Note on sums of almost orthogonal operators.

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M. COTLAR has recently developed an interesting „unified theory" of Hilbert transforms and ergodic theorems [1]. He makes essential use of the following theorem:

Let $T_1+T_2+\cdots+T_n$ be a sum of permutable Hermitian operators in Hilbert space, satisfying the conditions

(1) $$\|T_i\| \leqq 1, \qquad \|T_i T_j\| \leqq \varepsilon^{|j-i|} \qquad (1 \leqq i,\ j \leqq n)$$

where $0 \leqq \varepsilon < 1$. Then we have

(2) $$\|T_1+T_2+\cdots+T_n\| \leqq c(\varepsilon)$$

where $c(\varepsilon)$ is a finite constant not depending on n and the particular choice of the T_i.

In the case $\varepsilon=0$ we have $T_i T_j = O$ for $i \neq j$, i. e. the terms of the sum are "orthogonal". In the case $\varepsilon>0$ we may say that the terms are "almost orthogonal", the quantity ε being a measure for the discrepancy from strict orthogonality. Putting $S=T_1+T_2+\cdots+T_n$ we have, in the case $\varepsilon=0$, $S^k=T_1^k+T_2^k+\cdots+T_n^k$ for $k=1,2,\ldots$, thus

$$\|S^k\| \leqq n, \qquad \lim_{k\to\infty} \|S^k\|^{\frac{1}{k}} \leqq \lim_{k\to\infty} n^{\frac{1}{k}} = 1.$$

Since S is hermitian, we have $\|S^k\|=\|S\|^k$, thus $\|S\| \leqq 1$, i. e. for $\varepsilon=0$ the theorem is true with $c(0)=1$, and this is obviously the best constant.

For a proof in the case $\varepsilon>0$ reference is made to a previous paper of the same author [2], where it is based on a rather complicated combinatorial problem. We shall give here a very simple proof by reducing the problem to the one-dimensional case i. e. to a problem on scalars.

This reduction is made possible by the following well-known representation theorem for commutative operator algebras (see f. i. [3]): Any commutative algebra A, with real scalars, of bounded Hermitian operators T in Hilbert space, may be mapped isomorphically and isometrically into the algebra of all real-valued continuous functions $u(M)$ on some compact Hausdorff space $\mathfrak{M}$, i. e. in such a way that if T, $T' \in A$ and

$$T \to u(M), \quad T' \to u'(M),$$

then $cT \to cu(M)$ for any real scalar c, $T + T' \to u(M) + u'(M)$, $TT' \to u(M)u'(M)$, and

$$\|T\| = \sup_{M \in \mathfrak{M}} |u(M)|.$$

Choose in particular A as the (commutative) algebra with real scalars generated by the operators $T_1, \ldots, T_n$. If $T_i \to u_i(M)$ $(i = 1, \ldots, n)$, then the hypotheses (1) mean that

$$|u_i(M) \leqq 1, \quad |u_i(M)u_j(M)| \leqq \varepsilon^{|j-i|}$$

for all $M \in \mathfrak{M}$, and we have to show that these imply

$$|u_1(M) + \cdots + u_n(M)| \leqq c(\varepsilon)$$

for all $M \in \mathfrak{M}$.

We shall show even more, namely that for any sum

$$s = v_1 + \cdots + v_n$$

of real numbers with $|v_i| \leqq 1$, $|v_i v_j| \leqq \varepsilon^{|j-i|}$, we have $|s| \leqq c(\varepsilon)$.

For any real number $\lambda \geqq 0$ denote by $n(\lambda)$ the number of those terms v_i in this sum for which $|v_i| \geqq \lambda$. Then we have

$$\sum_1^n |v_i| = -\int_0^{1+0} \lambda\, dn(\lambda) = -[\lambda n(\lambda)]_0^{1+0} + \int_0^1 n(\lambda)\, d\lambda = \int_0^1 n(\lambda) d(\lambda),$$

because $n(\lambda) = 0$ for $\lambda > 1$. If, for a fixed λ, $n(\lambda)$ is $\geqq 1$, denote by i_1 and i_2 the first and the last indices i for which $|v_i| \geqq \lambda$; we have obviously

$$n(\lambda) \leqq 1 + i_2 - i_1.$$

On the other hand we have

$$\lambda^2 \leqq |v_{i_1} v_{i_2}| \leqq \varepsilon^{i_2 - i_1},$$

thus

$$i_2 - i_1 \leqq 2 \log \lambda / \log \varepsilon, \quad \text{i. e.} \quad i_2 - i_1 \leqq a(\lambda) = [2 \log \lambda / \log \varepsilon],$$

where $[\alpha]$ denotes the greatest integer $\leqq \alpha$.

So it results the inequality

$$n(\lambda) \leqq 1 + a(\lambda)$$

and this is valid for $0 \leqq \lambda \leqq 1$ obviously also if $n(\lambda) = 0$. Thus we have

$$|s| \leqq \sum_1^n |v_i| \leqq \int_0^1 (1 + a(\lambda))\, d\lambda = 1 + \sum_1^\infty m\left(\varepsilon^{\frac{m}{2}} - \varepsilon^{\frac{m+1}{2}}\right),$$

since $a(\lambda)$ assumes the value m on the interval $\varepsilon^{\frac{m+1}{2}} < \lambda \leqq \varepsilon^{\frac{m}{2}}$ $(m = 0, 1, \ldots)$.

This achieves the proof, yielding the following value of the constant:

$$c(\varepsilon)=1+(1-\sqrt{\varepsilon})\sum_{1}^{\infty} m(\sqrt{\varepsilon})^m=1+\frac{(1-\sqrt{\varepsilon})\sqrt{\varepsilon}}{(1-\sqrt{\varepsilon})^2}=\frac{1}{1-\sqrt{\varepsilon}}=\frac{1+\sqrt{\varepsilon}}{1-\varepsilon}.$$

The problem of the best constant is left open. Considering the sums

$$S_{2n+1}=\sum_{\nu=-n}^{n}\varepsilon^{|\nu|}=\frac{1+\varepsilon}{1-\varepsilon}-2\frac{\varepsilon^n}{1-\varepsilon}$$

one sees, however, that the best constant $c^*(\varepsilon)$ must satisfy the inequality

$$\frac{1+\varepsilon}{1-\varepsilon}\leqq c^*(\varepsilon)\leqq\frac{1+\sqrt{\varepsilon}}{1-\varepsilon},$$

in particular

$$3\leqq c^*\left(\frac{1}{2}\right)\leqq 2+\sqrt{2}<3{,}5.$$

$\Big($M. Cotlar shows only that $c^*\left(\frac{1}{2}\right)\leqq 8.\Big)$

References.

[1] M. Cotlar, A unified theory of Hilbert transforms and ergodic theorems, *Revista Mat. Cuyana*, **1** (1955), 105—167.
[2] M. Cotlar, A combinatorial lemma and its applications to L^2-spaces, *ibidem*, **1** (1955), 41—55.
[3] I. Gelfand, Normierte Ringe, *Mat. Sbornik*, **9** (51) (1941), 3—23.

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