$ onto $\tilde{\mathfrak{Q}}$.

Now pause to find when this is valid. Surely not in general, for $\dim \mathfrak{P} = \dim \mathfrak{Q}$ is necessary for the existence of an isometry of $\mathfrak{P}$ onto $\mathfrak{Q}$; and even this is far from sufficient in infinite-dimensional $\mathfrak{H}$, see below. The formal proof above does not require C^{-1} to be bounded, but does require it to be densely defined, that is, $\mathfrak{N}(C) = 0$. (Sufficient but not necessary is for C to have a positive lower bound, or equivalently (see (2. 31), (2.4)), $\|P - Q\| < 1$). Can something be done even when $\mathfrak{N}(C) \neq 0$? Clearly yes. Still (3. 12) gives a unitary U on $\tilde{\mathfrak{N}}(C)$. If

(3. 2) $$\dim(\mathfrak{P} \cap \tilde{\mathfrak{Q}}) = \dim(\tilde{\mathfrak{P}} \cap \mathfrak{Q}),$$

then define U on $\mathfrak{P} \cap \tilde{\mathfrak{Q}}$ as some isometry V onto $\tilde{\mathfrak{P}} \cap \mathfrak{Q}$; define U on $\tilde{\mathfrak{P}} \cap \mathfrak{Q}$ as $-V^*$.[3]) Extend the domain of U to all $\mathfrak{H}$ by linearity.

Define subspaces $\mathfrak{P}$ and $\mathfrak{Q}$ to be *equivalently positioned* provided (3. 2) holds. What has been proved is the following strengthening of a theorem of Sz.-Nagy [12, § 105].[4])

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[3]) The U constructed here is a little special, for later convenience. Namely, on $\mathfrak{N}(C)$, $U^2 = -1$. The reader may verify that this special property is consistent with — indeed is implied by — the paragraph following Theorem 4. 3.

[4]) It may be easily verified that the operator he uses agrees, on $\mathfrak{P}$, with (3.12).

Theorem 3.1. *If $\mathfrak{P}$ and $\mathfrak{Q}$ are equivalently positioned, there is a unitary W on $\mathfrak{H}$ taking $\mathfrak{P}$ onto $\mathfrak{Q}$ and $\tilde{\mathfrak{P}}$ onto $\tilde{\mathfrak{Q}}$.*

That is, (3.2) implies $\dim \mathfrak{P} = \dim \mathfrak{Q}$ and $\dim \tilde{\mathfrak{P}} = \dim \tilde{\mathfrak{Q}}$. But all three of these equations are equivalent in finite-dimensional $\mathfrak{H}$; the construction of U may appear to have yielded meager results. The following theorem, and especially § 7, defend its introduction more convincingly.

Theorem 3.2. *$\mathfrak{P}$ and $\mathfrak{Q}$ are equivalently positioned if and only if there exists a unitary operator W on $\mathfrak{H}$ taking $\mathfrak{P}$ onto $\mathfrak{Q}$ and $\tilde{\mathfrak{P}}$ onto $\tilde{\mathfrak{Q}}$, such that $W \leftrightarrow C(P, Q)$.*

Proof. If $W \leftrightarrow C$, $W\mathfrak{R}(C) = \mathfrak{R}(C)$. By (2.5), $W(\mathfrak{P} \cap \tilde{\mathfrak{Q}}) \subseteq \mathfrak{R}(C) \cap W\mathfrak{P} = \mathfrak{R}(C) \cap \mathfrak{Q} = \tilde{\mathfrak{P}} \cap \mathfrak{Q}$. Similar treatment of W^* completes the proof of (3.2). The converse, almost proved above, calls for one more thing: the proof that U defined above commutes with C. It takes $\mathfrak{R}(C)$ onto $\mathfrak{R}(C)$, so only (3.12) demands attention. By (2.25), $QPC = QPQP = CQP$, and similarly $\tilde{Q}\tilde{P} \leftrightarrow C$, therefore $U \leftrightarrow C$.

The conditions of the theorem do not force $W = U$, even on $\tilde{\mathfrak{R}}(C)$. They are satisfied by a wider class of W, which this paper could treat but will not.

The relation "equivalently positioned" between subspaces $\mathfrak{P}$ and $\mathfrak{Q}$ is superfluous in the finite-dimensional case, as already mentioned, because equivalent to $\dim \mathfrak{P} = \dim \mathfrak{Q}$. In the infinite-dimensional case it has a different drawback: it is not transitive.[5] Here is a simple example where $\mathfrak{P}$ and $\mathfrak{R}$ are equivalently positioned, likewise $\mathfrak{Q}$ and $\mathfrak{R}$, but $\mathfrak{P}$ and $\mathfrak{Q}$ are not. Let $\mathfrak{H}$ be generated by the countable orthogonal set $\{\ldots, x_{-1}, x_0, x_1, x_2, \ldots\}$; in symbols, $\mathfrak{H} = [\ldots, x_{-1}, x_0, x_1, x_2, \ldots]$. Let $\mathfrak{P} = [x_1, x_2, \ldots]$, $\mathfrak{Q} = [x_0, x_1, x_2, \ldots]$, $\mathfrak{R} = [\ldots, x_{-2}, x_{-1}]$. Then $\mathfrak{P} \cap \tilde{\mathfrak{R}}$, $\tilde{\mathfrak{P}} \cap \mathfrak{R}$, $\mathfrak{Q} \cap \tilde{\mathfrak{R}}$, $\tilde{\mathfrak{Q}} \cap \mathfrak{R}$ are all denumerable-dimensional; but $\dim (\mathfrak{P} \cap \tilde{\mathfrak{Q}}) = 0 \neq 1 = \dim (\tilde{\mathfrak{P}} \cap \mathfrak{Q})$.

(This has exhibited also a pair of subspaces, $\mathfrak{P}$ and $\mathfrak{Q}$, which are not equivalently positioned in spite of the existence of a unitary W as described by Theorem 3.1 — namely, the "bilateral shift" $Wx_i = x_{i-1}$.)

However, in case $\dim \mathfrak{P}$ and $\dim \mathfrak{Q}$ are equal *and finite*, $\mathfrak{P}$ and $\mathfrak{Q}$ *are* necessarily equivalently positioned, of course, and the relation is transitive in this special case. Indeed this case does not differ in any important way from the still more special case where $\dim \mathfrak{H}$ is finite.

[5]) The reader is reminded that the relation between $\mathfrak{P}$ and $\mathfrak{Q}$ of satisfying $\dim (\mathfrak{P} \cap \tilde{\mathfrak{Q}}) = \dim (\tilde{\mathfrak{P}} \cap \mathfrak{Q}) = 0$ is not transitive even in 2-space.

4. The angle bisector. Before finding the unitary invariants, generalize some more trigonometry, for later use. Assuming for the moment $\mathfrak{N}(C)=0$, define

$$X=X(P,Q)=C^{-1/2}(P-\tilde{Q})=C^{-1/2}(P+Q-1). \tag{4.1}$$

$X=X^*$ is obvious, by (2.24). $X^2=1$ follows most easily from (2.4) with (2.32). One calculates at once that $\tilde{Q}XP=0=\tilde{P}XQ$. Summing up, X is a symmetry on $\mathfrak{H}$ which exchanges $\mathfrak{P}$ with $\mathfrak{Q}$ (hence also $\tilde{\mathfrak{P}}$ with $\tilde{\mathfrak{Q}}$).

Accordingly,

$$Y=Y(P,Q)=\frac{1}{2}(1+X) \tag{4.2}$$

is the projection onto a subspace which may be named the *angle bisector* of $\mathfrak{P}$ and $\mathfrak{Q}$. Even in 2-space, angle bisectors are not unique, but $\mathfrak{Y}$ here is essentially "the angle bisector of the acute angle", as appears from Theorem 4.1 below.

If now $\mathfrak{N}(C)\neq 0$, but $\mathfrak{P}$ and $\mathfrak{Q}$ equivalently positioned, then define X on $\mathfrak{N}(C)$ as an arbitrary symmetry exchanging $\mathfrak{P}\cap\tilde{\mathfrak{Q}}$ with $\tilde{\mathfrak{P}}\cap\mathfrak{Q}$ (cf. §. 3). Keep (4.2). If $\mathfrak{P}$ and $\mathfrak{Q}$ not equivalently positioned, X is not defined on all of $\mathfrak{H}$.

Lemma 4.1. *If $\mathfrak{P}$ and $\mathfrak{Q}$ are equivalently positioned, PQP and QPQ have the same spectrum.* (Cf. Lemma 5.2.)

Proof. $QPQ=XPQPX$.

Theorem 4.1. *Let $\mathfrak{N}(C(P,Q))=0$. Then $X(P,Q)$ is the unique symmetry V which exchanges $\mathfrak{P}$ with $\mathfrak{Q}$ (hence also $\tilde{\mathfrak{P}}$ with $\tilde{\mathfrak{Q}}$) and satisfies $PVP\geqq 0$.*

Proof. One computes $PXP=C^{-1/2}(PQP)=(PQP)^{1/2}\geqq 0$.

For the converse, let Z be the projection $\frac{1}{2}(V+1)$. Decomposing $\mathfrak{H}$ as $\mathfrak{Z}\oplus\tilde{\mathfrak{Z}}$ gives matrix expressions[6])

$$V=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},\qquad P=\begin{pmatrix}P_{11} & P_{12}\\ P_{21} & P_{22}\end{pmatrix},\qquad Q=\begin{pmatrix}P_{11} & -P_{12}\\ -P_{21} & P_{22}\end{pmatrix}. \tag{4.31}$$

The matrix for P is general, while that for Q is derived from $Q=VPV$. We know more about the P_{ij}: by the idempotence of P, $P_{12}P_{21}=P_{11}-P_{11}^2$ and $P_{21}P_{12}=P_{22}-P_{22}^2$. Substituting these into the matrix for $C(P,Q)$ com-

[6]) The assigning of a canonical isomorphism between $\mathfrak{Z}$ and $\tilde{\mathfrak{Z}}$ is avoidable in this proof; so, for example, P_{12} may be regarded merely as an operator from one Hilbert space $\tilde{\mathfrak{Z}}$, onto another, $\mathfrak{Z}$. See Theorem 6.2.

puted from (2. 22), one gets

$$\begin{pmatrix} 1-4P_{11}+4P_{11}^2 & 0 \\ 0 & 1-4P_{22}+4P_{22}^2 \end{pmatrix}. \tag{4.32}$$

What is to be proved is that $V=X(P,Q)$, or, by (4. 1), that $C^{1/2}$ is equal to

$$V(P+Q-1)=\begin{pmatrix} 2P_{11}-1 & 0 \\ 0 & 1-2P_{22} \end{pmatrix}. \tag{4.33}$$

For this the last hypothesis of the theorem will be essential, for there are many symmetries exchanging $\mathfrak{P}$ with $\mathfrak{Q}$ [3], and the proof so far applies to any of them.

On the subspace $\mathfrak{N}(C(P,Z))$, which is $\mathfrak{N}(P-\tilde{Z})$ by (2. 4) with (2. 32), we have $V=\tilde{P}-P$, $PVP=-P\geqq 0$. Therefore $\mathfrak{N}(C(P,Z))=0$, and Lemma 4. 1 applies to $\mathfrak{P}$, $\mathfrak{Z}$. Now the hypothesis $PVP\geqq 0$ implies $PZP\geqq \frac{1}{2}P$, which implies by Lemma 4. 1 $ZPZ\geqq\frac{1}{2}Z$, or $P_{11}\geqq\frac{1}{2}$. Similarly $P_{22}\leqq\frac{1}{2}$. Hence (4. 33) is the positive square root of (4. 32), as predicted.

The next theorems show that the particular "rotations" singled out in the last section are related to the particular angle bisectors singled out in this section in a way which generalizes the 2-space facts. First come lemmas leading to the half-angle formula. Some of the results are ambiguous in general; so for now, assume $\mathfrak{N}(C(P,Q))=0$.

Lemma 4. 2. $YP+\tilde{Y}\tilde{P}=\frac{1}{2}(1+U)$.

Proof. Using definitions (4. 2) then (4. 1),

$$2YP+2\tilde{Y}\tilde{P}-1=XP-\tilde{X}\tilde{P}=C^{-1/2}(QP+\tilde{Q}\tilde{P})=U.$$

Lemma 4. 3. $U+U^*=2C^{1/2}$.

Proof. Define for the moment

$$F=C^{1/2}(U+U^*)=QP+\tilde{Q}\tilde{P}+PQ+\tilde{P}\tilde{Q}.$$

To prove $F=2C$, $PFP=2PQP=2PCP$, $PF\tilde{P}=P(Q+\tilde{Q})\tilde{P}=0=$ $=2PC\tilde{P}$, and similarly for $\tilde{P}FP$ and $\tilde{P}F\tilde{P}$.

Lemma 4. 4. $C(P,Y)=\frac{1}{2}(1+C(P,Q)^{1/2})=C(Y,Q)$.

Proof. Use (3.11) applied to P, Y; then Lemma 4.2, then Lemma 4.3:

$$C(P,Y)=(PY+\tilde{P}\tilde{Y})(YP+\tilde{Y}\tilde{P})=\frac{1}{4}(2+U+U^*)=\frac{1}{2}+\frac{1}{2}C(P,Q)^{1/2}.$$

Symmetrically for $C(Y,Q)$.

Theorem 4.2. $U(P, Y)=U(Y, Q)$.

Proof. Expressing both sides by (3.12), and using $C(P, Y)=C(Y, Q)$ (Lemma 4.4), the problem is reduced to proving $YP+\tilde{Y}\tilde{P}=QY+\tilde{Q}\tilde{Y}$. But by Lemma 4.2 applied twice,

$$YP+\tilde{Y}\tilde{P}=\frac{1}{2}(1+U(P, Q))=\frac{1}{2}(1+U(Q, P)^*)=(YQ+\tilde{Y}\tilde{Q})^*.$$

Theorem 4.3. $U(P, Y)^2=U(P, Q)$.

Proof. First,

$$U(P, Y)^2=U(Y, Q)\,U(P, Y)=C(P, Y)^{-1}(QY+\tilde{Q}\tilde{Y})(YP+\tilde{Y}\tilde{P})=$$
$$=C(P, Y)^{-1}(QYP+\tilde{Q}\tilde{Y}\tilde{P}).$$

Now $Y=\frac{1}{2}(1+C(P, Q)^{-1/2}(1-\tilde{P}-Q))$ by (4.2), (4.1), so $QYP=$ $=\frac{1}{2}(1+C(P, Q)^{-1/2})\,QP=C(P, Q)^{-1}\,C(P, Y)\,QP$. Similarly, from $\tilde{Y}=$ $=\frac{1}{2}(1-C(P, Q)^{-1/2}(-1+P+Q))$ follows $\tilde{Q}\tilde{Y}\tilde{P}=C(P, Q)^{-1}\,C(P, Y)\,\tilde{Q}\tilde{P}$. Substituting these in the above expression for $U(P, Y)^2$ makes it exactly the expression (3.12) for $U(P, Q)$.

It should be observed that if $\mathfrak{N}(C(P, Q))\neq 0$ but $\mathfrak{P}$ and $\mathfrak{Q}$ equivalently positioned, the last results remain true, provided the (hitherto not unique) definition of $U(P, Q)$ on $\mathfrak{N}(C)$ is made to accord with the definition of $X(P, Q)$ there by *assuming* the conclusion of Theorem 4.3 there.

Theorem 4.4. $X(P, Q)(P-\tilde{P})=U(P, Q)$.

Proof. By definitions, each side is equal to

$$C(P, Q)^{-1/2}\{(-\tilde{P}+Q)P-(P-\tilde{Q})\tilde{P}\}.$$

Another (equally obvious) version of the theorem: $X(P, Q)P=$ $=U(P, Q)P$.

The theorem shows that any $U(P, Q)$ is the product of two symmetries. The following theorem implies as much of a converse as is true: the product of the symmetries leaving fixed subspaces $\mathfrak{P}$, $\mathfrak{Q}$ respectively is a U if and only if $S(P, Q)\leqq\frac{1}{2}$. See also Theorem 6.3.

Theorem 4.5. *$S(P, Q)\leqq\frac{1}{2}$ is necessary and sufficient for the existence of $\mathfrak{R}$ such that $P=Y(R, Q)$.*

That is, $\mathfrak{P}$ and $\mathfrak{Q}$ may not be "too far apart", cf. [2]; in particular, $\mathfrak{N}(C(P, Q)) = 0$.

Proof. Necessity is essentially the first sentence in the proof of Theorem 4. 1. Sufficiency will be proved from the hard half of Theorem 4. 1 (its uniqueness assertion).

Define $R = (P - \tilde{P})\,Q(P - \tilde{P})$. To show that $P = Y(R, Q)$, that is, that $P - \tilde{P} = X(R, Q)$, requires no argument on $\mathfrak{N}(C(R, Q))$, because there $X(R, Q)$ is simply *any* symmetry exchanging $\mathfrak{P}$ with $\mathfrak{Q}$, and $P - \tilde{P}$ will do. On $\tilde{\mathfrak{N}}(C(R, Q))$, it must be shown in addition that $Q(P - \tilde{P})Q \geqq 0$; but this follows immediately from $S(P, Q) \leqq \frac{1}{2}$.

(The definition $R = U(P, Q)^* PU(P, Q)$ would, by Theorems 4.2 and 4. 4, have been equivalent.)

5. Unitary invariants for a pair of subspaces. The purpose in this section is to give a complete set of unitary invariants for $\mathfrak{P}, \mathfrak{Q}$ in terms of the spectral multiplicity function of the bounded self-adjoint $C(P, Q)$. (I assume familiarity with spectral multiplicity theory [7, III].) In the 4-dimensional case described in the introduction, C has two 2-dimensional eigenspaces with corresponding eigenvalues $\cos^2\theta_i$; so this section generalizes what was said there.

What is required is to assign a set of objects to any pair of subspaces $\mathfrak{P}, \mathfrak{Q}$; to show $\mathfrak{P}, \mathfrak{Q}$ can be carried onto $\mathfrak{P}', \mathfrak{Q}'$ by an isometry of $\mathfrak{H}$ if and only if the same set of objects was assigned to $\mathfrak{P}'$, $\mathfrak{Q}'$ as to $\mathfrak{P}$, $\mathfrak{Q}$; and to say exactly what sets of objects can arise.

Lemma 5. 1. *The following are unitary invariants for* $\mathfrak{P}, \mathfrak{Q}$:

$$\dim(\mathfrak{P} \cap \mathfrak{Q}),\ \dim(\tilde{\mathfrak{P}} \cap \tilde{\mathfrak{Q}}),\ \dim(\mathfrak{P} \cap \tilde{\mathfrak{Q}}),\ \dim(\tilde{\mathfrak{P}} \cap \mathfrak{Q}).$$

This is obvious. Only $(\mathfrak{N}(C) \cup \mathfrak{N}(S))^{\sim}$ requires study.

Lemma 5. 2. *If* $\mathfrak{N}(C) = \mathfrak{N}(S) = 0$, *then* $PQP = PCP$ *and* $\tilde{P}\tilde{Q}\tilde{P} = \tilde{P}C\tilde{P}$ *have the same spectrum.*

Proof. In this case $\mathfrak{P}$ and $\mathfrak{Q}$ are equivalently positioned and so are $\mathfrak{Q}$ and $\tilde{\mathfrak{P}}$. A unitary operator commuting with C and exchanging $\mathfrak{P}$ with $\tilde{\mathfrak{P}}$ is $U(Q, \tilde{P})U(P, Q)$.

Lemma 5. 3. *If* $\mathfrak{N}(C) = \mathfrak{N}(S) = 0$, *the spectrum of* C *is of even multiplicity, and is on* $[0, 1]$ *with zero multiplicity at the endpoints, but is otherwise arbitrary.*

Proof. To prove the spectrum of C is of even multiplicity (that is, the values of its spectral multiplicity function are even integers or infinite), I will specify two orthogonal projections commuting with C such that the restrictions of C to their subspaces are isomorphic; it is not hard to see that this is equivalent.

Namely, the projections are P and $\tilde{P}$. The isomorphism results from Lemma 5.2.

The other statements about the spectrum of C are obvious.

Conversely, suppose given an operator (call it A) with such a spectrum. $\mathfrak{N}(A)=\mathfrak{N}(1-A)=0$ by assumption. It must be shown that $A=C(P,Q)$ for suitable P,Q. Now (to rephrase the first paragraph of the proof) the even multiplicity of the spectrum of A is equivalent to the possibility of representing A in the matrix form $\begin{pmatrix} B & 0 \\ 0 & B \end{pmatrix}$ (with respect to some expression of $\mathfrak{H}$ in the form $\mathfrak{H}_1 \oplus \mathfrak{H}_2$ and some canonical isomorphism $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ of $\mathfrak{H}_1$ onto $\mathfrak{H}_2$). Here $0 \leqq B \leqq 1$, $\mathfrak{N}(B)=\mathfrak{N}(1-B)=0$.

Use the construction of MICHAEL [15, § 2]: Define

$$P=\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad Q=\begin{pmatrix} B & (B-B^2)^{1/2} \\ (B-B^2)^{1/2} & 1-B \end{pmatrix}. \tag{5.1}$$

That P is a projection is obvious; also $Q=Q^*$, and one calculates $Q^2=Q$. Evaluating $C(P,Q)$ according to the definition (2.11), one obtains A.

For the theorem, nothing substantial is lacking but the main assertion: If $\mathfrak{P}, \mathfrak{Q}$ have the same associated invariants as do $\mathfrak{P}', \mathfrak{Q}'$, then there is unitary equivalence. The only non-trivial part of that is the following

Lemma 5.4. *Let $\mathfrak{N}(C)=\mathfrak{N}(S)=0$, and let $C(P',Q')$ be unitary equivalent to $C(P,Q)$. Then there is some unitary V on $\mathfrak{H}$ taking $\mathfrak{P}$ onto $\mathfrak{P}'$, $\mathfrak{Q}$ onto $\mathfrak{Q}'$.*

Proof. It is enough to show (i) that P and Q may be represented in the form (5.1), by suitably choosing the complementary subspaces $\mathfrak{H}_1$ and $\mathfrak{H}_2$ and the canonical isomorphism, and (ii) that the isomorphism type of B in (5.1) depends only on the isomorphism type of $C(P,Q)$. Because then P' and Q' will have isomorphic matrix expressions (5.1), and the construction of V will be obvious.

Let $\mathfrak{H}_1=\mathfrak{P}$ and $\mathfrak{H}_2=\tilde{\mathfrak{P}}$. The canonical isomorphism $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ is $U(Q,\tilde{P})\,U(P,Q)\,P$. It has already been proved (Lemma 5.2) that then $C(P,Q)=\begin{pmatrix} B & 0 \\ 0 & B \end{pmatrix}$, with the spectrum of B determined by that of C; and the matrix form of P is as required. By (2.25),

$$Q=\begin{pmatrix} B & D \\ D^* & 1-B \end{pmatrix}$$

for some D. Because $Q^2=Q$, $DD^*=B-B^2$. But is $D\geqq 0$? Or equivalently, is the operator $(PQ\tilde{P})\,U(Q,\tilde{P})\,U(P,Q)=(CS)^{-1/2}\,PQ\tilde{P}QP$ positive? Clearly yes.

The results may be summed up as follows.

Theorem 5.1. *The following is a complete set of unitary invariants for* $\mathfrak{P},\mathfrak{Q}$*:* (i) *four cardinal numbers, the dimensionalities of* $\mathfrak{P}\cap\mathfrak{Q}$, $\tilde{\mathfrak{P}}\cap\tilde{\mathfrak{Q}}$, $\mathfrak{P}\cap\tilde{\mathfrak{Q}}$, $\tilde{\mathfrak{P}}\cap\mathfrak{Q}$; *any 4-tuple of cardinals may occur;* (ii) *a spectral multiplicity function on measures on* $[0,1]$, *that of* $C(P,Q)$ *restricted to* $(\mathfrak{N}(C)\cup\mathfrak{N}(S))^{\sim}$; *it has even values, and is zero on the point measures at* 0 *and* 1, *but is otherwise arbitrary.*

Remark. The essential point in the proof is the fact that an operator B on a Hilbert space $\mathfrak{H}_1$ which satisfies $0\leqq B\leqq 1$ is of the form PQP, where P and Q are projections (in some constructed Hilbert space $\mathfrak{H}\supseteq\mathfrak{H}_1$). A much more general theorem was proved in 1940 by NAIMARK; namely, B is replaced by an increasing family of positive operators and Q by a resolution of the identity [15, § 2]. I used MICHAEL's construction here because it is the same sort of generalized trigonometry that is used throughout the present paper. Actually, it can be extended, though less easily, to prove NAĬMARK's theorem.

6. Other characterization theorems. Another complete set of unitary invariants for $\mathfrak{P},\mathfrak{Q}$ can be got from the spectrum of $P-Q$ instead of $C(P,Q)$ [3, §§ V, VI]. To avoid repetition, I will give a slightly different statement of the idea.

Theorem 6.1. *The following are necessary and sufficient conditions on an operator* A *in order that it be the difference of two projections:* $-1\leqq A\leqq 1$; *and on* $\tilde{\mathfrak{N}}(1-A^2)$ *there exists a unitary* W *such that* $AW=-WA$.

Proof. If $A=P-Q$, so that $1-A^2=C(P,Q)$, take $W=X(P,Q)$ as in § 4; for, when restricted to $\tilde{\mathfrak{N}}(C)$, $\mathfrak{P}$ and $\mathfrak{Q}$ are equivalently positioned. The conditions are evidently satisfied.

Conversely, let A, W satisfy the conditions. Since it is clear what to do about $\mathfrak{N}(A)$ and $\mathfrak{N}(1-A^2)$, let us for simplicity hereafter take them to be zero. That is, $\mathfrak{H}=\mathfrak{H}_+\oplus\mathfrak{H}_-$, where $\mathfrak{H}_+$ is the closure of the range of A^+, and $\mathfrak{H}_-$ that of A^-. Setting $V=W$ on $\mathfrak{H}_+$ and $V=W^*$ on $\mathfrak{H}_-$ determines V as an operator on $\mathfrak{H}$, which one may verify to be a symmetry satisfying $AV=-VA$. Notice that $V\leftrightarrow 1-A^2$.

Define operators

$$P=\frac{1}{2}(1+A+V(1-A^2)^{1/2}),\qquad Q=\frac{1}{2}(1-A+V(1-A^2)^{1/2}).$$

They are hermitian (since V is $-W$ would not do here). Their difference is A. The theorem is proved if they are shown to be idempotent. This is a simple matter of multiplying out, remembering the properties of V and A.

By the way, $V = X(P, Q)$ here (as one checks most easily by noting again that $A = P - Q$ implies $1 - A^2 = C$).

Theorem 6.2. *Let $\mathfrak{P}, \mathfrak{Q}$ be equivalently positioned, and $\mathfrak{Q}, \tilde{\mathfrak{P}}$ also equivalently positioned. Then with some F there exist matrix representations*

$$P = \begin{pmatrix} F & -(F\tilde{F})^{1/2} \\ -(F\tilde{F})^{1/2} & \tilde{F} \end{pmatrix}, \qquad Q = \begin{pmatrix} F & (F\tilde{F})^{1/2} \\ (F\tilde{F})^{1/2} & \tilde{F} \end{pmatrix}.$$

The condition $\frac{1}{2} \leqq F \leqq 1$ may be imposed; F is otherwise arbitrary; the spectral multiplicity function of F is then a complete set of unitary invariants for $\mathfrak{P}, \mathfrak{Q}$.

This is readily proved (independently of Theorems 5.1 and 6.1) by pursuing the study of the matrix representation (4.31), using $\mathfrak{Y}(P, Q)$ for $\mathfrak{Z}$ and $U(Q, \tilde{P})\,U(P, Q)$ for $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Details may be omitted. In this representation the other associated operators take simple forms too, for instance

$$U(P, Y) = \begin{pmatrix} F^{1/2} & -F^{1/2} \\ F^{1/2} & F^{1/2} \end{pmatrix}.$$

Slight modifications can avoid the special hypothesis about $\mathfrak{P}$ and $\mathfrak{Q}$, giving still another substitute set of invariants instead of Theorem 5.1.

The first part of the next theorem may be compared to known results [3], [11].

Theorem 6.3. *Unitary W is the product of some pair of symmetries if and only if* (i) *its spectrum is symmetric (multiplicity counted) with respect to the real axis. $W = U(P, Q)$ for some $\mathfrak{P}, \mathfrak{Q}$ if and only if its spectrum, beside having the above property* (i), *lies in the (closed) right half plane.*

Proof. In any case the spectrum of W^* is the image of that of W with respect to the real axis; so (i) is equivalent to saying W^* unitary equivalent to W. By much the same familiar argument as above, we may put $ZWZ = W^*$, Z a symmetry, without loss of generality. (Namely — if $Z'^* WZ' = W^*$, Z' unitary, then Z' exchanges $(i(W - W^*))^+ \mathfrak{H}$ with $(i(W - W^*))^- \mathfrak{H}$; set $Z = Z'^*$ on the first space, $Z = Z'$ on the second, and $Z = 1$ on $\mathfrak{N}(W - W^*)$.) But for unitary W and symmetry Z, the following are evidently equivalent:[7] $ZWZ = W^*$; $(ZW)^2 = 1$; $ZW = Z_1$ is a symmetry; $W = ZZ_1$ with Z and Z_1 both symmetries.

[7]) This computation has occurred in proving different results [9, Corollaries 3, 4].

The spectrum of W lies in the right half plane if and only if $W+W^* \geqq 0$. $U(P,Q)$ has this property (Lemma 4.3). Conversely, let a product of two symmetries, $W=(Q-\tilde{Q})(P-\tilde{P})$, have the property $0 \leqq P(W+W^*)P = 2P(Q-\tilde{Q})P$, that is $P\tilde{Q}P \leqq \frac{1}{2}P$, and similarly $\tilde{P}Q\tilde{P} \leqq \frac{1}{2}\tilde{P}$; adding, $S(P,Q) \leqq \frac{1}{2}$. By Theorems 4.5 and 4.4, there exists $\mathfrak{R}$ for which $W=U(R,Q)$.

7. Extremal properties of U. Not only does $U(P,Q)$ of § 3 carry $\mathfrak{P}$ onto $\mathfrak{Q}$ and $\tilde{\mathfrak{P}}$ onto $\tilde{\mathfrak{Q}}$, if $\mathfrak{P}$ and $\mathfrak{Q}$ are equivalently positioned, it does so "as economically as possible." The theorems of this section make this vague assertion precise. Throughout, W will mean any unitary operator such that $W\mathfrak{P}=\mathfrak{Q}$, $W\tilde{\mathfrak{P}}=\tilde{\mathfrak{Q}}$. Then the vague assertion might mean

$$\|1-U\| \leqq \|1-W\|. \tag{7.1}$$

Such theorems are of interest both because they emphasize the suitability of U for applications to perturbation theory [12, § 136], and because of their relevance to metrics [2].

The norm involved in the first theorem is the usual Hilbert norm or bound:

$$\|A\| = \sup\{\|Ax\|: \|x\|=1\}; \text{ for } A \geqq 0,\ \|A\| = \sup\{(Ax,x): \|x\|=1\}.$$

Theorem 7.1 $\|1-U\| \leqq \|1-W\|$.

Proof. It is enough to discuss $\|(1-W^*)(1-W)\|$, because it is equal to $\|1-W^*\|\cdot\|1-W\| = \|1-W\|^2$. Now

$$\begin{aligned}\|(1-W^*)(1-W)\| &\geqq \|P(1-W^*)(1-W)P\| = \|(1-W)P\|^2 \\ &= \sup_{\substack{\|x\|=1\\ x\in\mathfrak{P}}} \|x-Wx\|^2 \geqq \sup_{\substack{\|x\|=1\\ x\in\mathfrak{P}}} \inf_{\substack{\|y\|=1\\ y\in\mathfrak{Q}}} \|x-y\|^2.\end{aligned} \tag{7.2}$$

For each x, a minimizing unit vector y in the last expression is $Qx/\|Qx\|$[8]); I give the well-known proof. Suppose a unit vector $y_0 \in \mathfrak{Q}$ such that

$$\|x-y_0\|^2 < \left\|x - \frac{Qx}{\|Qx\|}\right\|^2.$$

Expanding gives $\operatorname{Re}(x,y_0) > (Qx,x)/\|Qx\| = \|Qx\|$. But the $y \in \mathfrak{Q}$ which minimizes $\|x-y\|^2$ without restriction on $\|y\|$ is of course $y=Qx$; in particular,

$$\|x - \|Qx\|y_0\|^2 \geqq \|x-Qx\|^2.$$

Expanding *this* gives $\operatorname{Re}(x,y_0) \leqq \|Qx\|$, a contradiction.

[8]) Unless $x \in \tilde{\mathfrak{Q}}$. But if such x exists the right-hand member of (7.2) is 2, and the rest of the proof is simplified. The uniqueness of the minimizing y is of no concern here.

In this derivation I mentioned that $\|Qx\|^2 = (Qx, x)$, since Q is a projection. I mention further that since $x \in \mathfrak{P}$,

$$\|Qx\|^2 = (Qx, x) = (PQPx, x) = (Cx, x) = \|C^{1/2}x\|^2.$$

The right-hand member of (7. 2) can now be rewritten as (x restricted as before)

$$\sup \left\| x - \frac{Qx}{\|Qx\|} \right\|^2 = \sup\ (2 - 2\|Qx\|) = \sup (2 - 2\|C^{1/2}x\|) =$$
$$= \sup (2 - 2\,(C^{1/2}x, x)) = \|P(2 - 2\,C^{1/2})\,P\| = \|2 - 2\,C^{1/2}\| = \|(1 - U^*)\,(1 - U)\|.$$

(The last equality is by Lemma 4. 3. The one before it is by Lemma 5. 2 (and $C \longleftrightarrow P$).) What has been proved is

$$\|1 - W\|^2 = \|(1 - W^*)\,(1 - W)\| \geqq \|(1 - U^*)\,(1 - U)\| = \|1 - U\|^2,$$

that is, the theorem.

Beside this norm,

(7. 31) $$\|A\| = \|A\|_{II} = \sup\{\|Ax\| \ : \ \|x\| = 1\},$$

certain other norms will be considered, such as the Frobenius norm

(7. 32) $$\|A\|_2 = (\mathrm{tr}\, A^* A)^{1/2}.$$

In the infinite-dimensional case the other norms do not in general exist; but (7. 1) would still make sense if one or both sides was infinite. Also questions concerning eigenvalues may be handled in some cases. For completely continuous hermitian $B \geqq 0$, denote the eigenvalues, multiplicity counted, by $\lambda_k(B)$, ordered as $\lambda_1(B) \geqq \lambda_2(B) \geqq \ldots$. In particular, when dim $\mathfrak{H} = n$ is finite, let Φ be any symmetric gauge function of n variables, and consider the norm

(7. 33) $$\|A\|_\Phi = \Phi\big(\lambda_1(|A|), \ldots, \lambda_n(|A|)\big),$$

where $|A| = (A^* A)^{1/2} \geqq 0$. This norm is unitary invariant: for any unitary V, $\|VA\|_\Phi = \|A\|_\Phi = \|AV\|_\Phi$. It is known [16], [13, pp. 84—88] that every unitary invariant norm is of this form.[9]) Included in (7. 33) are the norms

(7. 34) $$\|A\|_p = \big(\lambda_1(|A|)^p + \cdots + \lambda_n(|A|)^p\big)^{1/p}, \qquad p \geqq 1.$$

(7. 32) is the special case $p = 2$ and (7. 31) is the limiting case $p \to \infty$.

The eigenvalues are given, for hermitian $B \geqq 0$, by

(7. 4) $$\lambda_k(B) = \inf_{\mathfrak{N}_{k-1}} \sup_{\substack{\|x\|=1 \\ x \perp \mathfrak{N}_{k-1}}} (Bx, x).$$

Here $\mathfrak{N}_{k-1}$ denotes an arbitrary $k-1$-dimensional subspace [12, p. 235]. (7. 4)

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[9]) Normalize, as usual, by requiring one-dimensional projections to have norm 1.

may, and will, be used as the definition of λ_k even if B is not completely continuous, thus λ_k need not then be an eigenvalue. (All λ_k from some k on will be equal the max of the continuous spectrum.)

Before discussing alternate interpretations of (7. 1), I give a very strong extremal property of U.

Theorem 7. 2. $\lambda_k(P(1-U^*)(1-U)P) \leqq \lambda_k(P(1-W^*)(1-W)P)$.

Proof. Use the expression (7. 4), observing that the minimax certainly can be confined to $\mathfrak{P}$. Analogously to (7. 2),

$$\lambda_k(P(1-W^*)(1-W)P) \geqq \inf_{\mathfrak{M}_{k-1}\subseteq \mathfrak{P}} \sup_{\substack{\|x\|=1 \\ x\in \tilde{\mathfrak{M}}_{k-1}\cap \mathfrak{P}}} \inf_{\substack{\|y\|=1 \\ y\in \mathfrak{Q}}} \|x-y\|^2.$$

Reasoning parallel to that which proved the preceding theorem now proves the right-hand member equal to

$$\lambda_k(P(2-2C^{1/2})P) = \lambda_{2k}(2-2C^{1/2}) = \lambda_{2k}((1-U^*)(1-U)) = \\ = \lambda_k(P(1-U^*)(1-U)P).$$

This proves the theorem.

Theorem 7. 3. $\|1-U\|_2 \leqq \|1-W\|_2$.

Proof. In case of pure point spectrum, and in case the indicated sums converge,

$$\|1-W\|_2^2 = \mathrm{tr}((1-W^*)(1-W)) = \\ = \mathrm{tr}(P(1-W^*)(1-W)P) + \mathrm{tr}(\tilde{P}(1-W^*)(1-W)\tilde{P}),$$

and the W which minimizes this is U, by Theorem 7. 2. In case $\|1-U\|_2$ is infinite, the same reasoning shows that $\|1-W\|_2$ is also. The possibility remains that $\|1-W\|_2$ may be infinite but not $\|1-U\|_2$; this too is satisfactory.

Theorem 7. 4. *In the finite-dimensional case,* $\|1-U\|_p \leqq \|1-W\|_p$ *for* $p \geqq 2$.

Proof. $\|1-W\|_p^p = \|(1-W^*)(1-W)\|_{p/2}^{p/2}$, so it is more than enough to show that

$$\|(1-U^*)(1-U)\| \leqq \|(1-W^*)(1-W)\|$$

for every unitary invariant norm. For this, it is known [5, Thm. 4] to be sufficient to prove that $S_k((1-U^*)(1-U)) \leqq S_k((1-W^*)(1-W))$ for $k=1, \ldots, n$; where by definition $S_k = \Sigma_{j=1}^k \lambda_j$.

This general fact will be used [6, Thm. 3], [10, § 2]: for hermitian B,

$$S_k(B) \geqq S_k(PBP + \tilde{P}B\tilde{P}).$$

Though the following proof is written for even $k=2m$, the restriction is not essential.

$$S_{2m}((1-W^*)(1-W)) \geqq$$
$$\geqq \max_{l \leqq 2m} \{S_l(P(1-W^*)(1-W)P)+S_{2m-l}(\tilde{P}(1-W^*)(1-W)\tilde{P})\} \geqq$$
$$\geqq S_m(P(1-W^*)(1-W)P)+S_m(\tilde{P}(1-W^*)(1-W)\tilde{P}).$$

But taking $W=U$ minimizes the right-hand member (Theorem 7. 2) and also makes both inequalities become equalities.

This completes the positive work of § 7, but I add remarks on some natural conjectures.

I see no reason to doubt that Theorem 7. 4 is true for dim $\mathfrak{H}$ infinite.[10]) Then Theorem 7. 2 would become a special case, Theorem 7. 1 a limiting case.

The bound on p in Theorem 7. 4 cannot be improved. In 2-space, let $\mathfrak{P}$ and $\mathfrak{Q}$ be nearly orthogonal 1-subspaces; $\|1-U\|_p^p$ is nearly $2(2^{p/2})$, and if $p<2$ this can be greater than $\|1-X\|_p^p$.

However, it is a plausible conjecture that if $\mathfrak{P}$ and $\mathfrak{Q}$ are assumed to be close in the sense that $S(P, Q) \leqq \frac{1}{2}$ then (7. 1) holds for all unitary invariant norms.

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[10]) For the norm constructed like $\| \ \|_p$ using only $\lambda_1, \ldots, \lambda_N$ in (7. 34), with some finite N, the analog of Theorem 7. 4 is true in infinite-dimensional space. Some further generalizations of the same character are possible.

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Added in proof: See also

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