On the compound Poisson distribution.

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A probability distribution is called a compound Poisson distribution if its characteristic function can be represented in the form

$$(1)\qquad \varphi(u)=\exp\left\{i\gamma u+\int_{-\infty}^{0}(e^{iux}-1)\,dM(x)+\int_{0}^{\infty}(e^{iux}-1)\,dN(x)\right\}$$

where γ is a constant, $M(x)$ and $N(x)$ are defined on the intervals $(-\infty, 0)$ and $(0, \infty)$, respectively, both are monotone non-decreasing, $M(-\infty)=$ $=N(\infty)=0$, further the integrals

$$\int_{-1}^{0} x\,dM(x),\quad \int_{0}^{1} x\,dN(x)$$

exist. We shall prove that under certain conditions we obtain (1) as a limit distribution of double sequences of independent and infinitesimal random variables and apply this theorem to stochastic processes with independent increments.

Theorem 1. *Let $\xi_{n1}, \xi_{n2}, \ldots, \xi_{nk_n}$ $(n=1, 2, \ldots)$ be a double sequence of random variables. Suppose that the random variables in each row are independent, they are infinitesimal, i.e. for every $\varepsilon>0$*

$$\lim_{n\to\infty}\max_{1\leqq k\leqq k_n}\mathbf{P}(|\xi_{nk}|>\varepsilon)=0,$$

finally, there exists a finite-valued, non-negative random variable η such that

$$\sum_{k=1}^{k_n}|\xi_{nk}|\leqq\eta \qquad (n=1,2,\ldots)$$

with probability 1. (This last condition means that the sums of the absolute values of the sample summands are uniformly bounded.) Suppose, moreover, that the sequence of probability distributions of the variables

$$\zeta_n=\xi_{n1}+\xi_{n2}+\cdots+\xi_{nk_n}$$

converges to a limiting distribution. Then this is a compound Poisson distribution.

Proof. Let us define the functions $f^+(x), f^-(x)$ as follows:

$$f^+(x)=\begin{cases} x & \text{if} \quad x \geqq 0, \\ 0 & \text{if} \quad x<0, \end{cases} \qquad f^-(x)=\begin{cases} 0 & \text{if} \quad x \geqq 0, \\ x & \text{if} \quad x<0. \end{cases}$$

Clearly $f^+(x)f^-(x) \equiv 0$ and

$$\zeta_n^+ = \sum_{k=1}^{k_n} f^+(\xi_{nk}) \leqq \eta \qquad (n=1,2,\ldots),$$

$$-\zeta_n^- = -\sum_{k=1}^{k_n} f^-(\xi_{nk}) \leqq \eta \qquad (n=1,2,\ldots),$$

with probability 1. Hence it follows that for every $K>0$ the relations

$$\mathbf{P}(\zeta_n^+ \geqq K) \leqq \mathbf{P}(\eta \geqq K) \qquad (n=1,2,\ldots),$$
$$\mathbf{P}(\zeta_n^- \leqq -K) \leqq \mathbf{P}(\eta \geqq K) \qquad (n=1,2,\ldots)$$

hold. This imply that the distributions of the sequences ζ_n^+ and ζ_n^- are compact sets. Let $F_n^+(x)$ and $F_n^-(x)$ denote the distribution functions of the variables ζ_n^+ and ζ_n^-, respectively. Let us choose a sequence of integers $n_1, n_2, \ldots$ for which

$$\text{(2)} \qquad \begin{aligned} &\lim_{i\to\infty} F_{n_i}^+(x) = F^+(x), \\ &\lim_{i\to\infty} F_{n_i}^-(x) = F^-(x) \end{aligned}$$

(where $F^+(x)$ and $F^-(x)$ are distribution functions) at every point of continuity of the latters. Let τ be a positive number such that the functions $F^+(x)$ and $F^-(x)$ are continuous at τ and $-\tau$, respectively. Since the random variables in the double sequences

$$f^+(\xi_{n1}), f^+(\xi_{n2}), \ldots, f^+(\xi_{nk_n}),$$
$$f^-(\xi_{n1}), f^-(\xi_{n2}), \ldots, f^-(\xi_{nk_n})$$

are infinitesimal and independent in each row, moreover the relations (2) hold, we conclude that if $F_{nk}^+(x) = \mathbf{P}(f^+(\xi_{nk})<x)$, $F_{nk}^-(x) = \mathbf{P}(f^-(\xi_{nk})<x)$, then the sequences

$$\sum_{k=1}^{k_{n_i}} \int_{0<x<\tau} x\,dF_{n_ik}^+(x), \qquad \sum_{k=1}^{k_{n_i}} \int_{-\tau<x<0} x\,dF_{n_ik}^-(x)$$

are convergent (see [1] § 25, Theorem 4, Remark). This implies that

$$\lim_{i\to\infty} \sum_{k=1}^{k_{n_i}} \left(\int_{0<x<\tau} x\,dF_{n_ik}^+(x) \right)^2 = \lim_{i\to\infty} \sum_{k=1}^{k_{n_i}} \left(\int_{-\tau<x<0} x\,dF_{n_ik}^-(x) \right)^2 = 0.$$

Thus if

$$\varphi_{nk}^{+}(t)=\int_{-\infty}^{\infty} e^{itx}\,dF_{nk}^{+}(x),\qquad \varphi_{nk}^{-}(t)=\int_{-\infty}^{\infty} e^{itx}\,dF_{nk}^{-}(x),$$

then from the inequality

$$\left|\int_{-\infty}^{\infty}(e^{itx}-1)\,dG(x)\right|\leqq|t|\int_{|x|<\tau}|x|\,dG(x)+2\int_{|x|\geqq\tau}dG(x),$$

valid for every distribution function $G(x)$ and every $\tau>0$, it follows (using Theorem 4 of [1] § 25) that

$$\lim_{i\to\infty}\sum_{k=1}^{k_{n_i}}|\varphi_{n_ik}^{+}(t)-1|^2=\lim_{i\to\infty}\sum_{k=1}^{k_{n_i}}|\varphi_{n_ik}^{-}(t)-1|^2=0.$$

Hence the conditions of Theorem 2 of [2] are fulfilled and thus the variables $\zeta_{n_i}^{+}$ and $\zeta_{n_i}^{-}$ are asymptotically independent, i. e.

$$\lim_{i\to\infty}\mathbf{P}(\zeta_{n_i}^{+}<x,\ \zeta_{n_i}^{-}<y)=F^{+}(x)\,F^{-}(y). \tag{3}$$

Let $F(x)$ denote the limiting distribution of the random variables ζ_n. Since $\zeta_n=\zeta_n^{+}+\zeta_n^{-}$, we get from (3)

$$F(x)=F^{+}(x)*F^{-}(x). \tag{4}$$

The laws $F(x), F^{+}(x), F^{-}(x)$ are infinitely divisible. In LÉVY's formula

$$i\gamma u-\frac{\sigma^2u^2}{2}+\int_{-\infty}^{0}\left(e^{iux}-1-\frac{iux}{1+x^2}\right)dM(x)+\int_{0}^{\infty}\left(e^{iux}-1-\frac{iux}{1+x^2}\right)dN(x)$$

there correspond to $F(x), F^{+}(x)$ and $F^{-}(x)$ constants and functions, which we denote by $\gamma', \gamma_1, \gamma_2;\ \sigma^2, \sigma_1^2, \sigma_2^2;\ M(x), M_1(x), M_2(x);\ N(x), N_1(x), N_2(x)$, respectively. According to (4)

$$\gamma'=\gamma_1+\gamma_2,\ \sigma^2=\sigma_1^2+\sigma_2^2,$$
$$M(x)=M_1(x)+M_2(x),\qquad N(x)=N_1(x)+N_2(x).$$

If $\sigma^2>0$, then at least one of σ_1^2 and σ_2^2 is positive too. This is, however, impossible, since $F^{+}(x)=0$ if $x\leqq 0$ and $F^{-}(x)=1$ if $x>0$.

We have therefore only to prove that the integrals

$$\int_{-1}^{0}x\,dM(x),\quad \int_{0}^{1}x\,dN(x)$$

exist. We prove the existence of the second integral, the existence of the first one can be proved similarly. We know that if τ_1 is a point of continuity of $N(x)$, then

$$\int_0^{\tau_1} x\, d \sum_{k=1}^{k_{n_i}} F_{n_i k}(x) \tag{5}$$

converges ([1], § 25, Theorem 4) hence it is bounded. If

$$\int_0^1 x\, dN(x) = \infty$$

then we can choose such a number τ $(0 < \tau < \tau_1)$ that

$$\int_\tau^{\tau_1} x\, dN(x) > L, \tag{6}$$

where L is the upper bound of the terms in the sequence (5) and $N(x)$ is continuous at the point τ. But we know from the limiting distribution theorems (cf. [1] § 25, Theorem 4) that

$$\lim_{i\to\infty} \sum_{k=1}^{k_{n_i}} (F_{n_i k}(x) - 1) = N(x) \qquad (x > 0)$$

at every point of continuity of $N(x)$, whence

$$\lim_{i\to\infty} \int_\tau^{\tau_1} x\, d \sum_{k=1}^{k_{n_i}} F_{n_i k}(x) = \int_\tau^{\tau_1} x\, dN(x). \tag{7}$$

Obviously (6) and (7) contain a contradiction.

Let us separate in LÉVY's formula the terms

$$iu \int_{-\infty}^{0} \frac{x}{1+x^2}\, dM(x), \quad iu \int_0^{\infty} \frac{x}{1+x^2}\, dN(x)$$

and unite them with $i\gamma' u$, then we obtain the required form of the limiting distribution. Thus our theorem is completely proved.

In the sequel we apply our result to the theory of stochastic processes with independent increments. We say that a stochastic process with independent increments ξ_t is weakly continuous if for every $\varepsilon > 0$

$$\mathbf{P}(|\xi_{t+h} - \xi_t| > \varepsilon) \to 0$$

when $h \to 0$, uniformly in t. We suppose that $\mathbf{P}(\xi_0 = 0) = 1$.

Theorem 2. *Let us suppose that the stochastic process with independent increments ξ_t is weakly continuous and its sample functions are of bounded variation with probability* 1 *in every finite time interval. If $\varphi(u, t)$ is the* *characteristic function of the random variable* ξ_t *then it has the form*

$$(8)\qquad \varphi(u,t)=\exp\left\{i\gamma(t)u+\int_{-\infty}^{0}(e^{iux}-1)\,dM(x,t)+\int_{0}^{\infty}(e^{iux}-1)\,dN(x,t)\right\}$$

where $\gamma(t)$ *is a continuous function of bounded variation in every finite time interval,* $M(x,t)$ *and* $N(x,t)$ *are continuous functions of the variable* t *and the integrals*

$$\int_{-1}^{0}x\,dM(x,t),\qquad \int_{0}^{1}x\,dN(x,t)$$

exist for every t.

P r o o f. According to our suppositions the double sequence of independent random variables

$$\xi_{\frac{t}{n}},\ \xi_{\frac{2t}{n}}-\xi_{\frac{t}{n}},\ \ldots,\ \xi_t-\xi_{\frac{n-1}{n}t}$$

satisfies all the conditions of Theorem 1. Moreover, for every n

$$\xi_t=\sum_{k=1}^{n}\left(\xi_{\frac{k}{n}t}-\xi_{\frac{k-1}{n}t}\right),$$

hence we have only to prove the assertion regarding the functions $\gamma(t)$, $M(x,t)$, $N(x,t)$. The continuity in t of these functions follows at once from the weak continuity of the process ξ_t and the convergence theorems of infinitely divisible distributions (see e.g. [1] Chapter 3).

Now we show that for every $T>0$ $\gamma(t)$ is of bounded variation in the interval $0\leqq t\leqq T$. Let us consider the sequence of subdivisions

$$I_k^{(n)}=\left[\frac{k-1}{2^n}T,\ \frac{k}{2^n}T\right]\qquad (k=1,2,\ldots,2^n;\ n=1,2,\ldots)$$

of the interval $[0,T]$ and let us denote the distribution function of the random variable $\xi_{\frac{k}{2^n}T}-\xi_{\frac{k-1}{2^n}T}$ by $F(x,I_k^{(n)})$. We know from the limiting distribution theorems that

$$\gamma\left(\frac{k}{2^n}T\right)-\gamma\left(\frac{k-1}{2^n}T\right)+\int_{-\tau}^{0}x\,d\left(M\left(x,\frac{k}{2^n}T\right)-M\left(x,\frac{k-1}{2^n}T\right)\right)+$$

$$+\int_{0}^{\tau}x\,d\left(N\left(x,\frac{k}{2^n}T\right)-N\left(x,\frac{k-1}{2^n}T\right)\right)=\lim_{N\to\infty}\sum_{I_j^{(N)}\subseteq I_k^{(n)}}\ \int_{|x|<\tau}x\,dF(x,I_j^{(N)})$$

(cf. [1] § 25, Theorem 4), hence

$$(9)\qquad \sum_{k=1}^{2^n}\left|\gamma\left(\frac{k}{2^n}T\right)-\gamma\left(\frac{k-1}{2^n}T\right)\right| \leqq \int_0^{\varepsilon} x\,dN(x,T)-\int_{-\varepsilon}^{0} x\,dM(x,T)+$$

$$+\lim_{N\to\infty}\sum_{k=1}^{2^N}\int_{|x|<\varepsilon}|x|\,dF(x,I_k^{(N)}).$$

The boundedness of the sequence on the right-hand side of (9) is a consequence of the fact that the non-decreasing sequence

$$\sum_{k=1}^{2^N}\left|\xi_{\frac{k}{2^N}T}-\xi_{\frac{k-1}{2^N}T}\right|$$

converges with probability 1, and of Theorem 4 of [1] § 25. Since $\gamma(t)$ is continuous, this implies that it is of bounded variation. Thus Theorem 2 is proved.

Bibliography.

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